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APPROXIMATE CONTROLLABILITY PROPERTIES OF THE SEMILINEAR HEAT EQUATION WITH LUMPED CONTROLS

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In this article, we study the global controllability properties of a one-dimensional semilinear heat equation with sublinear reaction term, governed in a bounded domain by internal lumped controls. We prove that it is possible to *exactly* control any finite dimensional portion of its solution (when expanded along the sequence of the eigenfunctions of the associated Laplacian), provided that the truncated linear equation is approximately controllable in $L^2(0,1)$. We also describe a certain topology (weaker than $L^2(0,1)$) in which this system is, in fact, globally approximately controllable at any positive time. Some extensions to the case of several dimensions are also given.

Keywords: semilinear heat equation, controllability, internal lumped control

1. Introduction

1.1. Problem formulation and motivation

We consider the following homogeneous Dirichlet problem for the semilinear one dimensional heat equation:

$$u_t = u_{xx} + f(u) + v(t)\chi_{(l_1,l_2)}(x) \text{ in } Q_T = (0,1)\times(0,T), \quad v \in L^2(0,T), \tag{1a}$$

$$u(0,t) = u(1,t) = 0, \quad u|_{t=0} = u_0 \in L^2(0,1),$$

where $\chi_{(l_1,l_2)}(x)$ is the characteristic function of a given subinterval $(l_1,l_2) \subset (0,1)$. We assume that $f(u)$ is globally Lipschitz and is such that for some $C > 0$ and $\alpha \in (0,1]$:

$$|f(p)| \leq C(1 + |p|^{1-\alpha}), \quad \forall p \in \mathbb{R}. \tag{1b}$$

It is well-known (see, e.g., Ladyzhenskaya *et al.*, 1968) that (1) admits a unique generalized solution from the space $C([0,T];L^2(0,1)) \bigcap H_0^{1,0}(Q_T)$, where $H_0^{1,0}(Q_T) = \{\phi \mid \phi, \phi_x \in L^2(Q_T), \phi|_{x=0,1} = 0\}$.

It is said that (1) is *approximately controllable* in a given phase-space H at time T if the range of its solution mapping $L^2(0,T) \ni v \to u(\cdot,T)$ is dense in H. In this

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article, we are concerned with the following question: *For what nonlinear terms f and spaces H does this property hold, provided that it holds for the truncated linear system (4)?*

The approximate controllability problem was thoroughly studied (in several space dimensions) in (Fabre *et al.*, 1992; 1995; Fernandez and Zuazua, 1999) in the case of *locally-distributed* controls (i.e., when $v = v(x,t)$, not $v = v(t)$ as in (1a)) and globally Lipschitz reaction-convection nonlinear terms. The methods of these works involve unique continuation and fixed point techniques, combined with a variational approach (see, e.g., Lions, 1990). More recently, in (Zuazua, 1997) it was shown that, in fact, any finite-dimensional portion of the solution to this equation, otherwise approximately controllable, can also be controlled exactly (by locally- distributed controls, i.e., *not* lumped like in this paper). This property is further regarded as the *finite* exact controllability.

By the method of Carleman's estimates, in (Fursikov and Imanuvilov, 1996) the approximate and exact null-controllability property were established for a class of semilinear reaction-diffusion equations with varying coefficients and, also, with globally Lipschitz nonlinear terms.

Several results, both positive and negative, on the controllability and reachability properties of the parabolic semilinear equations are also available for the superlinear terms, e.g., in (Fursikov and Imanuvilov, 1996; Khapalov, 1995; 1999b; 1999c; Fernandez-Cara, 1997; Zuazua, 1997). In particular, in (Khapalov, 1999c) a class of globally approximately controllable heat equations with superlinear time-dependent terms, governed by locally distributed controls, was described.

The lumped controls which we consider in this paper are strongly motivated by various applications. They can be regarded as a degenerated class of locally distributed ones, and, hence, their study usually requires quite different methods. Generally, one cannot expect equally strong results for these two types of control. For example, the study of controllability with locally distributed controls is essentially based on the unique continuation property of solutions to the linear parabolic equation from an open set. This result, obviously, cannot be associated with the case of lumped controls, which are the functions of time only. This explains, in particular, the reason why we focus primarily on the 1-D case (see also Section 4 for further discussion in this respect).

While for the standard linear heat equation this problem is well-understood by now (Fattorini and Russell, 1974; Mizel and Seidman, 1969; Sakawa, 1974), see also the references therein), little is known regarding the semilinear case. Among early works in this area we can mention only (Zhou, 1982), dealing with uniformly bounded Lipschitz nonlinearity.

The method which we use in this article is quite different from the classical fixed point or implicit function arguments. It is based on the idea to solve the controllability problem in an asymptotically short time in order to 'suppress' in this way the effect of nonlinearity. This method was introduced in (Khapalov, 1995) (also see Khapalov, 1999a). Using it, various results on approximate controllability of the semilinear reaction-diffusion-convection equations, governed by *lumped* controls, were obtained in (Khapalov, 1995; 1999a), assuming the logarithmic-type growth condition on the nonlinear term. In (Khapalov, 1999a) we also gave an example of a globally approximately controllable heat equation with superlinear time-dependent term. (We also showed in Khapalov, 1999a) that the result of (Zhou, 1982) follows by this asymptotic method in an obvious way.)

In this article, we are primarily concerned with the finite exact controllability of (1a), that is, when H is finite dimensional. In other words, we focus on the question: *For what f can, at least, 'finite portions' of solutions to (1a) be exactly controllable?* In that respect, it is somewhat surprising to see that the finite exact controllability holds for *any* sublinear term as in (1b).

By its 'spirit', this article is very close to (Khapalov, 1999b), where we studied the superlinear terms and locally-distributed controls. Here we are interested in much weaker ('singular') lumped controls. To cope with this principal complication, we combine the above-mentioned asymptotic method of (Khapalov, 1995) with the traditional Riesz basis approach relevant to the linear boundary problems with pointwise controls (see Fattorini and Russell, 1974; Mizel and Seidman, 1969; Sakawa, 1974). (We remind the reader that $\{e^{-(\pi k)^2 t} \mid k = 1, \dots\}$ form a Riesz basis in $L^2(0,T)$ for any $T > 0$.)

1.2. Main results

Denote by $\{\lambda_k = (\pi k)^2, \omega_k(x) = \sqrt{2}\sin \pi k x,\ k = 1, \dots\}$ the eigenvalues and orthonormalized in $L^2(0,1)$ eigenfunctions of the spectral problem: $-\omega_{xx} = \lambda\omega, \omega \in H_0^1(0,1) = \{\phi \mid \phi, \phi_x \in L^2(0,1), \phi|_{x=0,1} = 0\}$.

Our goal in this article is to prove the following Theorems 1 and 2. A possible extension of Theorem 1 to the case of several dimensions is given by Theorem 3 in Section 4. Set $L_K^2(0,1) = \{\phi \mid \phi(x) = \sum_{i=1}^K \alpha_i\omega_i(x), \alpha_i \in \mathbb{R}\}$ and denote by Π_K the operator of the orthogonal projection in $L^2(0,1)$ onto $L_K^2(0,1)$.

Theorem 1. (Finite exact controllability in $H = L_K^2(0,1)$) *Let $l_2 \pm l_1$ be irrational numbers. Given $T > 0$, for every $K = 1, \dots,$ $u_0 \in L^2(0,1)$, and $u_T \in L_K^2(0,1)$ there is a control $v \in L^2(0,T)$ such that for the corresponding solution to (1)*

$$\Pi_K u(\cdot, T) = u_T. \tag{2}$$

Let $c_1, \dots, c_k, \dots$ be a non-increasing sequence of positive numbers. Denote by W the Banach space of functions $\{\phi \mid \phi(x) = \sum_{k=1}^\infty \alpha_k\omega_k(x), \sum_{k=1}^\infty \alpha_k^2 c_k < \infty\}$, endowed with the norm

$$\|\phi\|_W = \Big(\sum_{k=1}^{\infty} \alpha_k^2 c_k\Big)^{1/2}.$$

Note that $L^2(0,1)$ is continuously embedded into W, so $u \in C([0,T]; W)$.

Theorem 2. (Approximate controllability in $H = W$) *Let $l_2 \pm l_1$ be irrational numbers and $T > 0$ be given. There is a monotone decreasing sequence of positive numbers*

$\{c_k\}_{k=1}^{\infty}$, *defining* W *(see (35) below), such that for every* $u_0 \in L^2(0,1)$, $u_T \in W$, *and* $\varepsilon > 0$ *there is a control* $v \in L^2(0,T)$ *for which*

$$\|u(\cdot,T) - u_T\|_W \leq \varepsilon. \tag{3}$$

2. Proof of Theorem 1

Fix any natural number K and $u_0 \in L^2(0,1)$, $u_T \in L^2_K(0,1)$. We intend to find a control v which ensures (2).

Step 1. Consider the following truncated linear boundary problem:

$$\frac{\partial u_L}{\partial t} = u_{Lxx} + \chi_{(l_1,l_2)} v(t) \ \text{ in } \ Q_T, \tag{4}$$

$$u_L(0,t) = u_L(1,t) = 0, \quad u_L|_{t=0} = u_0, \quad v \in L^2(0,T).$$

(Here we use the subscript '$_L$' to emphasize the connection between (1a) and (41).) It is well-known (Sakawa, 1974) that if $l_2 \pm l_1$ are irrational numbers, then (4) is approximately controllable in $L^2(0,1)$ at any positive time T. Since (4) is a particular case of (1), this explains our assumptions on l_1 and l_2 in Theorem 1 and 2.

Given v, set $z = u - u_L$. Then it follows from (1) and (4) that

$$\frac{\partial z}{\partial t} = z_{xx} + f(u) \ \text{ in } \ Q_T,$$

$$z(0,t) = z(1,t) = 0, \quad z|_{t=0} = 0 \ \text{ in } \ (0,1). \tag{5}$$

Multiplication of this equation by z and further integration by parts over Q_T yield the classical energy estimate

$$\|z\|_{C([0,T];L^2(0,1))} + \Big(\int_0^T\int_0^1 z_x^2(x,\tau)\,\mathrm{d}x\,\mathrm{d}\tau\Big)^{1/2} \leq 2\sqrt{T}\Big(\int_0^T\int_0^1 f^2(u(x,\tau))\,\mathrm{d}x\,\mathrm{d}\tau\Big)^{1/2}. \tag{6}$$

By Hölder's inequality and (1b), using the inequality $(a+b)^2 \leq 2a^2 + 2b^2$, we obtain

$$\int_0^1 f^2(u(x,\tau))\,\mathrm{d}x \leq C^2 \int_0^1 \big(1 + |u(x,\tau)|^{1-\alpha}\big)^2\,\mathrm{d}x \leq 2C^2 + 2C^2 \int_0^1 |u(x,\tau)|^{2(1-\alpha)}\,\mathrm{d}x$$

$$\leq 2C^2 + 2C^2\Big(\int_0^1 |u(x,\tau)|^2\,\mathrm{d}x\Big)^{1-\alpha} \leq 2C^2 + 2C^2\|u\|^{2(1-\alpha)}_{C([0,T];L^2(0,1))}.$$

From here, by using the elementary inequality $\sqrt{a+b} \leq a+b, (a,b \geq 0)$, we derive from (6) that

$$\|z\|_{C([0,T];L^2(0,1))} + \Big(\int_0^T\int_0^1 z_x^2(x,\tau)\,\mathrm{d}x\,\mathrm{d}\tau\Big)^{1/2}$$
$$\leq cT\Big(1+\|u\|^{1-\alpha}_{C([0,T];L^2(0,1))}\Big), \quad c = 2\sqrt{2}C. \tag{7}$$

Step 2. Denote by $S(t)$ the semigroup associated with (4). Then we may write

$$u_L(\cdot,t) = S(t)u_0 + \int_0^t S(t-\tau)v(\tau)\chi_{(l_1,l_2)}\,\mathrm{d}\tau. \tag{8}$$

Set

$$V_0(0,T) = \Big\{p \in L^2(0,T) \mid \int_0^t S(t-\tau)p(\tau)\chi_{(l_1,l_2)}\,\mathrm{d}\tau = 0\Big\} \subset L^2(0,T).$$

Represent control v in (1a) as a sum $v = v^* + v_*$, where v^* is assumed to be fixed (it will be selected later in (31) to ensure (2)), and v_* ranges over $V_0^{\perp}(0,T)$ (v_* will be employed to 'neutralize' the effect of nonlinearity). This presentation will be used below to introduce eqn. (27) to implement the fixed-point argument.

Then we can write

$$u_L = u_{L*} + u_*, \quad u_{L*}(\cdot,t) = S(t)u_0 + \int_0^t S(t-\tau)v^*(\tau)\chi_{(l_1,l_2)}\,\mathrm{d}\tau,$$

$$u_*(\cdot,t) = \int_0^t S(t-\tau)v_*(\tau)\chi_{(l_1,l_2)}\,\mathrm{d}\tau. \tag{9a}$$

Accordingly,

$$u = u_{L*} + u_* + z = u_{L*} + u_* + Q(u_*). \tag{9b}$$

Here, given u_{L*}, we introduced the nonlinear operator $Q : u_* \to z$ as follows:

$$Q : U \ni u_* \to z$$
$$= Q(u_*) = \int_0^t S(t-\tau)\big(f(u)\big)(\cdot,\tau)d\tau \in C\big([0,T];L^2(0,1)\big), \tag{10}$$

where u is the solution to (1) with $v = v^* + v_*$. This operator is defined on the linear manifold

$$U = \Big\{p \,\Big|\, \int_0^t S(t-\tau)p(\tau)\chi_{(l_1,l_2)}\,\mathrm{d}\tau, t \in [0,T], p \in V_0^{\perp}(0,T)\Big\} \subset C\big([0,T];L^2(0,1)\big),$$

which we further equip with the norm of the latter space.

Let us show that the operator Q is continuous on U. Indeed, let $u_{**}, u_* \in U$. By employing the Lipschitz property of $f(u)$ (i.e., $|f(p_1) - f(p_2)| \le C_*|p_1 - p_2|$ for some $C_* > 0$), from the energy estimate like (6), we conclude that

$$\begin{aligned}\|Q(u_{**}) - Q(u_*)\|_{C([0,T];L^2(0,1))} &\le 2\sqrt{T}\Big(\int_0^T\int_0^1 \big((f(u(u_{**})) - f\big(u(u_*)\big)^2(x,\tau)\,\mathrm{d}x\,\mathrm{d}\tau\Big)^{1/2}\\ &\le 2TC_*\|u(u_{**}) - u(u_*)\|_{C([0,T];L^2(0,1))}, \qquad (11)\end{aligned}$$

where $u(u_{**})$ and $u(u_*)$ are the solutions to (1), for the same 'fixed' u_{L*}, corresponding to u_{**} and u_*. By (9b),

$$\begin{aligned}&\|u(u_{**}) - u(u_*)\|_{C([0,T];L^2(0,1))}\\ &\qquad \le \|u_{**} - u_*\|_{C([0,T];L^2(0,1))} + \|Q(u_{**}) - Q(u_*)\|_{C([0,T];L^2(0,1))}. \qquad (12)\end{aligned}$$

Combining (11) and (12) under the assumption that T is sufficiently small, namely

$$T \le \frac{1}{4C_*}, \qquad (13)$$

yields

$$\|Q(u_{**}) - Q(u_*)\|_{C([0,T];L^2(0,1))} \le \|u_{**} - u_*\|_{C([0,T];L^2(0,1))}, \qquad (14)$$

implying the desirable continuity of Q.

We further assume that (13) holds up to Step 9, in which we show how our results can be extended to larger T's.

Inequality (14) also allows us to conclude that if $\{u_{*i}\}_{i=1}^{\infty}$ is a Cauchy sequence in $L^2(Q_T)$, so is $\{Q(u_{*i})\}_{i=1}^{\infty}$. Therefore, the nonlinear operator $Q(u_*)$ can be extended continuously to the closure (we further denote it by $\bar{U}$) of U in $C([0,T];L^2(0,1))$. We denote this extension by

$$\bar{Q} : \bar{U} \ni u_* \to \bar{Q}(u_*) \in C([0,T];L^2(0,1)).$$

Step 3. It is well-known that the general solution to (4) admits the following representation:

$$u_L(x,t) = \sum_{k=1}^{\infty} e^{-\lambda_k t}\Big(\int_0^1 u_0(r)\omega_k(r)\,\mathrm{d}r\Big)\omega_k(x)$$
$$+\sum_{k=1}^{\infty}\int_0^t e^{-\lambda_k(t-\tau)}\Big(\int_0^1 v(\tau)\chi_{[l_1,l_2]}(r)\omega_k(r)\,\mathrm{d}r\Big)\,\mathrm{d}\tau\ \omega_k(x), \quad (15)$$

where the series converges in the $L^2(0,1)$-norm uniformly over $t \geq 0$.

Let $\{q_k\}_{k=1}^{\infty}$ be a biorthogonal sequence to $\{e^{\lambda_k \tau}\}_{k=1}^{\infty}$ in $L^2(0,T)$ (Fattorini and Russell, 1974; Mizel and Seidman, 1969):

$$\int_0^T e^{-\lambda_k \tau} q_l(\tau)\,\mathrm{d}\tau = \begin{cases} 1 & \text{if } k = l, \\ 0 & \text{if } k \neq l. \end{cases} \quad (16)$$

Without loss of generality we can assume that all q_k's lie in $V_0^{\perp}(0,T)$. Set

$$v_k(\tau) = q_k(T-\tau)\Big(\sqrt{2}\int_{l_1}^{l_2} \sin \pi k x\,\mathrm{d}x\Big)^{-1}, \quad \tau \in (0,T), \quad (17a)$$

so that

$$\int_0^T\int_0^1 e^{-\lambda_k(T-\tau)} v_l(\tau)\chi_{[l_1,l_2]}(r)\omega_k(r)\,\mathrm{d}r\,\mathrm{d}\tau = \begin{cases} 1 & \text{if } k = l, \\ 0 & \text{if } k \neq l. \end{cases} \quad (17b)$$

Denote by $V_*(0,T)$ the set of all the finite linear combinations of (linear independent) $v_k, k = 1,\ldots,$ and also set

$$V_{*K}(0,T) = \operatorname{span}\{v_k\}_{k=1}^{K}.$$

Since ω_k's form a basis in $L^2(0,1)$, the formulas (15)–(17) allow us to conclude that (4) is approximately controllable in $L^2(0,1)$ at any time $T > 0$, by using controls from $V_*(0,T)$ only.

Step 4. Set

$$U_K = \Big\{p \,\Big|\, \int_0^t S(t-\tau)p(\tau)\chi_{(l_1,l_2)}\,\mathrm{d}\tau,\ t \in [0,T],\ p \in V_{*K}(0,T)\Big\} \subset U.$$

From Step 3 it follows that for every $u_* \in U_K$ there exists a $v_* \in V_{*K}(0,T)$ such that

$$\Pi_K \bar{Q}(u_*)|_{t=T} = \int_0^T S(t-\tau)v_*(\tau)\chi_{(l_1,l_2)}\,\mathrm{d}\tau \in L_K^2(0,1). \quad (18a)$$

Namely, if

$$\Pi_K \bar{Q}(u_*)|_{t=T} = \sum_{k=1}^{K} \alpha_k \omega_k, \tag{18b}$$

then

$$v_* = \sum_{k=1}^{K} \alpha_k v_k \in V_{*K}(0,T). \tag{18c}$$

Introduce a nonlinear operator

$$\mathcal{R}_K : U_K \ni u_* \to \mathcal{R}_K(u_*) = \int_0^t S(t-\tau)v_*(\tau)\chi_{(l_1,l_2)}\, \mathrm{d}\tau \in U_K$$
$$\subset C\big([0,T]; L^2(0,1)\big), \tag{19}$$

where v_* is from (18c).

Note that $\mathcal{R}_K(u_*)$ is continuous. Indeed, by Step 2, $\Pi_K \bar{Q}(u_*)$ is a continuous operator on $\bar{U}$ into $C([0,T]; L^2(0,1))$. In particular, its *finite dimensional* trace at $t = T$ in $L^2(0,1)$, i.e., $\Pi_K \bar{Q}(u_*)|_{t=T}$, defined by a finite set of $\alpha_1, \ldots, \alpha_K$ in (18), depends continuously upon u_*. In view of (18c), v_* then also depends continuously (via those $\alpha' s$, because $v_k, k = 1, \ldots, K$ were fixed in (17)) in the norm of $L^2(0,T)$ upon u_*. On the other hand, the well-known classical regularity result (similarly to (6)) states that

$$\|\mathcal{R}_K(u_*)\|_{C([0,T];L^2(0,1))} = \max_{t\in[0,T]} \|\int_0^t S(t-\tau)v_*(\tau)\chi_{(l_1,l_2)}\, \mathrm{d}\tau\|_{L^2(0,1)}$$
$$\leq 2\sqrt{T}\|v_*\|_{L^2(0,T)}, \tag{20}$$

which gives the continuous dependence of $\mathcal{R}_K(u_*)$ on v_*. This, along with all the above, implies the desired continuity of $\mathcal{R}_K$ (as a superposition of two continuous mappings).

Step 5. From (7) and (9b), and by the elementary inequalities $a^{1-\alpha} \leq 1 + a$ and $(a+b)^{1-\alpha} \leq a^{1-\alpha} + b^{1-\alpha}$, $(a, b > 0, \alpha \in (0,1])$, and by the triangle inequality, we have for any $u_* \in \bar{U}$:

$$\begin{aligned}
\|z\|_{C([0,T];L^2(0,1))} &= \|Q(u_*)\|_{C([0,T];L^2(0,1))} \le cT\Big(1 + \|u\|^{1-\alpha}_{C([0,T];L^2(0,1))}\Big) \\
&\le cT\Big(1 + \|u_{L*} + u_* + Q(u_*)\|^{1-\alpha}_{C([0,T];L^2(0,1))}\Big) \\
&\le cT\Big(1 + \big(\|u_{L*}\|_{C([0,T];L^2(0,1))} + \|u_*\|_{C([0,T];L^2(0,1))}\big)^{1-\alpha}\Big) \\
&\quad + cT\|Q(u_*)\|^{1-\alpha}_{C([0,T];L^2(0,1))} \\
&\le cT\Big(1 + \big(\|u_{L*}\|_{C([0,T];L^2(0,1))} + \|u_*\|_{C([0,T];L^2(0,1))}\big)^{1-\alpha}\Big) \\
&\quad + cT\Big(1 + \|Q(u_*)\|_{C([0,T];L^2(0,1))}\Big).
\end{aligned}$$

Again, if for this c from (7)

$$T \le \frac{1}{2c}, \tag{21}$$

then

$$\|Q(u_*)\|_{C([0,T];L^2(0,1))} \le 2 + \big(\|u_{L*}\|_{C([0,T];L^2(0,1))} + \|u_*\|_{C([0,T];L^2(0,1))}\big)^{1-\alpha}. \tag{22}$$

We further assume that (21) holds up to Step 9, in which we show how our results can be extended to larger T's.

Note that for every fixed u_{L*}, (22) gives the following estimate:

$$\|Q(u_*)\|_{C([0,T];L^2(0,1))} \le (2 + 2^{1-\alpha})\|u_*\|^{1-\alpha}_{C([0,T];L^2(0,1))}, \tag{23a}$$

for u_* such that

$$\|u_*\|_{C([0,T];L^2(0,1))} \ge \max\big\{1, \|u_{L*}\|_{C([0,T];L^2(0,1))}\big\}. \tag{23b}$$

(23) will be used in Step 7.

Step 6. Taking into account that

$$\|Q(u_*)\|_{C([0,T];L^2(0,1))} \ge \|Q(u_*)|_{t=T}\|_{L^2(0,1))} \ge \|\Pi_K Q(u_*)|_{t=T}\|_{C([0,T];L^2(0,1))}$$

and that, by (18b),

$$\|\Pi_K Q(u_*)|_{t=T}\|_{C([0,T];L^2(0,1))} = \left(\sum_{k=1}^{K} \alpha_k^2\right)^{1/2},$$

we deduce that

$$\|Q(u_*)\|_{C([0,T];L^2(0,1))} \ge \left(\sum_{k=1}^{K} \alpha_k^2\right)^{1/2}. \tag{24}$$

In turn, given T, satisfying (13) and (21), formula (18c) implies the existence of a constant $M(K,T)$, as given explicitly in (34) below, such that for the corresponding v_* in (18c) we have

$$\|v_*\|_{L^2(0,T)} \le M(K,T) \left(\sum_{k=1}^{K} \alpha_k^2 \right)^{1/2}. \tag{25}$$

Combining (25) and (24) along with (20) yields

$$\|\mathcal{R}_K(u_*)\|_{C([0,T];L^2(0,1))} \le 2\sqrt{T}\|v_*\|_{L^2(0,T)} \le 2\sqrt{T}M(K,T) \left(\sum_{k=1}^{K} \alpha_k^2 \right)^{1/2}$$

$$\le 2\sqrt{T}M(K,T)\|Q(u_*)\|_{C([0,T];L^2(0,1))}, \quad \forall u_* \in U_K,$$

where v_* is defined by (18). Then, by (23a)

$$\|\mathcal{R}_K(u_*)\|_{C([0,T];L^2(0,1))} \le (2+2^{1-\alpha})2\sqrt{T}M(K,T)\|u_*\|^{1-\alpha}_{C([0,T];L^2(0,1))} \tag{26}$$

under the condition (23b).

Step 7. Consider the following equation in the *finite dimensional* linear space U_K (recall that $\mathcal{R}_K$ is continuous from U_K into U_K), endowed with the $C([0,T];L^2(0,1))$-norm:

$$u_* + \mathcal{R}_K(u_*) = 0. \tag{27}$$

By Altman's fixed point theorem, see (Schwartz, 1969, p.97) eqn. (27) has a solution if for some $L(K,T)$ the following estimate holds:

$$\|\mathcal{R}_K(u_*) + u_*\|^2_{C([0,T];L^2(0,1))} \ge \|\mathcal{R}_K(u_*)\|^2_{C([0,T];L^2(0,1))} - \|u_*\|^2_{C([0,T];L^2(0,1))}$$

$$\forall u_* \in U_K : \|u_*\|_{C([0,T];L^2(0,1))} = L(K,T). \tag{28}$$

The $C([0,T];L^2(0,1))$-norm of this solution does not exceed this $L(K,T)$. It suffices then to establish the existence of an $L(K,T)$ for which the expression on the right in (28) is non-positive. This immediately follows from (26), provided we take $\|u_*\|_{C([0,T];L^2(0,1))}$ sufficiently large so that

$$(2+2^{1-\alpha})2\sqrt{T}M(K,T)\|u_*\|^{1-\alpha}_{C([0,T];L^2(0,1))} \le \|u_*\|_{C([0,T];L^2(0,1))},$$

that is, taking into account (23b), for

$$L(K,T) = \max\left\{1, \left((2+2^{1-\alpha})2\sqrt{T}M(K,T)\right)^{1/\alpha}, \|u_{L*}\|_{C([0,T];L^2(0,1))}\right\}, \tag{29}$$

where T satisfies (13) and (21).

Step 8. Denote by $\hat{u} \in \bar{U}$ the solution of (27). Note that, in fact, by (27) and (19), $\hat{u} \in U$, that is, it is generated by some control as a solution to (4). Substitute $\hat{u}$ as u_* into (9b). Then (27) yields

$$u = u_{L*} + \hat{u} + Q(\hat{u}) = u_{L*} - \mathcal{R}_K(\hat{u}) + Q(\hat{u}). \tag{30}$$

Based on (15)–(17), select now v^* so that

$$\Pi_K u_{L*}(\cdot, T) = u_T. \tag{31}$$

Then, since $\Pi_K(Q(u_*) - \mathcal{R}_K(u_*))|_{t=T} = 0$ (see (18) and (19)), (30) and (31) imply

$$\Pi_K u(\cdot, T) = \Pi_K u_{L*}(\cdot, T) = u_T, \tag{32}$$

or (2) at time T satisfying (13) and (21).

Remark 1. With K increasing and/or T decreasing the value of $M(K,T)$, found in Step 8, tends to infinity.

Step 9. To extend the above result to T larger than it is given in (13) and (21), we need to derive (14) and (22) for these T. To do this, by using the time-invariantness of (1a) and (4), we can apply the same strategy as leading to (14) and (22), but on the interval $(T - T_*, T)$, where $T - T_*$ satisfies both (13) and (21), while putting $v \equiv 0$ on $(0, T - T_*)$ and with $u(\cdot, T - T_*)$ in place of u_0. Accordingly, we can, e.g., define $\bar{U}$ as a subset of $C([T - T_*, T]; L^2(0,1))$. This completes the proof of Theorem 1. ■

3. Proof of Theorem 2

The argument of Step 7 implies the existence of a constant $L(k,T), k = 1, \dots, T > 0$ in (29) such that

$$\begin{aligned} ||\hat{u}||_{C([0,T];L^2(0,1))} &\leq L(k,T) \\ &= \max\left\{1, \left((2 + 2^{1-\alpha})2\sqrt{T}M(k,T)\right)^{1/\alpha}, ||u_{L*}||_{C([0,T];L^2(0,1))}\right\}. \end{aligned} \tag{33}$$

Consider (25) in more detail. Note that we can set

$$\begin{aligned} M(K,T) = K \max_{i,j=1,\dots,K} &\left\{ \left(\sqrt{2}\int_{l_1}^{l_2} \sin \pi i x \, dx\right)^{-1} \right. \\ &\left. \times \left(\sqrt{2}\int_{l_1}^{l_2} \sin \pi j x \, dx\right)^{-1} \int_0^T q_i(\tau) q_j(\tau) \, d\tau \right\}. \end{aligned} \tag{34a}$$

Let

$$M_1(K) = \max_{i,j=1,\dots,K} \left\{ \left(\sqrt{2}\int_{l_1}^{l_2} \sin\pi i x \,\mathrm{d}x\right)^{-1} \left(\sqrt{2}\int_{l_1}^{l_2} \sin\pi j x \,\mathrm{d}x\right)^{-1} \right\}.$$

Then

$$M(K,T) \le K M_1(K) \max_{i,j=1,\dots,K} \left\{ \|q_i\|_{L^2(0,T)} \|q_j\|_{L^2(0,T)} \right\}.$$

In (Fattorini and Russell, 1974, Th. 1.5) it was shown that there is a $\delta > 0$ and an $M_2(T,\delta) > 0$ such that

$$\|q_i\|_{L^2(0,T)} \le M_2(T,\delta) e^{\lambda_i \delta}, \quad i = 1, \dots .$$

Therefore,

$$M(K,T) \le K M_1(K) M_2^2(T,\delta) e^{2\lambda_K \delta}, \quad K = 1, \dots . \tag{34b}$$

Select c_k's so that

$$\lim_{k\to\infty} c_{k+1}^{1/2} \left(k M_1(k) e^{2\lambda_k \delta}\right)^{1/\alpha} = 0. \tag{35}$$

Now, to prove Theorem 2, we need to show that for any $u_0 \in L^2(0,1)$ and $u_T \in W$ there is a sequence $\{u_k\}_{k=1}^{\infty}$ of solutions u_k to (1) converging to u_T in the W-norm. It is sufficient to do this for any $u_T \in \bigcup_{k=1}^{\infty} L_k^2(0,1)$.

Take any $\varepsilon > 0$ and

$$u_T = \sum_{i=1}^{m} a_i \omega_i \in L_m^2(0,1), \quad u_0 = \sum_{i=1}^{\infty} b_i \omega_i \in L^2(0,1).$$

Select

$$v^* = \sum_{i=1}^{I} \left(-e^{-\lambda_i T} b_i + a_i\right) v_i,$$

where we assume that $a_i = 0$ for $i > m$ and I is large enough to guarantee that

$$I \ge m \quad \text{and} \quad \left(\sum_{i=I+1}^{\infty} c_i b_i^2 e^{-2\lambda_i T}\right)^{1/2} < \varepsilon. \tag{36}$$

Then formulas (15)–(17) ensure that for the solution u_{L*I} to (4) corresponding to this v^* we have, similarly to (31), that

$$\Pi_I u_{L*I}(\cdot,T) = u_T \text{ and } u_{L*I}(\cdot,T) - u_T = \sum_{i=I+1}^{\infty} b_i e^{-\lambda_i T} \omega_i. \tag{37}$$

The argument of Section 2 implies (again, see (32)) that there is a sequence of solutions to (4): $u_{Lk} = u_{L*I} + u_{*k}, k = I, I+1, \dots$ such that for the corresponding solution u_k to (1) we have

$$\Pi_k u_k(\cdot,T) = \Pi_k u_{L*I}(\cdot,T), \quad k = I, I+1, \dots \tag{38}$$

and, by (9b), (22), (29) and (33),

$$\|u_k(\cdot,T)\|_{L^2(0,1)} \le 6L(k,T).$$

From (33)–(38) it follows that

$$\begin{aligned}\|u_k(\cdot,T)-u_T\|_W &\le \|u_k(\cdot,T)-\Pi_k u_k(\cdot,T)\|_W + \|\Pi_k u_k(\cdot,T)-u_T\|_W \\ &\le c_{k+1}^{1/2}\|u_k(\cdot,T)\|_{L^2(0,1)} + \|u_{L*I}(\cdot,T)-u_T\|_W \\ &\le 6c_{k+1}^{1/2}L(k,T) + \left(\sum_{i=I+1}^{\infty} c_i b_i^2 e^{-2\lambda_i T}\right)^{1/2} < \varepsilon\end{aligned}$$

as $k\to\infty$ (note that by (35) and (29) $6c_{k+1}^{1/2}L(k,T)\to 0$ as $k\to\infty$). This gives (3) required in Theorem 2. ■

4. Concluding Remarks

Let B be any finite-dimensional submanifold of the trace of the manifold U, defined in (10), in $L^2(0,1)$ at time T. Instead of v_k's as in (17), let us consider now any finite set of controls that generate solutions to (4) whose traces at $t=T$ form an orthonormalized basis in B (similar to (16)). In this way, it is not hard to see that, without any changes, the proof of Theorem 1 provides the exact controllability of (1) in B (i.e., in place of $L^2_K(0,1)$). This strategy can be extended to several space dimensions exactly in the same way, under the condition that the corresponding truncated linear problem (like (4)) is approximately controllable (for the latter see, e.g., (Sakawa, 1974)). More generally, this argument provides the following generalization of Theorem 1.

Consider the semilinear mixed problem

$$u_t = \Delta u + f(u) + \mathcal{P}v \ \text{ in } \ Q_T = \Omega\times(0,T), \quad v\in V, \tag{39}$$

$$u=0 \ \text{ in } \ \Sigma_T = \partial\Omega\times(0,T), \quad u|_{t=0}=u_0\in L^2(\Omega),$$

where Ω is a bounded domain of an n-dimensional Euclidean space $\mathbb{R}^n$ with boundary $\partial\Omega$ and $\mathcal{P}$ is an operator with the range in $L^2(Q_T)$, defined on a given set V of available controls v. Denote by $\mathcal{R}(T)$ the linear manifold generated by all the traces of solutions to (39) at time T, when v runs over V.

Theorem 3. *Let assumption (1b) hold. Given $T>0$, let $\mathcal{R}_*(T)$ be an arbitrary finite dimensional linear submanifold of $\mathcal{R}(T)$. Then for every $u_0\in L^2(\Omega)$, and $u_T\in\mathcal{R}_*(T)$ there is a control $v\in V$ such that for the corresponding solution to (39), (1b)*

$$\Pi_* u(\cdot,T) = u_T,$$

where Π_* denotes the operator of the orthogonal projection in $L^2(0,1)$ onto $\mathcal{R}_*(T)$.

On the other hand, the result of Theorem 2 cannot be extended to the general multidimensional case. Indeed, Theorem 2 makes use of the specific structure of the solution of the controllability problem for (4), exploiting Riesz's property of the sequence $\{e^{\lambda_k}, k = 1, \dots\}$. The latter does not hold for dimensions higher than 1.

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