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CONSTRAINED CONTROLLABILITY OF DYNAMIC SYSTEMS

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The present paper is devoted to a study of constrained controllability and controllability for linear dynamic systems if the controls are taken to be non-negative. By analogy to the usual definition of controllability it is possible to introduce the concept of positive controllability. We shall concentrate on approximate positive controllability for linear infinite-dimensional dynamic systems when the values of controls are taken from a positive closed convex cone and the operator of the system is normal and has pure discrete point spectrum. Special attention is paid to positive infinite-dimensional linear dynamic systems. General approximate constrained controllability results are then applied to distributed-parameter dynamic systems described by linear partial-differential equations of parabolic type with various kinds of boundary conditions. Several remarks and comments on the relationships between different concepts of controllability are given. Finally, a simple illustrative example is also presented.

Keywords: constrained controllability, infinite-dimensional systems, distributed-parameter systems, positive dynamic systems.

1. Introduction

Controllability is one of the fundamental concepts in mathematical control theory (Bensoussan *et al.*, 1993; Klamka, 1982; 1991; 1992; 1993a; 1993b). Roughly speaking, controllability generally means that it is possible to steer a dynamic system from an arbitrary initial state to an arbitrary final state using a set of admissible controls. In the literature there are many different definitions of controllability which depend on a given class of dynamic systems (Bensoussan *et al.*, 1993; Klamka, 1991; 1993a; 1993b; Schanbacher, 1989; Triggiani, 1975a; 1976; 1978).

Controllability problems for linear control systems defined in infinite-dimensional Banach spaces have attracted an intense interest over the past twenty years. For infinite-dimensional dynamic systems it is necessary to distinguish between the notions of approximate and exact controllability (Bensoussan *et al.*, 1993; Klamka, 1991; 1992; 1993b; McGlothin, 1978; Triggiani, 1975a; 1975b; 1976; 1977; 1978). This results directly from the fact that in infinite-dimensional spaces there exist linear subspaces which are not closed.

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So far, most of the literature in this area has been concerned, however, with unconstrained controllability, and little is known as for the case when the control is restricted to take on values in a given subset of the control space. Until now, scarce attention has been paid to the important case where the control of a system are non-negative. In this case controllability is possible only if the system is oscillating in some sense. Therefore, the most difficult case for constrained controllability is for dynamic systems with real eigenvalues (Son, 1990).

The present paper is devoted to a study of constrained approximate controllability (Peichl and Schappacher, 1986; Son, 1990) for linear normal infinite-dimensional dynamic systems if the controls are to be non-negative. By analogy to the usual definition of controllability it is possible to introduce the concept of approximate positive controllability (Schanbacher, 1989). For such dynamic systems a direct verification of constrained approximate controllability is rather difficult and complicated (Peichl and Schappacher, 1986). Therefore, we generally assume that the values of controls are taken from a positive closed convex cone (Son, 1990) and the operator of the system is normal and has pure discrete point spectrum (Triggiani, 1975a; 1976). Special attention is paid to positive infinite-dimensional linear dynamic systems, i.e. to dynamic systems preserving positivity (Schanbacher, 1989).

General constrained approximate controllability results are then applied to general distributed parameter dynamic systems described by linear partial-differential equations of parabolic type with various kinds of boundary conditions. Finally, as a simple illustrative example, the constrained approximate controllability of a one-dimensional heat equation with homogeneous Dirichlet boundary conditions and scalar non-negative control is also considered.

2. Notation and System Description

Throughout this paper we use X to denote an infinite-dimensional separable real Hilbert space. By $L^p([0,t],\mathbb{R}^m)$, $1 \le p \le \infty$ we denote the space of all p-integrable functions on $[0,t]$ with values in $\mathbb{R}^m$ and by $L^p_\infty([0,\infty),\mathbb{R}^m)$ the space of all locally p-integrable functions on $[0,\infty)$ with values in $\mathbb{R}^m$.

Following (Schanbacher, 1989; Smith, 1995) we define an order $\le$ in the space X such that $(X,\le)$ is a lattice and the ordering is compatible with the structure of X, i.e. X is an ordered vector space. This implies that the set $X^+ = \{x \in X : x \ge 0\}$ is a convex positive cone with vertex at zero. Moreover, it follows that $x_1 \le x_2$ if and only if $x_2 - x_1 \in X^+$. An element $x \in X^+$ is called positive and we write $x > 0$ if x is positive and different from zero. Moreover, an element $x^* \in X^+$ is called strictly positive, and we write $x^* \gg 0$, if $\langle x^*, x\rangle_X > 0$ for all $x > 0$. An ordered vector space X is called a vector lattice if any two elements x_1, x_2 in X have a supremum and an infimum denoted by $\sup\{x_1,x_2\}$ and $\inf\{x_1,x_2\}$ respectively. For an element x of the vector lattice we write $|x| = \sup\{x,-x\}$ and call it the absolute value of x. We call two elements x_1, x_2 of the vector lattice X orthogonal if $\inf\{|x_1|,|x_2|\} = 0$. A linear form $w \in X$ is called positive $(w \ge 0)$ if $\langle w,x\rangle_X \ge 0$ for all $x \ge 0$ and strictly positive $(w \gg 0)$ if $\langle w,x\rangle_X > 0$ for all $x > 0$. Relevant examples of vector lattices with a strictly positive linear form are given by the following spaces of practical interest: $\mathbb{R}^n$ and $L^2(\Omega, \mathbb{R})$, where Ω is a measurable subset of $\mathbb{R}^n$, with standard order relations (Smith, 1995).

A bounded linear operator F from a vector lattice X into a vector lattice V is called positive, i.e. $F \geq 0$, if $Fx \geq 0$ for $x \geq 0$. Therefore, a positive operator F maps the positive cone X^+ into the positive cone V^+. Let $S(t) : X \to X$, $t \geq 0$ be a strongly continuous semigroup of bounded linear operators. We call the semigroup positive i.e. $S \geq 0$, if X is a vector lattice and $S(t)$ is a positive linear operator for every $t \geq 0$.

For a set $M \subseteq X$, we define the polar cone by $M^0 = \{w \in X, \langle w, x\rangle_X \leq 0$ for all $x \in M\}$. The closure, convex hull and interior are denoted by $\mathrm{cl}\, M$, $\mathrm{co}\, M$ and $\mathrm{int}\, M$, respectively.

Let us consider linear a infinite-dimensional time-invariant control system of the following form:

$$x'(t) = Ax(t) + Bu(t) \tag{1}$$

Here $x(t) \in X$, the latter being an infinite-dimensional separable Hilbert space which constitutes a vector lattice with a strictly positive linear form.

Let B be a bounded linear operator from $\mathbb{R}^m$ into X. Therefore $B = [b_1, b_2, \dots, b_j, \dots, b_m]$ and

$$Bu(t) = \sum_{j=1}^{m} b_j u_j(t)$$

where $b_j \in X$, $j = 1, 2, \dots, m$ and $u(t) = [u_1(t), u_2(t), \dots, u_j(t), \dots, u_m(t)]^{\mathrm{tr}}$.

We would like to emphasize that the assumption that the linear operator B is bounded, rules out the application of our theory to boundary control problems, because in this situation B is typically unbounded.

Let $A : X \supset D(A) \to X$ be a normal, generally unbounded, linear operator with compact resolvent $R(s, A)$ for all s in the resolvent set $\rho(A)$. Then we have the following properties (Bensoussan *et al.*, 1993; Klamka, 1991; Triggiani, 1976; 1978):

1) A has only a pure discrete point spectrum $\rho_p(A)$ consisting entirely of isolated eigenvalues s_i, $i = 1, 2, \dots$. Moreover, each eigenvalue s_i has a finite multiplicity $n_i < \infty$, $i = 1, 2, \dots$ equal to the dimensionality of the corresponding eigenmanifold.

2) The eigenvectors $x_{ik} \in D(A)$, $i = 1, 2, \dots$, $k = 1, 2, \dots, n_i$ form a complete orthonormal set in the separable Hilbert space X.

3) A generates an analytic semigroup of bounded linear operators $S(t) : X \to X$, for $t > 0$.

Let $U^+ \subset \mathbb{R}^m$ be a positive cone in $\mathbb{R}^m$, i.e. $U^+ = \{u \in \mathbb{R}^m : u_j \geq 0$ for $j = 1, 2, \dots, m\}$. We define the set of admissible non-negative controls U_{ad} as follows:

$$U_{\mathrm{ad}} = \{u \in L^p_{\mathrm{loc}}([0, \infty), \mathbb{R}^m);\ u(t) \in U^+ \ \text{ a.e. on } [0, \infty)\}$$

It is well-known (Bensoussan *et al.*, 1993; Klamka, 1991; Triggiani, 1976) that for each $u \in U_{\rm ad}$ and $x(0) \in X$ there exists a unique so-called mild solution $x(t, x(0), u) \in D(A)$, $t \geq 0$ of eqn. (1) given by

$$x\big(t, x(0), u\big) = S(t)x(0) + \int_0^t S(t-s)Bu(s)\,\mathrm{d}s$$

We say that the dynamic system (1) is positive if the semigroup $S(t)$ and operator B are positive (Schanbacher, 1989). In this case the solution $x(t, x(0), u)$ for an initial condition $x(0) \in X^+$ and an admissible control $u \in U_{\rm ad}$ remains in X^+ for all $t \geq 0$.

We define the attainable or reachable set in time T (from the origin) by

$$K_T(U^+) = \left\{ \int_0^T S(T-s)Bu(s)\,\mathrm{d}s, \quad u \in U_{\rm ad} \right\}$$

The set $K_\infty(U^+) = \bigcup_{T>0} K_T(U^+)$ is called the attainable or reachable set in finite time.

Using the concept of the attainable set, we may define various kinds of controllability for the dynamic system (1). For infinite-dimensional dynamic systems it is necessary to introduce two fundamental notions of controllability, namely exact (strong) controllability and approximate (weak) controllability. However, since our dynamic system has an infinite-dimensional state space X and a finite-dimensional control space $\mathbb{R}^m$, then it is never exactly controllable in any sense (Triggiani, 1975b; 1977). Therefore, in the sequel we shall concentrate only on the approximate controllability with positive controls for (1).

Definition 1. (Bensoussan *et al.*, 1993; Klamka, 1991; 1993b). The dynamic system (1) is said to be approximately controllable with non-negative controls if $\mathrm{cl}\, K_\infty(U^+) = X$.

In the unconstrained case, i.e. when the control values are taken from the whole space $\mathbb{R}^m$, we simply say about the approximate controllability of the dynamic system (1).

The above notion of approximate controllability is defined in the sense that we want to reach a dense subspace of the entire state space. However, in many instances for positive systems with non-negative controls, it is known that all states are contained in a closed positive cone X^+ of the state space. In this case approximate controllability in the sense of the above definition is impossible, but it is interesting to know conditions under which the reachable states are dense in X^+. This observation leads to the concept of the so-called positive approximate controllability.

Definition 2. (Schanbacher, 1989) The dynamic system (1) is said to be approximately positive controllable if $\mathrm{cl}\, K_\infty(U^+) = X^+$.

Remark 1. From the above two definitions it follows directly that the approximate controllability with non-negative controls always implies approximate positive controllability. However, in general, the converse statement is not true.

Finally, we shall recall some fundamental theorems concerning the unconstrained and constrained approximate controllability of the dynamic system (1).

Using eigenvectors x_{ik}, $i = 1, 2, \ldots$, $k = 1, 2, \ldots, n_i$ we introduce for the operator B the following notation (Klamka, 1993b; Triggiani, 1976):

$$B_i = \left| \begin{array}{cccccc} \langle b_1, x_{i1} \rangle_X & \langle b_2, x_{i1} \rangle_X & \cdots & \langle b_j, x_{i1} \rangle_X & \cdots & \langle b_m, x_{i1} \rangle_X \\ \langle b_1, x_{i2} \rangle_X & \langle b_2, x_{i2} \rangle_X & \cdots & \langle b_j, x_{i2} \rangle_X & \cdots & \langle b_m, x_{i2} \rangle_X \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \langle b_1, x_{ik} \rangle_X & \langle b_2, x_{ik} \rangle_X & \cdots & \langle b_j, x_{ik} \rangle_X & \cdots & \langle b_m, x_{ik} \rangle_X \\ \langle b_1, x_{in_i} \rangle_X & \langle b_2, x_{in_i} \rangle_X & \cdots & \langle b_j, x_{in_i} \rangle_X & \cdots & \langle b_m, x_{in_i} \rangle_X \end{array} \right|$$

B_i, $i = 1, 2, \ldots$ are $n_i \times m$-dimensional constant matrices which play an important role in controllability investigations (Klamka, 1991; 1993b; Son, 1990; Triggiani, 1976; 1978). For the case when the eigenvalues s_i are simple, i.e. $n_i = 1$, $i = 1, 2, \ldots$, b^i's are m-dimensional row vectors of the following simple form:

$$b^i = \left[\langle b_1, x_i \rangle_X, \langle b_2, x_i \rangle_X, \ldots, \langle b_j, x_i \rangle_X, \ldots, \langle b_m, x_i \rangle_X \right] \qquad \text{for } i = 1, 2, \ldots$$

For simplicity, let us write $b_{ikj} = \langle b_j, x_{ik} \rangle_X$ for $i = 1, 2, \ldots$, $k = 1, 2, \ldots, n_i$ and $j = 1, 2, \ldots, m$. Therefore we may express matrices B_i and vectors b^i in a more convenient form:

$$B_i = \left| \begin{array}{cccccc} b_{i11} & b_{i12} & \cdots & b_{i1j} & \cdots & b_{i1m} \\ b_{i21} & b_{i22} & \cdots & b_{i2j} & \cdots & b_{i2m} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ b_{ik1} & b_{ik2} & \cdots & b_{ikj} & \cdots & b_{ikm} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ b_{in_i1} & b_{in_i2} & \cdots & b_{in_ij} & \cdots & b_{in_im} \end{array} \right| \qquad \text{for } i = 1, 2, \ldots$$

$$b^i = \left[b_{i1},\ b_{12},\ \ldots,\ b_{ij},\ \ldots,\ b_{im} \right] \qquad \text{for } i = 1, 2, \ldots$$

Since the operator A is normal, then using the above notation it is possible to express the solution $x(t, x(0), u)$ as follows:

$$x\big(t, x(0), u\big) = \sum_{i=1}^{\infty} \sum_{k=1}^{n_i} v_{ik}^0(t) x_{ik} + \sum_{i=1}^{\infty} \sum_{k=1}^{n_i} v_{ik}^u(t) x_{ik} \tag{2}$$

where

$$v_{ik}^0(t) = \exp(s_i t) \, \langle x(0), x_{ik} \rangle_X$$

$$v_{ik}^u(t) = \int_0^t \exp\big(s_i(t - \tau)\big) \left(\sum_{j=1}^{m} b_{ikj} u_j(\tau) \right) \mathrm{d}\tau$$

for $i = 1, 2, \ldots$ and $k = 1, 2, \ldots, n_i$. We start with the well-known (see e.g. Klamka, 1991; 1993b; Triggiani, 1976 or 1978) for details) necessary and sufficient conditions for the approximate controllability with unconstrained controls.

Theorem 1. (Triggiani, 1976) *The dynamic system (1) is approximately controllable if and only if* rank $B_i = n_i$ *for every* $i = 1, 2, \ldots$.

Corollary 1. (Triggiani, 1976) *Let* $m = 1$. *Then the dynamic system (1) is approximately controllable if and only if every vector* $b^i \in \mathbb{R}^m$, $i = 1, 2, \ldots$ *contains at least one non-zero element.*

Now we recall a known (see (Son, 1990) for details) necessary and sufficient condition for the approximate controllability with non-negative controls for the dynamic system (1).

Theorem 2. (Son, 1990) *The dynamic system (1) is approximately controllable with non-negative controls if and only if* rank $B_i = n_i$ *for every* $i = 1, 2, \ldots$ *and the columns of these matrices* B_i, $i = 1, 2, \ldots$ *which correspond to the real eigenvalues, form positive bases in the space* $\mathbb{R}^m$.

Remark 2. The above result implies, in particular, that the number of positive controls required for the approximate controllability with non-negative controls is at least that of the highest multiplicity of the eigenvalues plus one. Therefore, the dynamic system (1) with one scalar non-negative control is never approximately controllable (Son, 1990). Moreover, it should be stressed that in a general case of multiple eigenvalues it is not so easy to verify the hypothesis that the set of given vectors forms a positive basis in the Euclidean space.

Remark 3. Using the concept of the polar cone C^0, the results stated in the above theorems can be extended to constrained controls which take their values in a given closed compact cone C with non-empty interior $\operatorname{int} C \in U_{\text{ad}}$ (Son, 1990).

3. Constrained Controllability

In this section we shall present results concerning the constrained approximate controllability for the dynamic system (1). We start with the following result on approximate positive controllability:

Theorem 3. *If there exist* p *and* q *such that the eigenvalue* $s_p \in \mathbb{R}$ *and coefficients* b_{pqj} *have the same sign for every* $j = 1, 2, \ldots, m$, *then the dynamic system (1) is not approximately positive controllable.*

Proof. In order to prove this theorem, it is sufficient to indicate a final state $x_f \in X^+$ which cannot be reached approximately from a given initial state $x_0 \in X^+$. We shall prove this in two steps.

First, let us assume that $x_{pq} \notin X^+$. Let us take $x(0) = 0$. Therefore, by (2) we have $v^0_{ik}(t) = 0$ for $t \geq 0$ and every $i = 1, 2, \ldots,$ $k = 1, 2, \ldots, n_i$. Let us choose the final state

$$x_f = \sum_{i=1}^{\infty} \sum_{k=1}^{n_i} \langle x_f, x_{ik} \rangle_X \, x_{ik} = \sum_{i=1}^{\infty} \sum_{k=1}^{n_i} v^f_{ik} x_{ik} \in X^+$$

as follows:

$$x_f = \sup\{-x_{pq}, 0\} \in X^+ \quad \text{if } b_{pqj} > 0 \quad \text{for } j = 1, 2, \ldots, m$$

$$x_f = \sup\{x_{pq}, 0\} \in X^+ \quad \text{if } b_{pqj} < 0 \quad \text{for } j = 1, 2, \ldots, m$$

Therefore,

$$v^f_{pq} = \langle x_f, x_{pq} \rangle_X = \langle \sup\{-x_{pq}, 0\}, x_{pq} \rangle_X < 0 \ \text{ when } b_{pqj} > 0 \ \text{ for } j = 1, 2, \ldots, m$$

$$v^f_{pq} = \langle x_f, x_{pq} \rangle_X = \langle \sup\{x_{pq}, 0\}, x_{pq} \rangle_X > 0 \quad \text{when } b_{pqj} < 0 \ \text{ for } j = 1, 2, \ldots, m$$

Following (2), let us observe that for given p, q and $x(0) = 0$ we have

$$v_{pq}(t) = \int_0^t \exp\left(s_p(t-\tau)\right) \left(\sum_{j=1}^{m} b_{pqj} u_j(\tau) \right) \mathrm{d}\tau \tag{3}$$

Therefore, since the admissible controls are non-negative, i.e. $u_j(t) \geq 0$ for $j = 1, 2, \ldots, m$ and $t \geq 0$, from (3) it follows that

$$v_{pq}(t) > 0 \ \text{ for } t \geq 0 \ \text{ if } b_{pqj} > 0 \ \text{ for } j = 1, 2, \ldots, m$$

$$v_{pq}(t) < 0 \ \text{ for } t \geq 0 \ \text{ if } b_{pqj} < 0 \ \text{ for } j = 1, 2, \ldots, m$$

Taking into account the form of the solution $x(t, 0, u)$ given by (2), we have

$$\begin{aligned} \left\| x(t,0,u) - x_f \right\|_X &= \left(\sum_{i=1}^{\infty} \sum_{k=1}^{n_i} \left| v_{ik}(t) - v^f_{ik} \right|^2 \right)^{1/2} \\ &> \left| v_{pq}(t) - v^f_{pq} \right| > \text{const} > 0 \quad \text{for } t \geq 0 \end{aligned} \tag{4}$$

Therefore, by (4) the final state $x_f \in X^+$ cannot be reached approximately from zero in any time using non-negative controls.

Now, let us consider the case when an eigenfunction $x_{pq} \in X^+$ and $x(0) \neq 0$. Hence, just as in the first part of the proof, following (2) for given p and q, we have

$$v_{pq}(t) = \exp(s_p t) \langle x(0), x_{pq} \rangle_X + \int_0^t \exp\left(s_p(t-\tau)\right) \left(\sum_{j=1}^{m} b_{pqj} u_j(\tau) \right) \mathrm{d}\tau \tag{5}$$

Since x_{pq} is an orthonormal eigenvector, taking $x(0) = x_{pq} \in X^+$ we have

$$\langle x(0), x_{pq} \rangle_X = \langle x_{pq}, x_{pq} \rangle_X = 1$$

Therefore, since the admissible controls are non-negative, i.e. $u_j(t) \geq 0$ for $j = 1, 2, \ldots, m$ and $t \geq 0$, from (5) it follows that

$$v_{pq}(t) > 1 \text{ for } s_p > 0 \text{ and } b_{pqj} > 0 \text{ for } j = 1, 2, \ldots, m$$

$$v_{pq}(t) < 1 \text{ for } s_p > 0 \text{ and } b_{pqj} < 0 \text{ for } j = 1, 2, \ldots, m$$

Since we investigate the approximate positive controllability of the dynamic system (1), let us choose a final state $x_f \in X^+$ such that

$$v_{pq}^f > 1 \text{ for } s_p > 0 \text{ and } b_{pqj} > 0 \text{ for } j = 1, 2, \ldots, m$$

$$v_{pq}^f < 1 \text{ for } s_p > 0 \text{ and } b_{pqj} < 0 \text{ for } j = 1, 2, \ldots, m$$

Taking into account the form of the solution $x(t, 0, u)$ given by (2), we have

$$\begin{aligned} \|x(t,0,u) - x_f\|_X &= \left(\sum_{i=1}^{\infty} \sum_{k=1}^{n_i} \left| v_{ik}(t) - v_{ik}^f \right|^2 \right)^{1/2} \\ &> \left| v_{pq}(t) - v_{pq}^f \right| > \text{const} > 0 \quad \text{for } t \geq 0 \end{aligned} \tag{6}$$

Therefore, by (6) the final state $x_f \in X^+$ cannot be reached approximately from zero in any time using non-negative controls.

Now, let us consider the cases when $s_p > 0$, $b_{pqj} < 0$ and $s_p < 0$, $b_{pqj} > 0$. We choose the initial state $x(0) \in X^+$ and final state $x_f \in X^+$ such that $v_{pq}^0 = 0$ and $v_{pq}^f > 1$, for $s_p > 0$ and $b_{pqj} < 0$ for $j = 1, 2, \ldots, m$. In that case we have $v_{pq}(t) < 0$ for $t \geq 0$, and the final state $v_{pq}^f > 1$ cannot be reached by non-negative controls.

Finally, when $s_p < 0$ and $b_{pqj} > 0$ for $j = 1, 2, \ldots, m$, we choose $v_{pq}^0 = 0$, $v_{pq}^f = 0$, $v_{ik}^f > 1$ for $i, k = 1, 2, \ldots,$ $i \neq p$, $k \neq q$ and the uniformly stable dynamic system (1), and $v_{pq}^0 = 0$, $v_{pq}^f = 0$, $v_{ik}^f < 1$ for $i, k = 1, 2, \ldots,$ $i \neq p$, $k \neq q$ and non-uniformly stable dynamic system (1).

In both the cases the final state $x^f \in X^+$ cannot be reached by non-negative controls.

Hence the dynamic system (1) is not approximately positive controllable and our theorem follows. ■

From Theorem 3 and Remark 1 we obtain directly the next result concerning the approximate controllability of the dynamic system (1) with non-negative controls:

Corollary 2. *If the assumptions of Theorem 3 are satisfied, then the dynamic system (1) is not approximately controllable with non-negative controls.*

4. Positive Stationary Pairs

In Section 3 we obtained some negative results concerning the approximate positive controllability for the dynamic system (1). However, it is often not so important to reach the entire positive cone of the state space. It suffices to steer approximately the dynamic system to particular positive states and held constant by a non-negative control for all times. This observation directly leads to the concept of the so-called positive stationary pairs (Schanbacher, 1989). In this section, we generally assume that the dynamic system (1) is positive in the sense stated in Section 2.

Definition 3. (Schanbacher, 1989) We call a pair $\{x_s, u_s\} \in (X^+ \backslash \{0\}) \times U^+$ a positive stationary pair if $Ax_s + Bu_s = 0$. In this case $x(t, x_s, u_s) = x_s \in X^+$ is a non-zero constant solution of eqn. (1) for $t \geq 0$, $u(t) = u_s$ and $x_s = x(0)$.

Theorem 4. (Schanbacher, 1989) *Let the dynamic system (1) be positive and $S(t)$ be a uniformly exponentially stable positive semigroup. Then for each $u_s \in U^+ \backslash \ker B$ there exists exactly one $x_s = -A^{-1}Bu_s$ such that $\{x_s, u_s\}$ is a positive stationary pair. Moreover, if $\{x_s, u_s\}$ is a positive stationary pair, and we choose $x(0) \in X^+$ and $u(t) = u_s$, $t \geq 0$, then the solution of eqn. (1) tends to x_s as $t \to \infty$.*

Corollary 3. *Let* $\mathrm{Re}\,(s_i) \leq \mathrm{Re}\,(s_1) < 0$ *for* $i = 1, 2, \ldots$. *Then for each* $u_s \in U^+ \backslash \ker B$ *there exists exactly one*

$$x_s = \sum_{i=1}^{\infty} s_i^{-1} \sum_{k=1}^{n_i} \left\langle x_{ik}, \sum_{j=1}^{m} b_j u_{sj} \right\rangle_X x_{ik} \tag{7}$$

such that $\{x_s, u_s\}$ is a positive stationary pair.

Proof. Since the spectrum $\sigma(A)$ of the linear operator A is a pure discrete-point one, we conclude that the inequality $\mathrm{Re}\,(s_1) < 0$ is a necessary and sufficient condition for the so-called uniform stability of the linear dynamic system (see Bensoussan *et al.*, 1993; Schanbacher, 1989, for an exact definition of uniform stability). Therefore, using general spectral formula for the linear inverse operator A^{-1} and Theorem 4 stated above we obtain immediately (7). ∎

Remark 4. Many valuable remarks and comments on the relationships between different kinds of stability (the definitions of uniform exponential, strong and weak stabilities) of the linear abstract differential equation (1) and the conditions for the existence of positive stationary pairs for positive dynamic systems can be found in the paper (Schanbacher, 1989).

5. Constrained Controllability of Parabolic Dynamic Systems

In this section, we shall illustrate the general theorems and corollaries stated in Sections 3 and 4 for the case of linear distributed-parameter systems of parabolic type. We begin by describing the mathematical model of the considered distributed-parameter system. Let Ω be a bounded, open and connected subset of $\mathbb{R}^N$ with a smooth boundary $\partial\Omega$ and $\mathrm{cl}\,\Omega = \Omega \bigcup \partial\Omega$. Let Δ be the Laplacian operator on Ω and ∇ be the gradient operator on Ω. Let us consider a linear distributed-parameter dynamic system described by the following partial differential equation of parabolic type:

$$w_t(z,t) = Aw(z,t) + \sum_{j=1}^{m} b_j(z)u_j(t), \quad t > 0, \quad z \in \Omega \tag{8}$$

where $b_j \in L^2(\Omega)$, for $j = 1, 2, \dots, m$, and the admissible controls are non-negative, i.e. $u_j \in L^2_{\mathrm{loc}}([0,\infty), \mathbb{R}^+)$, $j = 1, 2, \dots, m$. The boundary conditions are assumed to be of the following form:

$$\alpha(z)w(z,t) + \beta(z)\frac{\partial w}{\partial v}(z,t) = 0, \quad t \geq 0, \quad z \in \partial\Omega \tag{9}$$

Suppose that $\alpha(z)$ and $\beta(z)$ are twice continuously differentiable on $\mathrm{cl}\,\Omega$, and are not identically zero simultaneously. The vector field $v(z)$ is the outer unit normal to $\partial\Omega$ at $z \in \partial\Omega$ and $\partial(\cdot)/\partial v = v\nabla$ denotes differentiation in the direction of the outward normal to Ω. Specifying $\alpha(z)$ and $\beta(z)$, we obtain Dirichlet, Neumann or Robin (mixed) boundary conditions.

The initial condition for eqn. (8) is given by

$$w(z,0) = w_0(z) \qquad z \in \Omega$$

The second-order uniformly elliptic differential operator has the following form:

$$A = \sum_{k,j=1}^{N} a_{kj}(z)D_k D_j + \sum_{k=1}^{N} a_k(z)D_k + a_0(z)I \tag{10}$$

where $z \in \mathbb{R}^N$, $a_{kj}(z) = a_{jk}(z)$, for $j,k = 1, 2, \dots, N$, $D_k = \partial/\partial z_k$, for $k = 1, 2, \dots, N$.

The domain $D(A)$ of the operator A is characterized explicitly by

$$D(A) = \Big\{ w \in L^2(\Omega) : Aw \in L^2(\Omega) \\ \text{and } \alpha(z)w(z,t) + \beta(z)\frac{\partial w}{\partial v}(z,t) = 0, \quad t \geq 0, \ z \in \partial\Omega \Big\}$$

The coefficients $a_{kj}(z)$, $a_k(z)$ and $a_0(z)$ are assumed to be twice continuously differentiable on Ω and $a_0(z) \geq 0$ for $z \in \Omega$. Moreover, since the operator A is uniformly elliptic, there exists a positive constant μ such that for all vectors $\xi \in \mathbb{R}^N$ we have

$$\sum_{k,j=1}^{N} a_{kj}(z)\xi_k\xi_j \geq \mu|\xi|^2 \quad \text{for } z \in \Omega$$

Various special cases of eqn. (8) can be considered. For example, the reaction-diffusion dynamic system

$$w_t(z,t) = d\Delta w(z,t) + aw(z,t) + \sum_{j=1}^{m} b_j(z)u_j(t), \quad t > 0, \quad z \in \Omega \tag{11}$$

where a and d are real constants, can be expressed in the form of the general differential equation (8).

It is well-known (see e.g. (Smith, 1995) for details) that the operator A generates an analytic positive semigroup of bounded compact operators $S(t) : X \to X$ for $t \geq 0$ (Smith, 1995). Moreover, since the set Ω is bounded, the operator A has pure discrete-point spectrum $\sigma(A) = \sigma_p(A) = \{s_1, s_2, s_3, \dots, s_i, \dots\}$, consisting entirely of isolated eigenvalues with finite multiplicities $n_i < \infty$, $i = 1, 2, \dots$. Moreover, the corresponding eigenfunctions $\{x_{ik},\ i = 1, 2, \dots,\ k = 1, 2, \dots, n_i\}$ form an orhonormal basis in the space $L^2(\Omega)$. An additional property of the operator A that will be important later is stated in the following lemma which is proved in (Smith, 1995).

Lemma 1. (Smith, 1995) *There exists a real eigenvalue* s_1 *of the operator* A *and the corresponding eigenvector* $x_1(z)$ *is a strictly positive element in the space* X*, i.e. it satisfies* $x_1(z) \gg 0$ *for all* $z \in \operatorname{cl}\Omega$ *in the case of Neumann or Robin (mixed) boundary conditions and for all* $z \in \Omega$ *in the case of Dirichlet boundary conditions. In the latter case, we also have*

$$\frac{\partial x_1}{\partial v}(z) < 0 \quad \text{for} \quad z \in \partial\Omega$$

Moreover, if s_i *is any other eigenvalue of the operator* A*, then the real part of* s_i*,* $\operatorname{Re}(s_i)$*, satisfies* $\operatorname{Re}(s_i) < s_1$ *for all* $i = 2, 3, \dots$.

Lemma 1 says that there exists a real eigenvalue of the operator A which is larger than the real parts of all other eigenvalues of the operator A. We call it the principal eigenvalue of the operator A. Moreover, Lemma 1 says that the associated eigenvector is positive and is called the principal eigenvector of the operator A.

We may express the dynamic system (8) with boundary conditions (9) as an abstract ordinary differential equation in the separable Hilbert space $X = L^2(\Omega)$. Since A given by (10) satisfies all the assumptions stated in the previous sections, it sufficies to substitute $x(t) = w(\cdot, t) \in L^2(\Omega) = X$.

Let us write

$$b_{1j} = \langle b_j, x_1 \rangle_{L^2(\Omega)} = \int_\Omega b_j(z) x_1(z)\, \mathrm{d}z \quad \text{for} \quad j = 1, 2, \dots, m \tag{12}$$

Now, using the general results stated in Section 3 we may formulate a theorem and corollaries on the positive approximate controllability for the distributed-parameter dynamic system (8) with normal operator A.

Theorem 5. *Let the linear operator* A *given by (10) be normal. Moreover, let us assume that the elements* b_{1j} *have the same sign for every* $j = 1, 2, \dots, m$*. Then the* *linear distributed-parameter dynamic system (8) is not approximately positive controllable.*

Proof. Let us observe that the distributed-parameter dynamic system (8) satisfies all the assumptions required in Theorem 3. Therefore, by Theorem 3 our dynamic system (8) is not approximately positive controllable. ■

Corollary 4. *If* $s_1 < 0$, *then for each* $u_s \in U^+ \backslash \ker B$ *there exists exactly one* x_s *such that* $\{x_s, u_s\}$ *is a positive stationary pair.*

6. Example

Let us consider the one-dimensional heat equation on a rod of unit length with non-insulated ends described by the following linear partial-differential equation of parabolic type:

$$w_t(z,t) = w_{zz}(z,t) + b(z)u(t), \quad 0 \le z \le 1, \quad t \ge 0 \tag{13}$$

with initial condition $w(z,0) = w_0(z)$ and Dirichlet-type homogeneous boundary conditions $w(0,t) = w(1,t) = 0$.

We wish to control the distributed-parameter system (13) by a non-negative scalar input $u \in L^2_{\text{loc}}([0,\infty), \mathbb{R}^+)$. We can interpret this control as an electrical heating input that for all times is proportional to a given heat distribution $b(z) \in L^2([0,1], \mathbb{R})$

We state this control problem as an abstract control problem on the separable Hilbert space $X = L^2([0,1], \mathbb{R})$. Write $w(z,t) = x(t) \in X$. Let $A = \mathrm{d}^2/\mathrm{d}z^2$ be the linear unbounded self-adjoint differential operator on X with domain $D(A) = \{w(z)X : w_{zz}(z) \in X, \quad w(0) = w(1) = 0\}$. It is known (Klamka, 1991) that the operator A has simple eigenvalues $s_i = -i^2\pi^2$ and the corresponding eigenfunctions $x_i(z) = \sqrt{2}\sin(i\pi z)$, $i = 1, 2, \ldots$ form an orthonormal basis in the space $X = L^2([0,1], \mathbb{R})$.

Since all the eigenvalues are real, then by Theorem 5 the dynamic system (13) is not approximately positive controllable for any $b \in X$. The same result was proved in (Schanbacher, 1989) but using quite different methods.

Moreover, let us observe that the operator A generates an analytic positive semigroup $S(t)$, $t \ge 0$ on X given by

$$S(t)x = \sum_{i=1}^{\infty} \exp(-i^2\pi^2 t)\langle x, x_i\rangle_{L^2} x_i$$

Now, let us assume that $b \in X^+ = L^2([0,1], \mathbb{R}^+)$. Therefore the distributed-parameter system (13) is positive. Following (Schanbacher, 1989) it should be stressed that the positive dynamic system (13) is not approximately positive controllable either.

However, since $\operatorname{Re}(s_1) = -\pi^2 < 0$, by Corollary 4 for each $u_s \in \mathbb{R}^+$ there exists exactly one element $x_s = -A^{-1}bu_s \in X^+$ given by

$$x_s = \sum_{i=1}^{\infty} \left(-i^2\pi^2\right)^{-1} \int_0^1 \sqrt{2}\sin(i\pi z)b(z)\,\mathrm{d}z\,\sqrt{2}\sin(i\pi z)u_s$$

such that $\{x_s, u_s\}$ is a positive stationary pair. Using the results stated in (Schanbacher, 1989), the element x_s can also be expressed as follows:

$$x_s(z) = \left(z\int_0^1\int_0^{\xi} b(\zeta)\,\mathrm{d}\zeta\,\mathrm{d}\xi - \int_0^{z}\int_0^{\xi} b(\zeta)\,\mathrm{d}\zeta\,\mathrm{d}\xi\right)u_s$$

Summarizing, the distributed-parameter dynamic system (13) is not approximately positive controllable and of course it is not approximately controllable with non-negative controls either. However, for the dynamic system (13) there exist stationary pairs.

7. Conclusions

The present paper contains several results on the constrained controllability for linear infinite-dimensional self-adjoint dynamic systems. Using spectral properties of normal generally unbounded linear operators with pure discrete-point spectra, conditions for various kinds of constrained controllability have been formulated and proved. General results have also been applied for constrained controllability considerations for linear distributed-parameter dynamic systems described by linear partial-differential equations of parabolic type with various kinds of boundary conditions. Finally, a simple illustrative example of a one-dimensional heat equation with homogeneous Dirichlet boundary conditions has been presented.

Some kinds of the presented results can be extended to cover the case of infinite-dimensional normal dynamic systems with discrete and continuous spectra. It is also possible to extend the results to second-order infinite-dimensional dynamic systems.

References

Bensoussan A., Da Prato G., Delfour M.C. and Mitter S.K. (1993): *Representation and Control of Infinite Dimensional Systems. Vols.I–II.* — Boston: Birkhäuser.

Klamka J. (1982): *Controllability, observability and stabilizability of distributed systems with infinite domain.* — Proc. 3-rd Symp. Control of Distributed Parameter Systems, Toulouse (France), pp.VI.7–VI.11.

Klamka J. (1991): *Controllability of Dynamical Systems.* — Dordrecht: Kluwer.

Klamka J. (1992): *Approximate controllability of second order dynamical systems.* — Appl. Math. Comp. Sci., Vol.2, No.1, pp.135–146.

Klamka J. (1993a): *Constrained controllability of linear retarded dynamical systems.* — Appl. Math. Comp. Sci., Vol.3, No.4, pp.647–672.

Klamka J. (1993b): *Controllability of dynamical systems — A survey.* — Arch. Contr. Sci., Vol.2, No.3/4, pp.281–307.

McGlothin (1978): *Controllability, observability, and duality in a distributed parameter system with continuous and point spectrum.* — IEEE Trans. Automat. Contr., Vol.23, No.4, pp.687–690.

Peichl G. and Schappacher W. (1986): *Constrained controllability in Banach spaces.* — SIAM J. Contr. Optim., Vol.24, No.6, pp.1261–1275.

Schanbacher T. (1989): *Aspects of positivity in control theory.* — SIAM J. Contr. Optim., Vol.27, No.3, pp.457–475.

Smith H.L. (1995): *Monotone Dynamical Systems. An Introduction to the Theory of Competitive and Cooperative Systems.* — Math. Surveys and Monographs, Vol.41, Providence: AMS.

Son N.K. (1990): *A unified approach to constrained approximate controllability for the heat equations and the retarded equations.* — J. Math. Anal. Applics., Vol.150, No.1, pp.1–19.

Triggiani R. (1975a): *Controllability and observability in Banach space with bounded operators.* — SIAM J. Contr. Optim., Vol.13, No.2, pp.462–491.

Triggiani R. (1975b): *On the lack of exact controllability for mild solutions in Banach space.* — J. Math. Anal. Applics., Vol.50, No.2, pp.438–446.

Triggiani R. (1976): *Extensions of rank conditions for controllability and observability in Banach space and unbounded operators.* — SIAM J. Contr. Optim., Vol.14, No.2, pp.313–338.

Triggiani R. (1977): *A note on the lack of exact controllability for mild solutions in Banach spaces.* — SIAM J. Contr. Optim., Vol.15, No.3, pp.407–11.

Triggiani R. (1978): *On the relationship between first and second order controllable systems in Banach spaces.* — SIAM J. Contr. Optim., Vol.16, No.6, pp.847–859.

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