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THE GRAM-SCHMIDT METHOD IN THE IDENTIFICATION OF A GENERALIZED CONTROL SYSTEM

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This article discusses the identification of a generalized linear control system described in the Bittner operational calculus by an abstract linear differential equation with constant coefficients. The identification problem leads to that of the best approximation in the vector space ℓ^2_m and is solved by using the Gram-Schmidt orthonormalization method. The classical Strejc method and the Shinbrot modulating function method are generalized here.

Keywords: operational calculus, derivative, integral, limit condition, control system, identification, orthonormalization method

1. Introduction

Control theory deals with systems whose dynamics are described, among others, by differential and difference equations. The analogies existing between them and their common features can be formulated in a uniform and unified way using Bittner (1974) operational calculus or Przeworska-Rolewicz (1988) algebraic analysis (cf. Sieńczewski, 1982).

In the paper, we consider the problem of identification of the coefficients $a_0, \ldots, a_n$ of the linear system

$$a_n S^n y + a_{n-1} S^{n-1} y + \cdots + a_1 S y + a_0 y = u \tag{1}$$

generated by means of the so-called abstract derivative S in the Bittner operational calculus. Due to the operational calculus model (i.e. the form of the operation S), (1) can be a differential or difference equation.

The basic notions of the Bittner operational calculus, which are used throughout the paper, are introduced in Section 2. In Section 3, the notion of the generalized control system described by the abstract differential eqn. (1) is introduced. In Section 4, the problem of the identification is carried over to determine the numbers $a_0, a_1, \ldots, a_n$, which quarantee a minimum of some functional $J(a_0, a_1, \ldots, a_n)$. The problem is solved as the best approximation problem in the vector space ℓ^2_m using the Gram-Schmidt orthonormalization. Section 5 includes a generalization of the classical

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method of the identification elaborated by Strejc (1961). A method which enables us to apply the Gram-Schmidt algorithm is obtained. In Section 6, it is shown how to use that algorithm for the identification of the system (1) by means of the modulating elements. This method was treated by the author and Zellma (1991) and it constitutes a generalization of the classical method of modulating functions presented by Shinbrot (1957).

The general theory elaborated in this paper is illustrated with a few numerical examples of various models of the operational calculus. They unify the identification of continuous and discrete, stationary and non-stationary, lumped- and distributed-parameter systems.

2. Operational Calculus

The *Bittner operational calculus* (Bittner, 1974) is a system

$$\mathrm{CO}(L^0, L^1, S, T_q, s_q, Q),$$

where L^0 and L^1 are linear spaces over the same field Γ of scalars and $L^1 \subset L^0$, the linear operation $S : L^1 \longrightarrow L^0$ (written as $S \in L(L^1, L^0)$), called the (abstract) *derivative*, is a surjection. Moreover, Q is a non-empty set of indices q for the operations $T_q \in L(L^0, L^1)$, $s_q \in L(L^1, L^1)$ called *integrals* and *limit conditions*, respectively, and such that $ST_q w = w$, $w \in L^0$, $T_q Sx = x - s_q x$, $x \in L^1$.

By induction we define a sequence of spaces L^n, $n \in \mathbb{N}$ such that

$$L^n := \{x \in L^{n-1} : Sx \in L^{n-1}\}.$$

Then $\quad \cdots \subset L^n \subset L^{n-1} \subset \cdots \subset L^1 \subset L^0 \quad$ and

$$S^n(L^{m+n}) = L^m,$$

where

$$L(L^n, L^0) \ni S^n := \underbrace{S \circ \cdots \circ S}_{n \text{ times}}, \quad n \in \mathbb{N}, \quad m \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}.$$

The kernel of S, i.e. the set $\operatorname{Ker} S := \{c \in L^1 : Sc = 0\}$, is called the *space of constants* for the derivative S.

For an element $x \in L^n$, $n \in \mathbb{N}$ the Taylor formula takes place:

$$x = s_q x + T_q s_q Sx + \cdots + T_q^{n-1} s_q S^{n-1} x + T_q^n S^n x, \quad q \in Q. \tag{2}$$

3. Generalized Control System

Let Γ be the field of reals $\mathbb{R}$. Consider the abstract differential equation

$$a_n S^n y + a_{n-1} S^{n-1} y + \cdots + a_1 Sy + a_0 y = u, \tag{3}$$

where $a_i \in \mathbb{R}$, $i \in \overline{0,n} := \{0, 1, \ldots, n\}$, $u \in L^0$, $y \in L^n$, $n \in \mathbb{N}$.

In control theory we are concerned with systems whose dynamics, in suitable models of operational calculus, is described by (3). The model (3) of those systems is called a *generalized linear differential stationary lumped-parameter control system* (briefly: *generalized control system*) (Bellert, 1975). The given element u and the unknown element y are called the *input signal* (*control*) and the *output signal* (*response*) of the system (3), respectively. The set Q is called the *instants set* (Wysocki and Zellma, 1991; 1993; 1995).

4. System Identification

Assume that from (3), after some operations which are described in what follows, we obtain a vector equation

$$\sum_{i=0}^{n} a_i \overline{v}_i = \overline{w},$$

where $\overline{w}$ and $\overline{v}_i$, $i \in \overline{0,n}$ are m-dimensional vectors depending on the control u and the response y, respectively, i.e.

$$\overline{w} = \overline{w}(u), \quad \overline{v}_i = \overline{v}_i(y), \quad i \in \overline{0,n} \tag{4}$$

such that

$$\overline{w}(0) = \overline{0}, \quad \overline{v}_i(0) = \overline{0}, \quad i \in \overline{0,n}.$$

The cases when

$$\overline{w}^*, \overline{v}_i^* \in (\mathrm{Ker}\, S)_m := \bigoplus_{\nu=1}^{m} \mathrm{Ker}\, S, \quad i \in \overline{0,n}, \quad m \geqslant n+1 \tag{5}$$

for a fixed *identifying pair* $(u^*, y^*) \in L^0 \times L^n$, where $\oplus$ means the direct sum[1], are discussed in (Wysocki and Zellma, 1991; 1993; 1995). In those papers, by *identification of a control system* (3) we understand the problem of choosing the coefficients of (3) with given elements u^*, y^* so that the functional

$$J(a_0, a_1, \dots, a_n) := \Big\| \sum_{i=0}^{n} a_i \overline{v}_i^* - \overline{w}^* \Big\| \tag{6}$$

(called the *identification performance index*) attains its minimum, where $\|\cdot\|$ is the norm induced by the inner product $(\cdot|\cdot)$ in a fixed Hilbert space H and $\overline{w}^*, \overline{v}_i^*, i \in \overline{0,n}$ are the vectors of the form (4) determined for u^* and y^*.

[1] If X is a linear space over Γ, then the *direct sum* $\bigoplus_{i=1}^{k} X$ means the set of all k-tuples $\overline{x} = (x_1, x_2, \dots, x_k)$ such that $x_i \in X, i \in \overline{1,k}$, with the usual operations of coordinate-wise addition and multiplication by scalars.

That problem has been solved by means of the orthogonal projection theorem (Luenberger, 1984, Th. 2). Solving the system of *normal equations*

$$\sum_{i=0}^{n} a_i^0 b_{ij} = c_j, \quad j \in \overline{0,n},$$

where $b_{ij} := (\overline{v}_i^* | \overline{v}_j^*)$, $c_j := (\overline{w}^* | \overline{v}_j^*)$, $j \in \overline{0,n}$, optimal coefficients $a_0^0, a_1^0, \ldots, a_n^0$ of (3) are obtained.

Now the problem of the identification of (3) will be transformed to the best approximation in the vector space ℓ_m^2. We solve that problem using the *Gram-Schmidt orthonormalization method* (Alexiewicz, 1969; Musielak, 1976). Accordingly, we assume that the condition $\overline{w}^*, \overline{v}_i^* \in \ell_m^2$, $i \in \overline{0,n}$, $m \geqslant n+1$ will be satisfied instead of (5).

The elements of the real space ℓ_m^2 are systems of m real numbers with common operations on vectors and with the inner product

$$(\overline{v}|\overline{w}) := \sum_{\nu=1}^{m} v_\nu w_\nu, \quad \overline{v}, \overline{w} \in \ell_m^2 \tag{7}$$

inducing the norm

$$\|\overline{v}\| = \sqrt{\sum_{\nu=1}^{m} v_\nu^2}, \quad \overline{v} \in \ell_m^2. \tag{8}$$

Assume that

$$\mathcal{B} := \{\overline{v}_0^*, \overline{v}_1^*, \ldots, \overline{v}_n^*\}$$

is the set of linearly independent vectors in ℓ_m^2. From

$$\sum_{i=0}^{n} a_i S^i y^* = 0$$

we obtain

$$\sum_{i=0}^{n} a_i \overline{v}_i^* = \overline{0},$$

which implies $a_0 = a_1 = \cdots = a_n = 0$. Hence we have to assume that $u^* \in L^0 \setminus \{0\}$.

From the Schmidt Theorem (Alexiewicz, 1969, Th. IX, 5.2) it follows that we can transform linearly the set $\mathcal{B}$ and obtain the orthonormal set

$$\mathcal{Z} = \{\, \overline{z}_0, \overline{z}_1, \ldots, \overline{z}_n \,\}.$$

The elements of the set have to be defined by means of the so-called *Gram-Schmidt orthonormalization process* in the following way:

$$\overline{z}_0 := \frac{\overline{v}_0^*}{\|\overline{v}_0^*\|}, \quad \overline{z}_i := \frac{\overline{v}_i^* - \sum_{j=0}^{i-1} (\overline{v}_i^* | \overline{z}_j) \overline{z}_j}{\|\overline{v}_i^* - \sum_{j=0}^{i-1} (\overline{v}_i^* | \overline{z}_j) \overline{z}_j\|}, \quad i \in \overline{1,n}. \tag{9}$$

From (9) it follows that the elements of $\mathcal{Z}$ can be expressed as

$$\overline{z}_i = p_{0i}\overline{v}_0^* + p_{1i}\overline{v}_1^* + \cdots + p_{ii}\overline{v}_i^*, \quad i \in \overline{0,n}. \tag{10}$$

Only one non-singular upper-triangular matrix

$$\hat{P}_{(n+1)\times(n+1)} := \begin{bmatrix} p_{00} & p_{01} & p_{02} & \cdots & p_{0n} \\ 0 & p_{11} & p_{12} & \cdots & p_{1n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & p_{nn} \end{bmatrix}$$

corresponds to the system (10).

Let $\hat{B}_{m\times(n+1)}$ and $\hat{Z}_{m\times(n+1)}$ be the matrices in which vectors $\overline{v}_0^*, \overline{v}_1^*, \ldots, \overline{v}_n^*$ and $\overline{z}_0, \overline{z}_2, \ldots, \overline{z}_n$ are the columns, respectively, i.e.

$$\hat{B} := [\overline{v}_0^*, \overline{v}_1^*, \ldots, \overline{v}_n^*], \quad \hat{Z} := [\overline{z}_0, \overline{z}_1, \ldots, \overline{z}_n].$$

With this notation, the system (10) can be represented in the matrix form

$$\hat{Z} = \hat{B}\hat{P}. \tag{11}$$

The elements of the columns of $\hat{P}$ are the coefficients of developments of the columns of $\hat{Z}$ in the basis $\mathcal{B}$ which is formed of the columns of $\hat{B}$.

From (11) we obtain

$$\hat{B} = \hat{Z}\hat{R}, \tag{12}$$

where $\hat{R} := \hat{P}^{-1}$. In this case the elements of the columns of $\hat{R}$ are the coefficients of developments of the columns of $\hat{B}$ in the basis $\mathcal{Z}$ which is formed of the columns of $\hat{Z}$.

In the numerical examples which will be presented in what follows, the Gram-Schmidt algorithm (Fortuna *et al.*, 1982) has been applied for the distribution (12) of $\hat{B}$ to the orthonormal matrix $\hat{Z}$ and the upper-triangular matrix $\hat{R}$.

The problem of determining $\hat{P} = \hat{R}^{-1}$ is reduced to that of solving $n+1$ linear algebraic equations

$$\hat{R}\overline{p}_j = \overline{e}_j, \quad j \in \overline{0,n}, \tag{13}$$

where $\overline{p}_j$ is a vector which is the j-th column of $\hat{P}$ and $\overline{e}_j$ is a vector which has unity in the j-th position and the other elements are equal to zero. Each of the dependencies (13) is the system of equations with the triangular matrix $\hat{R}$, which can be solved recursively (Fortuna *et al.*, 1982).

From the Schmidt Theorem it can also be concluded that $\mathrm{Lin}\,\mathcal{B} = \mathrm{Lin}\,\mathcal{Z}$.[2] Thus the criterion of choosing optimal coefficients of eqn. (3) can be presented in the equivalent form

$$\min\left\{ \Big\| \sum_{i=0}^{n} a_i \overline{v}_i^* - \overline{w}^* \Big\|^2 : \overline{v}_i^* \in \mathcal{B},\ a_i \in \mathbb{R},\ i \in \overline{0,n} \right\}$$
$$= \min\left\{ \|\overline{v} - \overline{w}^*\|^2 : \overline{v} \in \mathrm{Lin}\,\mathcal{B} \right\} = \min\left\{ \|\overline{z} - \overline{w}^*\|^2 : \overline{z} \in \mathrm{Lin}\,\mathcal{Z} \right\}. \quad (14)$$

Since $\mathcal{Z}$ is a set of orthonormal elements, from the Bessel Theorem it follows (Alexiewicz, 1969, Th. IX, 5.4) that the above-mentioned minimum is attained for

$$\overline{z}^0 = \sum_{i=0}^{n} b_i^0 \overline{z}_i \quad (15)$$

and it equals

$$\|\overline{w}^*\|^2 - \sum_{i=0}^{n} (b_i^0)^2, \quad (16)$$

where b_i^0 are the *Fourier coefficients* of the element $\overline{w}^*$ for the system $\mathcal{Z}$, i.e.

$$b_i^0 := \left(\overline{w}^* | \overline{z}_i\right), \quad i \in \overline{0,n}. \quad (17)$$

Using (15) and (10), we can represent the optimal vector $\overline{z}^0$ as an element of the linear span $\mathrm{Lin}\,\mathcal{B}$. Namely, we get

$$\overline{z}^0 = \sum_{i=0}^{n} b_i^0 \Big(\sum_{j=0}^{i} p_{ji} \overline{v}_j^* \Big) = \sum_{i=0}^{n} \Big(\sum_{j=i}^{n} b_j^0 p_{ij} \Big) \overline{v}_i^*.$$

According to the above remark and (14), we conclude that

$$a_i^0 = \sum_{j=i}^{n} b_j^0 p_{ij}, \quad i \in \overline{0,n} \quad (18)$$

are coefficients of (3) minimizing the identification performance index (6).

[2] If $Y = \{y_1, y_2, \ldots, y_k\} \subset X$ and X is a linear space over Γ, then $\mathrm{Lin}\,Y$ means the *linear span* of Y, i.e. $\mathrm{Lin}\,Y := \{x \in X : x = \sum_{i=1}^{k} \gamma_i y_i,\ \gamma_i \in \Gamma, y_i \in Y, i \in \overline{1,k}\}$.

It is obvious that

$$J(a_0^0, a_1^0, \ldots, a_n^0) = J(b_0^0, b_1^0, \ldots, b_n^0).$$

Moreover,

$$J(a_0^0, a_2^0, \ldots, a_n^0) = \sqrt{\sum_{\nu=1}^{m} \Big(\sum_{i=0}^{n} a_i^0 V_i^\nu - W^\nu \Big)^2} \tag{19}$$

and

$$J(b_0^0, b_1^0, \ldots, b_n^0) = \sqrt{\sum_{\nu=1}^{m} (W^\nu)^2 - \sum_{i=0}^{n} \Big(\sum_{\nu=1}^{m} W^\nu Z_i^\nu \Big)^2}, \tag{20}$$

where W^ν, V_i^ν, Z_i^ν mean the ν-th coordinates of vectors $\overline{w}^*$, $\overline{v}_i^*$, $\overline{z}_i$, respectively.

Formulae (19) and (20) follow from (6), (16) and from (7), (8), respectively. To calculate the values of J, (19) is applied in numerical examples in this paper.

In suitable models of operational calculus, the elements u^* and y^* mean approximates, on the basis of the values obtained from measurements on a real system, input and output signals of the system. Spline interpolation of the signals is discussed in (Wysocki and Zellma, 1993; 1995).

On account of the above remarks, it may be deduced that the identification of (3) based on the orthonormalization method comprises the following stages:

1. Determining the distribution (12) by means of the Gram-Schmidt algorithm.
2. Determining the matrix $\hat{P} = \hat{R}^{-1}$ by recurrently solving the system (13).
3. Determining the optimal coefficients b_i^0 from (17).
4. Determining the optimal coefficients a_i^0 from (18).

Now we shall generalize two classical identification methods of control systems. We get the generalizations pertaining to the vector equation

$$\sum_{i=0}^{n} a_i \overline{v}_i = \overline{w},$$

where $\overline{v}_i, \overline{w} \in \ell_m^2$. This enables us to appy the Gram-Schmidt method.

5. Strejc Methods

The classical *Strejc method* (Eykhoff, 1974; Strejc, 1961) concerns a continuous stationary lumped-parameter system and it depends on the multiple integration of the equation describing its dynamics. The problem of choosing the best model is not considered in the classical method.

5.1. One-Point Method

Let T_q be a fixed integral corresponding to the derivative S. Moreover, let $r_1, r_2, \dots, r_m \in \mathbb{N}_0$ be given such that $r_i \neq r_j$ for $i \neq j$, where $i, j \in \overline{1,m}$, $m \geqslant n+1$. For each pair of signals $(u,y) \in L^0 \times L^n$ satisfying (3) with given coefficients $a_0, a_1, \dots, a_n$ we have

$$a_n T_q^{r_\nu} S^n y + a_{n-1} T_q^{r_\nu} S^{n-1} y + \cdots + a_1 T_q^{r_\nu} S y + a_0 T_q^{r_\nu} y = T_q^{r_\nu} u, \quad \nu \in \overline{1,m}. \tag{21}$$

Let Ω be a given non-empty set. Consider an indexing family of linear functionals $\{F_\omega\}_{\omega \in \Omega}$ defined on L^0. The set Ω will be called the *observation set* of (3), while its elements the *points*. The examples given below indicate in which way it is easiest to define Ω and F_ω's.

Assume that a mapping F_ω is fixed. Then from (21) we get

$$\sum_{i=0}^{n} a_i \overline{v}_i = \overline{w},$$

where

$$\overline{v}_i := \begin{bmatrix} F_\omega T_q^{r_1} S^i y \\ \vdots \\ F_\omega T_q^{r_m} S^i y \end{bmatrix}, \quad \overline{w} := \begin{bmatrix} F_\omega T_q^{r_1} u \\ \vdots \\ F_\omega T_q^{r_m} u \end{bmatrix}, \quad i \in \overline{0,n}. \tag{22}$$

Using the Taylor formula (2), we can reduce the order of the derivatives of the output signal y in coordinates of vectors $\overline{v}_i$ and express them using limit conditions. Namely, for $i \geqslant r$ we have

$$T_q^r S^i y = T_q^r S^r (S^{i-r} y) = S^{i-r} y - \sum_{j=0}^{r-1} T_q^j s_q S^{i+j-r} y. \tag{23}$$

However, for $i < r$ we obtain

$$T_q^r S^i y = T_q^{r-i} (T_q^i S^i y) = T_q^{r-i} y - \sum_{j=0}^{i-1} T_q^{r-i+j} s_q S^j y. \tag{24}$$

Now, the identification problem of the control system (3) consists in minimizing the functional

$$J_{q,\overline{r},\omega}(a_0, a_1, \dots, a_n) = \Big\| \sum_{i=0}^{n} a_i \overline{v}_i^* - \overline{w}^* \Big\|, \tag{25}$$

where $\overline{v}_i^*, \overline{w}^* \in \ell_m^2$ are the vectors of the form (22) determined for the fixed elements $u^* \in L^0 \backslash \{0\}$, $y^* \in L^n \backslash \operatorname{Ker} S^n$. The identification performance index (25) depends on the point $q \in Q$ defining the integral T_q, on the vector $\overline{r} := [r_1, r_2, \dots, r_m]$ defining iterations of T_q and on the point $\omega \in \Omega$ defining the mapping F_ω. It is necessary to choose those quantities in a such way that $\mathcal{B} = \{\overline{v}_0^*, \overline{v}_1^*, \dots, \overline{v}_n^*\}$ be a set of linearly independent vectors in ℓ_m^2.

Example 1. In the operational calculus (Bittner and Mieloszyk, 1982) with the derivative

$$Sx := \frac{\partial x(t,z)}{\partial t} + \frac{\partial x(t,z)}{\partial z},$$

integrals

$$T_q w := \int_q^t w(\tau, z - t + \tau)\,\mathrm{d}\tau$$

and limit conditions

$$s_q x := x(q, z - t + q),$$

where $q \in Q := [t_0, t_k] \subset \mathbb{R}$, $w = w(t,z) \in L^0 := C^1(Q \times \mathbb{R}, \mathbb{R})$, $x = x(t,z) \in L^1 = \{x \in L^0 : Sx \in L^0\}$, the operational equation

$$a_1 Sy + a_0 y = u$$

takes the form of the quasi-linear partial differential equation

$$a_1 \left(\frac{\partial y(t,z)}{\partial t} + \frac{\partial y(t,z)}{\partial z} \right) + a_0 y(t,z) = u(t,z). \tag{26}$$

Equation (26) describes the dynamics of a stationary distributed-parameter system in the classical sense. The elements $u^* = u^*(t,z) \in L^0 \setminus \{0\}$, $y^* = y^*(t,z) \in L^1 \setminus \operatorname{Ker} S$ are such that $u^*|_\Omega$ and $y^*|_\Omega$ are respectively the input and output signals of the system (26) approximated in the rectangle $\Omega := [t_0, t_k] \times [z_0, z_k]$ (on the basis of the values obtained from the measurements on a real system). The point $\omega = (a, b) \in \Omega$ defines the functional

$$F_\omega y(t,z) := y(a,b), \quad y = y(t,z) \in L^0$$

uniquely. With $m = 2$, for F_ω so determined, from (22)–(24) we get

$$V_0^1 = \begin{cases} y^*(a,b) & \text{for} \quad r_1 = 0, \\ \displaystyle\int_q^a y^*(\tau, b - a + \tau)\,\mathrm{d}\tau & \text{for} \quad r_1 = 1, \end{cases} \tag{27}$$

$$V_0^2 = \begin{cases} \displaystyle\int_q^a y^*(\tau, b - a + \tau)\,\mathrm{d}\tau & \text{for} \quad r_2 = 1, \\ \displaystyle\int_q^a \int_q^\tau y^*(\xi, b - a + \xi)\,\mathrm{d}\xi\,\mathrm{d}\tau & \text{for} \quad r_2 = 2, \end{cases} \tag{28}$$

$$V_1^1 = \begin{cases} \dfrac{\partial y^*(a,b)}{\partial t} + \dfrac{\partial y^*(a,b)}{\partial z} & \text{for} \quad r_1 = 0, \\ y^*(a,b) - y^*(q, b - a + q) & \text{for} \quad r_1 = 1, \end{cases} \tag{29}$$

$$V_1^2 = \begin{cases} y^*(a,b) - y^*(q,b-a+q) & \text{for} \quad r_2 = 1, \\ (q-a)y^*(q,b-a+q) + \int_q^a y^*(\tau, b-a+\tau)\,\mathrm{d}\tau & \text{for} \quad r_2 = 2, \end{cases} \tag{30}$$

$$\left.\begin{array}{l} W^1 = \begin{cases} u^*(a,b) & \text{for} \quad r_1 = 0, \\ \int_q^a u^*(\tau, b-a+\tau)\,\mathrm{d}\tau & \text{for} \quad r_1 = 1, \end{cases} \\ \\ W^2 = \begin{cases} \int_q^a u^*(\tau, b-a+\tau)\,\mathrm{d}\tau & \text{for} \quad r_2 = 1, \\ \int_q^a \int_q^\tau u^*(\xi, b-a+\xi)\,\mathrm{d}\xi\,\mathrm{d}\tau & \text{for} \quad r_2 = 2, \end{cases} \end{array}\right\} \tag{31}$$

where $q \in [t_0, t_k]$ and $q \neq a$.

Table 1 contains numerical results of the identification of the system (26). In the example, the coordinates of vectors $\overline{v}_i^*$ and $\overline{w}^*$ are determined directly from the formulae (27)–(31) using the analytical forms of u^* and y^* which are given in the table. The table also contains the absolute errors

$$\Delta_{q,\overline{r},\omega} y(t_j, z_j) = \left| y^*(t_j, z_j) - y(t_j, z_j) \right|, \quad (t_j, z_j) \in \Omega.$$

To define them, first we need to determine the system response $y(t,z)$ to the input signal $u^*(t,z)$. This is achieved by solving the equation

$$a_1^0 \left(\frac{\partial y(t,z)}{\partial t} + \frac{\partial y(t,z)}{\partial z} \right) + a_0^0 y(t,z) = u^*(t,z) \tag{32}$$

with the limit condition

$$y(q,z) = y^*(q,z). \tag{33}$$

The solution to the problem (32), (33)is expressed by the following formula (Mieloszyk, 1986):

$$y(t,z) = \exp\left[\frac{(q-t)a_0^0}{a_1^0}\right] \Bigg\{ y^*(q, z-t+q) + \frac{1}{a_1^0} \int_q^t u^*(\tau, z-t+\tau) \exp\left[\frac{(\tau-q)a_0^0}{a_1^0}\right] \mathrm{d}\tau \Bigg\}.$$

♦

Table 1. One-point Strejc method in identification of a partial differential equation of the first order.

$\Omega = [0,2] \times [0,2]$			$a_1 \left(\frac{\partial y}{\partial t} + \frac{\partial y}{\partial z} \right) + a_0 y = u$			
$q = 0$			$u^* = 3.98t + 6.04z + 5.03, \quad y^* = 2.01t + 3.04z - 4.97$			
$\overline{r} = [r_1, r_2]$			$r_1 = 0, \quad r_2 = 1$			
$\omega = (a,b)$			$a = 1, \quad b = 2$		$a = 2, \quad b = 2$	
a_0^0			1.98416		1.98416	
a_1^0			2.95038		2.94877	
$J_{q,\overline{r},\omega}$			8.33000×10^{-9}		3.79907×10^{-8}	
t	z	y^*	y	$\Delta_{q,\overline{r},\omega} y$	y	$\Delta_{q,\overline{r},\omega} y$
0.5	0.25	−3.205	−3.20647	1.46769×10^{-3}	−3.20529	2.93676×10^{-4}
1.0	0.50	−1.440	−1.44302	3.01952×10^{-3}	−1.44101	1.00690×10^{-3}
1.5	0.75	0.325	0.32043	4.57158×10^{-3}	0.32304	1.95988×10^{-3}
2.0	1.00	2.090	2.08392	6.08102×10^{-3}	2.08696	3.04131×10^{-3}
$\overline{r} = [r_1, r_2]$			$r_1 = 1, \quad r_2 = 2$			
$\omega = (a,b)$			$a = 1, \quad b = 2$		$a = 2, \quad b = 2$	
a_0^0			2.00000		2.00000	
a_1^0			2.94851		2.94851	
$J_{q,\overline{r},\omega}$			1.30385×10^{-8}		3.72529×10^{-8}	
t	z	y^*	y	$\Delta_{q,\overline{r},\omega} y$	y	$\Delta_{q,\overline{r},\omega} y$
0.5	0.25	−3.205	−3.19510	9.90471×10^{-3}	−3.19510	9.90468×10^{-3}
1.0	0.50	−1.440	−1.42633	1.36706×10^{-2}	−1.42633	1.36706×10^{-2}
1.5	0.75	0.325	0.33777	1.27688×10^{-2}	0.33777	1.27687×10^{-2}
2.0	1.00	2.090	2.09833	8.33180×10^{-3}	2.09833	8.33176×10^{-3}

Example 2. Let

$$L^n := \{ f(y) : \ f = f(t), \ y = y(t) \in C^n(\mathbb{R}, \mathbb{R}) \}, \quad n = 0, 1.$$

For any $f(y), g(y) \in L^1$ and $\alpha, \beta \in \mathbb{R}$ the operation defined by

$$Sf(y) := \frac{\mathrm{d}f(y)}{\mathrm{d}y} \frac{\mathrm{d}y(t)}{\mathrm{d}t} \tag{34}$$

satisfies the condition

$$S\big[\alpha f(y) + \beta g(y)\big] = \alpha S f(y) + \beta S g(y).$$

An analogous linearity condition is satisfied by the mappings

$$T_q g(y) := \int_q^t g\big[y(\tau)\big]\,\mathrm{d}\tau, \quad s_q f(y) := f\big[y(q)\big], \tag{35}$$

where $q \in Q := \mathbb{R}$ and $f(y) \in L^1$, $g(y) \in L^0$.

The operations (34), (35) form the so-called *pseudo-nonlinear model* of operational calculus (cf. Bellert, 1963). In that model the equation

$$a_1 SY + a_0 Y = u \tag{36}$$

describes a generalized pseudo-nonlinear control system in which $Y = f(y)$ is a function of the output signal y.

Consider the identification problem of the coefficients α_0, α_1 of the non-linear system

$$\alpha_1 y(t) y'(t) + \alpha_0 y^2(t) = u(t). \tag{37}$$

Let the elements $u^* = u^*(t) \in L^0 \setminus \{0\}$, $y^* = y^*(t) \in L^1 \setminus \operatorname{Ker} S$ be the identification pair for that problem, i.e. $u^*|_\Omega$ and $y^*|_\Omega$ are the input and output signals approximated in the interval $\Omega := [t_0, t_k]$ (based on the values obtained from the measurements). To use the outlined identification method, we convert (37) to the form (36) by letting $f(y) := y^2$ and $a_1 := 0.5\alpha_1$, $a_0 := \alpha_0$. For that case, $m = 2$ and

$$F_\omega f(y) := f\big[y(\omega)\big], \quad \omega \in \Omega, \quad f(y) \in L^0,$$

from (22)–(24) we get

$$V_0^1 = \begin{cases} \big[y^*(\omega)\big]^2 & \text{for } r_1 = 0, \\ \displaystyle\int_q^\omega \big[y^*(\tau)\big]^2\,\mathrm{d}\tau & \text{for } r_1 = 1, \end{cases} \tag{38}$$

$$V_0^2 = \begin{cases} \displaystyle\int_q^\omega \big[y^*(\tau)\big]^2\,\mathrm{d}\tau & \text{for } r_2 = 1, \\ \displaystyle\int_q^\omega\int_q^\tau [y^*(\xi)]^2\,\mathrm{d}\xi\,\mathrm{d}\tau & \text{for } r_2 = 2, \end{cases} \tag{39}$$

$$V_1^1 = \begin{cases} 2y^*(\omega)\big[y^*(\omega)\big]' & \text{for } r_1 = 0, \\ \big[y^*(\omega)\big]^2 - \big[y^*(q)\big]^2 & \text{for } r_1 = 1, \end{cases} \tag{40}$$

$$V_1^2 = \begin{cases} \left[y^*(\omega)\right]^2 - \left[y^*(q)\right]^2 & \text{for} \quad r_2 = 1, \\ (q-\omega)\left[y^*(q)\right]^2 + \displaystyle\int_q^\omega \left[y^*(\tau)\right]^2 \mathrm{d}\tau & \text{for} \quad r_2 = 2, \end{cases} \tag{41}$$

$$W^1 = \begin{cases} u^*(\omega) & \text{for} \quad r_1 = 0, \\ \displaystyle\int_q^\omega u^*(\tau)\,\mathrm{d}\tau & \text{for} \quad r_1 = 1, \end{cases} \tag{42}$$

$$W^2 = \begin{cases} \displaystyle\int_q^\omega u^*(\tau)\,\mathrm{d}\tau & \text{for} \quad r_2 = 1, \\ \displaystyle\int_q^\omega \int_q^\tau u^*(\xi)\,\mathrm{d}\xi\,\mathrm{d}\tau & \text{for} \quad r_2 = 2, \end{cases} \tag{43}$$

where $q \in [t_0, t_k]$ and $q \neq \omega$. Table 2 contains numerical results for the above model.

The coordinates of the vectors $\overline{v}_i^*$ and $\overline{w}^*$ are determined from (38)–(43) using the forms of u^* and y^* shown in the table. Solving the initial-value problem

$$\alpha_1^0 y(t) y'(t) + \alpha_0^0 y^2(t) = u^*(t), \quad y(t_0) = y^*(t_0),$$

the absolute errors

$$\Delta_{q,\overline{r},\omega} y(t_j) = \left|y^*(t_j) - y(t_j)\right|, \quad t_j \in \Omega \tag{44}$$

are also determined. ♦

5.2. Multi-Point Method

Let the integral T_q and the functionals $F_{\omega_1}, F_{\omega_2}, \ldots, F_{\omega_m}$, $m \geqslant n+1$ be fixed. Then for a fixed pair $(u, y) \in L^0 \times L^n$ satisfying (3) and for a fixed $r \in \mathbb{N}_0$ we have

$$a_n F_{\omega_\nu} T_q^r S^n y + a_{n-1} F_{\omega_\nu} T_q^r S^{n-1} y + \cdots + a_1 F_{\omega_\nu} T_q^r S y + a_0 F_{\omega_\nu} T_q^r y = F_{\omega_\nu} T_q^r u, \quad \nu \in \overline{1, m},$$

i.e.

$$\sum_{i=1}^{n} a_i \overline{v}_i = \overline{w},$$

where

$$\overline{v}_i := \begin{bmatrix} F_{\omega_1} T_q^r S^i y \\ \vdots \\ F_{\omega_m} T_q^r S^i y \end{bmatrix}, \quad \overline{w} := \begin{bmatrix} F_{\omega_1} T_q^r u \\ \vdots \\ F_{\omega_m} T_q^r u \end{bmatrix}, \quad i \in \overline{0, n}. \tag{45}$$

Table 2. One-point Strejc method in identification of an ordinary non-linear differential equation of the first order.

$\Omega = [0,3]$		$\alpha_1 yy' + \alpha_0 y^2 = u$			
$q = 0,\ \ \omega = 3$		$u^* = 2.97t^2 - 3.03,\ \ y^* = 0.98t + 1.02$			
$\overline{r} = [r_1, r_2]$		$r_1 = 0,\ \ r_2 = 1$		$r_1 = 1,\ \ r_2 = 2$	
α_0^0		3.06880		3.05826	
α_1^0		−6.29347		−6.26359	
$J_{q,\overline{r},\omega}$		1.59275×10^{-7}		2.16964×10^{-7}	
t	y^*	y	$\Delta_{q,\overline{r},\omega} y$	y	$\Delta_{q,\overline{r},\omega} y$
0.5	1.51	1.50668	3.32324×10^{-3}	1.50802	1.98046×10^{-3}
1.0	2.00	1.99518	4.82482×10^{-3}	1.99779	2.20607×10^{-3}
1.5	2.49	2.48405	5.94941×10^{-3}	2.48800	1.99556×10^{-3}
2.0	2.98	2.97261	7.38697×10^{-3}	2.97805	1.94912×10^{-3}
2.5	3.47	3.46029	9.70998×10^{-3}	3.46746	2.53579×10^{-3}
3.0	3.96	3.94640	1.35969×10^{-3}	3.95571	4.28992×10^{-3}

Now the identification performance index (6) depends on the point $q \in Q$ defining the integral T_q, on the number $r \in \mathbb{N}_0$ defining its iteration and on the vector $\overline{\omega} := [\omega_1, \omega_2, \ldots, \omega_m]$ defining the mappings F_{ω_ν}, where $\omega_\nu \in \Omega$. Thus

$$J_{q,r,\overline{\omega}}(a_0, a_1, \ldots, a_n) = \Big\| \sum_{i=0}^{n} a_i \overline{v}_i^* - \overline{w}^* \Big\|,$$

where $\overline{v}_i^*, \overline{w}^* \in \ell_m^2$ are vectors of the form (45) determined for fixed elements $u^* \in L^0 \setminus \{0\}$ and $y^* \in L^n \setminus \operatorname{Ker} S^n$. To obtain the coordinates of $\overline{v}_i^*$, we use (23) and (24).

Example 3. In the classical model of operational calculus in which

$$L^n := C^n(Q, \mathbb{R}), \quad n \in \mathbb{N}_0, \quad q \in Q := [t_0, t_k] \subset \mathbb{R}$$

and

$$S := \frac{\mathrm{d}}{\mathrm{d}t}, \quad T_q := \int_q^t, \quad s_q := |_{t=q}$$

the second-order equation

$$a_2 S^2 y + a_1 S y + a_0 y = u \tag{46}$$

takes the form

$$a_2 y''(t) + a_1 y'(t) + a_0 y(t) = u(t) \tag{47}$$

and describes the dynamics of a linear stationary lumped-parameter system. For

$$F_{\omega_\nu} y(t) := y(\omega_\nu), \omega_\nu \in \Omega := [t_0, t_k], \quad \nu \in \overline{1,m}, \quad m \geqslant 3, \quad y = \{y(t)\} \in L^0,$$

from (45) and (23) we obtain

$$V_0^\nu = \begin{cases} y^*(\omega_\nu) & \text{for} \quad r = 0, \\ \displaystyle\int_q^{\omega_\nu} y^*(\tau)\,\mathrm{d}\tau & \text{for} \quad r = 1, \end{cases} \tag{48}$$

$$V_1^\nu = \begin{cases} \dot{y}^*(\omega_\nu) & \text{for} \quad r = 0, \\ y^*(\omega_\nu) - y^*(q) & \text{for} \quad r = 1, \end{cases} \tag{49}$$

$$V_2^\nu = \begin{cases} \ddot{y}^*(\omega_\nu) & \text{for} \quad r = 0, \\ \dot{y}^*(\omega_\nu) - \dot{y}^*(q) & \text{for} \quad r = 1, \end{cases} \tag{50}$$

$$W^\nu = \begin{cases} u^*(\omega_\nu) & \text{for} \quad r = 0, \\ \displaystyle\int_q^{\omega_\nu} u^*(\tau)\,\mathrm{d}\tau & \text{for} \quad r = 1, \end{cases} \tag{51}$$

$q \neq \omega_\nu, \quad \nu \in \overline{1,m}$, where $u^* \in L^0 \setminus \{0\}, y^* \in L^2 \setminus \operatorname{Ker} S^2$ and functions $u^*|_\Omega$ and $y^*|_\Omega$ have the same meaning as before.

To assess the applied method by means of the absolute errors (44), first we have to determine the model signal $y(t)$ by solving the initial-value problem

$$a_2^0 y''(t) + a_1^0 y'(t) + a_0^0 y(t) = u^*(t), \quad y(t_0) = y^*(t_0), \quad y'(t_0) = \dot{y}^*(t_0).$$

Table 3 shows numerical results of the identification of (47) in which V_i^ν, W^ν are determined directly from (48)–(51) by using the analytical forms of u^* and y^*. ♦

Example 4. Consider the difference equation

$$a_2 y(k+2) + a_1 y(k+1) + a_0 y(k) = u(k) \tag{52}$$

describing the relationship between the input and output signals of a discrete stationary lumped-parameter system.

Let $C(\mathbb{N}_0)$ be the real linear space of real sequences $y(k)$, $k \in \mathbb{N}_0$ (with usual operations on sequences). The difference eqn. (52) is a particular case of the abstract differential eqn. (46) if we consider the operational calculus with the derivative

$$Sy(k) := y(k+1),$$

Table 3. Multi-point Strejc method in identification of an ordinary differential equation of the second order.

$\Omega = [0,1]$		$a_2y'' + a_1y' + a_0y = u$			
$q = 0$		$u^* = 0.9t^3 + 6.2t^2 - 5.3t + 4.1, \quad y^* = t^3 + t + 2$			
r		$r = 0$			
ω_ν		0.25 0.5 1		0.25 0.5 0.75 1	
a_0^0		1.05135		1.05156	
a_1^0		1.97838		1.97196	
a_2^0		−1.03649		−1.03223	
$J_{q,r,\varpi}$		6.39704×10^{-8}		3.17772×10^{-3}	
t	y^*	y	$\Delta_{q,r,\varpi}y$	y	$\Delta_{q,r,\varpi}y$
0.25	2.26563	2.26524	3.83224×10^{-4}	2.26509	5.33074×10^{-4}
0.50	2.62500	2.62382	1.17932×10^{-3}	2.62333	1.67065×10^{-3}
0.75	3.17188	3.16930	2.57758×10^{-3}	3.16836	3.51706×10^{-3}
1.00	4.00000	3.99466	5.34326×10^{-3}	3.99310	6.89621×10^{-3}
r		$r = 1$			
a_0^0		1.04641		1.04996	
a_1^0		2.00261		1.99206	
a_2^0		−1.04706		−1.04326	
$J_{q,r,\varpi}$		2.07898×10^{-8}		4.23165×10^{-4}	
t	y^*	y	$\Delta_{q,r,\varpi}y$	y	$\Delta_{q,r,\varpi}y$
0.25	2.26563	2.26559	3.97973×10^{-5}	2.26553	9.85786×10^{-5}
0.50	2.62500	2.62496	4.05535×10^{-5}	2.62486	1.36968×10^{-4}
0.75	3.17188	3.17169	1.87866×10^{-4}	3.17168	1.98998×10^{-4}
1.00	4.00000	3.99943	5.70748×10^{-4}	3.99955	4.53703×10^{-4}

integral

$$T_0y(k) := \begin{cases} 0 & \text{for} \quad k = 0, \\ y(k-1) & \text{for} \quad k > 0, \end{cases}$$

and limit condition

$$s_0y(k) := y(0)\delta_0^k,$$

where $y(k) \in L^0 = L^1 := C(\mathbb{N}_0), q = 0 \in Q := \{0\}$ and δ_0^k is the Kronecker symbol.

Let $\Omega := \overline{0,l}, l \in \mathbb{N}$. The sequences $u^* = u^*(k), y^* = y^*(k) \in L^0$ are such that $u^*|_\Omega$ and $y^*|_\Omega$ denote the observed values of the input and output signals of the real system for $k \in \Omega$, respectively. For

$$F_{\omega_\nu} y(k) := y(\omega_\nu), \quad \omega_\nu \in \Omega, \quad \nu \in \overline{1,m}, \quad m \geqslant 3, \quad y(k) \in L^0,$$

based on (45), we get

$$V_0^\nu = \begin{cases} y^*(\omega_\nu) & \text{for} \quad r = 0, \\ y^*(\omega_\nu - 1) & \text{for} \quad r = 1, \end{cases} \qquad V_1^\nu = \begin{cases} y^*(\omega_\nu + 1) & \text{for} \quad r = 0, \\ y^*(\omega_\nu) & \text{for} \quad r = 1, \end{cases}$$

$$V_2^\nu = \begin{cases} y^*(\omega_\nu + 2) & \text{for} \quad r = 0, \\ y^*(\omega_\nu + 1) & \text{for} \quad r = 1, \end{cases} \qquad W^\nu = \begin{cases} u^*(\omega_\nu) & \text{for} \quad r = 0, \\ u^*(\omega_\nu - 1) & \text{for} \quad r = 1, \end{cases}$$

where $\omega_\nu \in \overline{1,l}, \nu \in \overline{1,m}$.

Using the recurrent formula

$$y(k+2) = \frac{1}{a_2^0}\left[u^*(k) - a_0^0 y(k) - a_1^0 y(k+1)\right], \quad y(0) = y^*(0), \quad y(1) = y^*(1),$$

we determine the model signal $y(k)$ as the response of the system to the input signal $u^*(k)$.

We shall use the above model in identification of eqn. (52) describing the motion of a submarine. In a submarine the change in the increment $H(t) := h(t_0) - h(t)$ in the draught depth of the vessel depends, among other things, on the change in the trimming moment $M(t)$. In (Stępień, 1981) one can find results of the measurements of $H(t)$ versus $M(t)$ caused by the weight of the water translocated in the trimming tanks from the bow to the stern with constant speed $v = 2$ m/s. Those measurements are shown in Table 4, where

$$u(k) := M(10k), \quad y(k) := H(10k), \quad k \in \overline{0,23}.$$

This table also contains the corresponding identification results. ♦

From the examples it follows that the Strejc methods can be applied to various models of operational calculus. The examples impose no limits on those models. In the *modulating-element method*, which we are going to describe, the operations S and s_q require some additional conditions.

6. Modulating-Element Method

According to (23) and (24), in Strejc methods it is necessary to know the limit conditions for the derivative of the output signal. Practically determining the derivatives is rather difficult. To avoid it we can apply the modulating-element method (Wysocki and Zellma, 1993). The classical modulating-function method (Eykhoff, 1974; Loeb and Cahen, 1963; Shinbrot, 1957) concerns a continuous stationary lumped-parameter

Table 4. Multi-point Strejc method in identification of a difference equation of the second order describing the draught depth increment of a submarine depending on the trimming moment.

$\Omega = \overline{0,23}$			$a_2y(k+2)+a_1y(k+1)+a_0y(k)=u(k)$			
			$u^*(k)=M(10k)$ – trimming moment $y^*(k)=H(10k)$ – draught depth increment			
r			$r=0$		$r=1$	
ω_ν			$\omega_\nu=\nu+3,\ \nu\in\overline{1,19}$			
a_0^0			−3.44701		−3.42016	
a_1^0			−1.00670		−1.11835	
a_2^0			5.02896		5.10437	
$J_{r,\varpi}$			5.61696		5.63108	
k	u^* [Tm]	y^* [m]	y [m]	$\Delta_{r,\varpi}y$ [m]	y [m]	$\Delta_{r,\varpi}y$ [m]
0	0.0	0.0	0.0	0.0	0.0	0.0
1	1.6	0.1	0.1	0.0	0.1	0.0
2	2.8	0.2	0.0	0.2	0.0	0.2
3	4.0	0.5	0.4	0.1	0.4	0.1
4	5.2	0.9	0.6	0.3	0.6	0.3
5	6.6	1.7	1.2	0.5	1.2	0.5
6	8.1	2.1	1.7	0.4	1.7	0.4
7	8.5	2.7	2.5	0.2	2.5	0.2
8	9.5	3.3	3.3	0.0	3.3	0.0
9	10.6	4.1	4.0	0.1	4.0	0.1
10	12.2	4.7	4.9	0.2	4.9	0.2
11	13.3	5.7	5.9	0.2	5.9	0.2
12	14.0	6.9	7.0	0.1	7.0	0.1
13	16.1	8.0	8.1	0.1	8.1	0.1
14	17.6	9.1	9.2	0.1	9.2	0.1
15	18.9	10.1	10.6	0.5	10.6	0.5
16	20.3	11.3	11.9	0.6	11.9	0.6
17	21.5	12.7	13.4	0.7	13.4	0.7
18	23.0	14.1	14.9	0.8	14.9	0.8
19	23.7	15.9	16.4	0.5	16.5	0.6
20	24.6	17.9	18.1	0.2	18.1	0.2
21	25.4	19.5	19.6	0.1	19.6	0.1
22	25.4	21.2	21.2	0.0	21.2	0.0
23	25.4	22.6	22.7	0.1	22.8	0.2

system and depends on the integration by parts of the equation describing the system dynamics when multiplied by the so-called *modulating function*. The problem of choosing the best model is not considered in the classical method.

Based on (Wysocki and Zellma, 1993) we review how to get the vector equation

$$\sum_{i=0}^{n} a_i \overline{v}_i = \overline{w}.$$

For this purpose we suppose that

(A1) Q has more than one element,

(A2) L^0 is a real algebra and L^1 is its subalgebra,

(A3) the derivative S satisfies the *Leibniz condition*

$$S(x \cdot y) = Sx \cdot y + x \cdot Sy, \quad x, y \in L^1,$$

(A4) the operations s_{q_1}, s_{q_2}, $q_1, q_2 \in Q$ satisfy the *multiplication condition*

$$s_{q_i}(x \cdot y) = s_{q_i}x \cdot s_{q_i}y, \quad i = 1, 2, \quad x, y \in L^1.$$

Applying (A3), by induction on $k \in \mathbb{N}$ we prove (Wysocki, 1994; Wysocki and Zellma, 1991), the formula of *integration by parts*:

$$I_{q_1}^{q_2}(x \cdot S^k y) = \sum_{i=0}^{k-1} (-1)^i R_{q_1}^{q_2}(S^i x \cdot S^{k-i-1} y) + (-1)^k I_{q_1}^{q_2}(S^k x \cdot y), \tag{53}$$

where $S^0 x := x$, $q_1, q_2 \in Q$, $x, y \in L^k$, $k \in \mathbb{N}$ and the operations

$$I_{q_1}^{q_2} \in L(L^0, \operatorname{Ker} S), \quad R_{q_1}^{q_2} \in L(L^1, \operatorname{Ker} S)$$

are defined by

$$I_{q_1}^{q_2} w := (T_{q_1} - T_{q_2}) w = s_{q_2} T_{q_1} w, \quad R_{q_1}^{q_2} x := (s_{q_2} - s_{q_1}) x, \quad w \in L^0, \ y \in L^1.$$

The mapping $I_{q_1}^{q_2}$ is called the *operation of definite integration* (cf. Przeworska-Rolewicz, 1988).

Assume that $f_1, f_2, \ldots, f_m \in L^n$ satisfy the conditions

$$f_\nu \notin \operatorname{Ker} S^n, \quad s_{q_i} S^j f_\nu = 0, \quad \nu \in \overline{1, m}, \quad i = 1, 2, \quad j \in \overline{0, n-1}. \tag{54}$$

Then $f_1, f_2, \ldots, f_m$ are called the *modulating elements* of the system (3) corresponding to $q_1, q_2 \in Q$.

If a pair $(u, y) \in L^0 \times L^n$ satisfies (3) with given coefficients $a_0, a_1, \ldots, a_n$, then for each modulating element $f_\nu, \nu \in \overline{1, m}$ we have

$$a_n f_\nu S^n y + a_{n-1} f_\nu S^{n-1} y + \cdots + a_1 f_\nu S y + a_0 f_\nu y = f_\nu u, \quad \nu \in \overline{1, m}. \tag{55}$$

Applying $I_{q_1}^{q_2}$ to both sides of (55) and then using (53), (54) and (A4), we obtain

$$\sum_{i=0}^{n}(-1)^i a_i I_{q_1}^{q_2}(S^i f_\nu \cdot y) = I_{q_1}^{q_2}(f_\nu \cdot u), \quad \nu \in \overline{1,m}. \tag{56}$$

The system can be rewritten in the form

$$\sum_{i=0}^{n} a_i \overline{v}_i = \overline{w}, \tag{57}$$

where

$$\overline{v}_i = \begin{bmatrix} (-1)^i I_{q_1}^{q_2}(S^i f_1 \cdot y) \\ \vdots \\ (-1)^i I_{q_1}^{q_2}(S^i f_m \cdot y) \end{bmatrix}, \quad \overline{w} = \begin{bmatrix} I_{q_1}^{q_2}(f_1 \cdot u) \\ \vdots \\ I_{q_1}^{q_2}(f_m \cdot u) \end{bmatrix}, \quad i \in \overline{0,n}. \tag{58}$$

From the definition of $I_{q_1}^{q_2}$ it follows that

$$\overline{v}_i, \overline{w} \in (\operatorname{Ker} S)_m, \quad i \in \overline{0,n}.$$

Now, if $\operatorname{Ker} S \simeq \mathbb{R}$, then $\overline{v}_i, \overline{w}$ can be treated as elements of ℓ_m^2. Otherwise, from (56) we obtain the system (57) such that

$$V_i^\nu := (-1)^i F_\omega I_{q_1}^{q_2}(S^i f_\nu \cdot y), \quad W^\nu := F_\omega I_{q_1}^{q_2}(f_\nu \cdot u), \quad i \in \overline{0,n}, \ \nu \in \overline{1,m}, \tag{59}$$

where $F_\omega, \omega \in \Omega$ is a mapping from the family $\{F_\omega\}_{\omega\in\Omega}$. Moreover, $\overline{v}_i, \overline{w} \in \ell_m^2, i \in \overline{0,n}$.

In the method considered, the identification performance index (6), i.e. the functional

$$J_{\overline{f}}(a_0, a_1, \ldots, a_n) = \Big\| \sum_{i=0}^{n} a_i \overline{v}_i^* - \overline{w}^* \Big\|,$$

depends on the vector $\overline{f} := [f_1, f_2, \ldots, f_m]$ of modulating elements. Vectors $\overline{v}_i^*$ and $\overline{w}^*$ are elements of the form (58) or (59) determined for fixed $u^* \in L^0 \setminus \{0\}$ and $y^* \in L^n$.

Let

$$\mathrm{CO}(L^0, L^1, S', T_q', s_q', Q) \tag{60}$$

be an operational calculus satisfying the same assumptions as before. Moreover, let L^0 be an algebra with unity.

Consider the equation

$$a_1 p_1 S' y + a_0 p_0 y = p, \tag{61}$$

where $p_0, p_1 \in \operatorname{Inv}(L^0)$, $p \in L^0$, $y \in L^1$, $a_0, a_1 \in \mathbb{R}$ and $\operatorname{Inv}(L^0)$ denotes the set of all invertible elements in L^0.

To determine the coefficients a_0 and a_1 of (61) we can apply the foregoing identification method, but we have to introduce a new operational calculus in which

$$Sx := bS'x, \quad T_q w := T_q'(b^{-1}w), \quad s_q x := s_q' x, \tag{62}$$

where $b := p_0^{-1} p_1$. Then (61) is a particular form of (3), i.e.

$$a_1 Sy + a_0 y = u, \tag{63}$$

where $u := p_0^{-1} p$. Moreover, S satisfies the Leibniz condition and the operations $s_q, q \in Q$ are multiplicative (Mieloszyk, 1987).

From what has already been said, we are able to consider the identification problem for certain types of non-stationary lumped- or distributed-parameter systems.

Example 5. Let (60) be the classical model of operational calculus as in Example 3. Then (61) takes the form

$$a_1 p_1(t) y'(t) + a_0 p_0(t) y(t) = p(t), \tag{64}$$

where $p_0(t), p_1(t) \not\equiv 0$ for each $t \in [t_0, t_k]$.

Here (64) is the dynamic equation of a non-stationary lumped-parameter system. The modulating elements of the system corresponding to $q_1 = t_0, q_1 = t_k$ are arbitrary functions $0 \not\equiv f_\nu(t) \in C^1([t_0, t_k], \mathbb{R})$ satisfying

$$f_\nu(t_0) = f_\nu(t_k) = 0, \quad \nu \in \overline{1, m}, \quad m \geqslant 2. \tag{65}$$

Let $p^* = p^*(t) \in C^0([t_0, t_k], \mathbb{R})$ and $y^* = y^*(t) \in C^1([t_0, t_k], \mathbb{R})$ be functions approximating the input and output signals of the system (64) in $[t_0, t_k]$, respectively (on the basis of the measurements of a real system). Then $u^* = p^*(t)/p_0(t)$.

Using the operational calculus with S, T_q and s_q defined by (62), where $b = p_1(t)/p_0(t)$, we can determine the ν-th coordinates V_i^ν, W^ν of vectors $\overline{v}_i^*, \overline{w}^*$, respectively. Since $\operatorname{Ker} S \simeq \mathbb{R}$, we have

$$\begin{cases} V_0^\nu = \displaystyle\int_{t_0}^{t_k} \frac{f_\nu(t) p_0(t) y^*(t)}{p_1(t)} \, \mathrm{d}t, \\ V_1^\nu = -\displaystyle\int_{t_0}^{t_k} f_\nu'(t) y^*(t) \, \mathrm{d}t, \qquad \nu \in \overline{1, m}, \\ W^\nu = \displaystyle\int_{t_0}^{t_k} \frac{f_\nu(t) p^*(t)}{p_1(t)} \, \mathrm{d}t, \end{cases} \tag{66}$$

which results from (58).

To verify the applied method by means of absolute errors, first we have to solve the initial-value problem

$$a_1^0 p_1(t) y'(t) + a_0^0 p_0(t) y(t) = p^*(t), \quad y(t_0) = y^*(t_0),$$

where a_0^0 and a_1^0 are optimal coefficients of the system (64) obtained based on p^* and y^*.

Table 5 contains the corresponding numerical results, where k denotes the number of parts into which the integration interval $[t_0, t_k]$ is divided for the Simpson method of computing the definite integrals (66). ♦

Table 5. Modulating-function method in identification of an ordinary differential equation of the first order.

$t_0 = 0, t_1 = 1$	$a_1(0.5t^2 + t + 1)y' + a_0(t+1)y = p$				
$k = 128$	$p^* = t + 1$ $y^* = -0.001t^5 + 0.006t^4 + 0.105t^3 + 0.375t^2 + 0.5t$				
f_1 f_2	$\left[(t-t_0)(t-t_k)\right]^2$ $(t-t_k)\sin(t-t_0)$		$(t-t_0)(t-t_k)$ $(t-t_0)\sin(t-t_k)$		
a_0^0 a_1^0	-2.96517 1.98702		-3.00531 2.00098		
$J_{\overline{f}}$	1.17242×10^{-7}		6.75040×10^{-8}		
t	y^*	y	$\Delta_{\overline{f}} y$	y	$\Delta_{\overline{f}} y$
0.0	0	0	0	0	0
0.2	0.11585	0.11651	6.58398×10^{-4}	0.11581	3.82815×10^{-5}
0.4	0.26686	0.26813	1.26676×10^{-3}	0.26681	5.16484×10^{-5}
0.6	0.45838	0.46009	1.71447×10^{-3}	0.45834	4.30896×10^{-5}
0.8	0.69589	0.69773	1.84216×10^{-3}	0.69584	4.86281×10^{-5}
1.0	0.98500	0.98644	1.43772×10^{-3}	0.98486	1.42849×10^{-4}

Example 6. Let (60) be the operational calculus of Example 1. Then (61) takes the form of a quasi-linear partial differential equation

$$a_1 p_1(t,z)\left(\frac{\partial y(t,z)}{\partial t} + \frac{\partial y(t,z)}{\partial z}\right) + a_0 p_0(t,z) y(t,z) = p(t,z), \tag{67}$$

where $p_0(t,z), p_1(t,z) \neq 0$ for every $(t,z) \in [t_0, t_k] \times \mathbb{R}$. It (67) describes a non-stationary distributed parameter system. For $\Omega = [t_0, t_k] \times [z_0, z_k]$ the functions $p^* = p^*(t,z), y^* = y^*(t,z)$ have the same meaning as in Example 1.

Applying the operational calculus (62), where $b = p_1(t,z)/p_0(t,z)$, we bring (67) to the form (63), where $u = p(t,z)/p_0(t,z)$. The form of the limit conditions in this operational calculus implies that the modulating elements of the system corresponding to $q_1 = t_0, q_2 = t_k$ are arbitrary functions $0 \not\equiv f_\nu(t) \in C^2([t_0, t_k], \mathbb{R})$ satisfying (65).

As the kernel of $S = \partial/\partial t + \partial/\partial z$ is not isomorphic to $\mathbb{R}$, in order to determine the coordinates V_i^ν, W^ν of $\overline{v}_i^*, \overline{w}^*$ we have to use (59). For

$$F_\omega y(t,z) := y(\omega_1, \omega_2), \quad \omega = (\omega_1, \omega_2) \in \Omega, \quad y = y(t,z) \in L^0$$

we get

$$V_0^\nu = \int_{t_0}^{t_k} \frac{f_\nu(\tau) p_0(\tau, \omega_1 - \omega_2 + \tau) y^*(\tau, \omega_1 - \omega_2 + \tau)}{p_1(\tau, \omega_1 - \omega_2 + \tau)} \, \mathrm{d}\tau, \tag{68}$$

$$V_1^\nu = -\int_{t_0}^{t_k} f_\nu'(\tau) y^*(\tau, \omega_1 - \omega_2 + \tau) \, \mathrm{d}\tau, \tag{69}$$

$$W^\nu = \int_{t_0}^{t_k} \frac{f_\nu(\tau) p^*(\tau, \omega_1 - \omega_2 + \tau)}{p_1(\tau, \omega_1 - \omega_2 + \tau)} \, \mathrm{d}\tau, \quad \nu \in \overline{1,m}. \tag{70}$$

In the numerical example whose results are shown in Table 6, the integrals (68)–(70) are calculated by means of the Simpson method. ♦

Table 6. Modulating-function method in identification of a partial differential equation of the first order.

$\Omega = [0,1] \times [0,1]$	$a_1(t^2z^2+1)\left(\frac{\partial y}{\partial t} + \frac{\partial y}{\partial z}\right) + a_0(2t^2+3)y = p(t,z)$				
$t_0 = 0, t_k = 1$, $k = 128$	$p^* = 4.1t^3 + 2.9t^2z^2 + 8.2t^2z + 6.2t + 11.8z + 3$, $y^* = t + 2z$				
f_1, f_2	ω_1	ω_2	a_0^0	a_1^0	$J_{\overline{f},\omega}$
$(t-t_0)(t-t_k)$	0.5	0.5	1.96848	1.06161	4.30212×10^{-6}
$\sin(t-t_0)\sin(t-t_k)$	0.5	1.0	2.05497	1.00506	9.83337×10^{-8}
$\sin(t-t_0)(t-t_k)$	0.5	0.5	1.98078	1.04219	2.28549×10^{-6}
$\sin(t-t_0)\sin(t-t_k)$	0.5	1.0	2.05371	1.00590	4.94313×10^{-7}
$\left[(t-t_0)(t-t_k)\right]^2$	0.5	0.5	1.97701	1.04814	6.92734×10^{-8}
$\sin(t-t_0)\sin(t-t_k)$	0.5	1.0	2.05472	1.00523	1.52077×10^{-7}
$(t-t_0)(t-t_k)$	0.5	0.5	2.01455	0.98895	5.16449×10^{-8}
$\left[\exp(t-t_0)-1\right](t-t_k)$	0.5	1.0	2.01132	1.03414	1.30904×10^{-8}
$(t-t_0)(t-t_k)$	0.5	0.5	2.01317	0.99112	2.80191×10^{-8}
$(t-t_0)\sin(t-t_k)$	0.5	1.0	2.01014	1.03493	3.83606×10^{-8}

Other identification methods of the generalized control system (3) by means of the modulating elements are described in (Wysocki and Zellma, 1991; 1995). We can avail of them in the Gram-Schmidt orthonormalization method.

7. Conclusions

In this paper, we have assumed that any system whose mathematical model belongs to the class described in the Bittner operational calculus by the abstract linear differential equations with the constant coefficients

$$a_n S^n y + a_{n-1} S^{n-1} y + \cdots + a_1 S y + a_0 y = u, \tag{71}$$

can be a system to be identified.

The identification system means fixing the model (71) with the accuracy of the parameters $a_0, a_1, \ldots, a_n$. The identification performance index

$$J(a_0, a_1, \ldots, a_n) = \Big\| \sum_{i=0}^{n} a_i \overline{v}_i^* - \overline{w}^* \Big\| \tag{72}$$

introduces an order in the considered class of models. Its value was adopted to assess the accuracy. The model with coefficients $a_0^0, a_1^0, \ldots, a_n^0$ for which the functional (72) takes the minimum value was accepted as the best model. The optimal parameters in this sense were obtained using the Gram-Schmidt orthonormalization in the space ℓ_m^2. This method involves simple and fast algorithms because it uses the procedures which are available in practically all numerical environments.

The basic requirement to define the identification performance index (72) is the following vector equation in ℓ_m^2:

$$\sum_{i=0}^{n} a_i \overline{v}_i = \overline{w}. \tag{73}$$

It was obtained by generalization of the classical Strejc and Shinbrot methods. The generalized Strejc methods can be theoretically used in any operational calculus model because they do not require any restrictions on the derivative S, the integrals T_q and the limit conditions s_q. However, there may exist some practical restrictions which arise from the required knowledge about the limit conditions on the derivatives of the output signal. Those derivatives are determined based on experimental data. We can avoid this inconvenience using the modulating-element method which constitutes a generalization of the classical identification method employing the Shinbrot modulating function. On the other hand, the modulating-element method restricts the number of the operational calculus models in which it can be applied. This is because the derivative has to satisfy the Leibniz condition and the limit conditions have to be multiplicative. Accordingly, the method can be applied in neither the non-linear system described in Example 2, nor the discrete one from Example 4.

Using operational calculus, we have obtained the possibility of the uniform performance of two identification methods for linear continuous (Examples 1–3, 5 and 6) and discrete (Example 4) systems, for stationary (Examples 1–4) and non-stationary ones (Examples 5 and 6), for lumped (Examples 2–5) or distributed (Examples 1 and 6) parameter systems, as well as for some non-linear systems (Example 2). From the above it follows that the operational notion of a linear differential stationary lumped-parameter control system is much wider than the classical one. It encompasses all the system descriptions which can be converted to the form (71), when assuming the models of operational calculus which satisfy the assumptions of the elaborated general identification methods.

In particular, if we consider a model with the ordinary derivative $S = \mathrm{d}/\mathrm{d}t$, the general methods discussed here constitute some modifications of the classical Strejc and Shinbrot identification methods of continuous stationary lumped-parameter systems. In classical methods the problem of choosing the best model is not discussed.

Each of the general methods presented here depends on one or some parameters p. In the Strejc methods the parameters $p = [q, \overline{r}, \omega]$ or $p = [q, r, \overline{\omega}]$ define the integral, its iterations and the points of the observation set determining the proper functionals. On the other hand, in the modulating-element method we have $p = [f_1, f_2, \dots, f_m]$, where $f_1, f_2, \dots, f_m$ are the modulating elements. The parameters of those methods are chosen in such a way that the *identification condition* of the system (71) is satisfied. This condition is the linear independence of the vectors $\overline{v}_0^*, \overline{v}_1^*, \dots, \overline{v}_n^*$. They have the form of the vectors existing in (73) and are determined for a fixed identifying pair (u^*, y^*).

Therefore the optimal coefficients $a_0^0, a_1^0, \dots, a_n^0$ of (71) define the best model only for the fixed identifying pair and for the fixed parameters of the selected method, i.e. $a_i^0 = a_i^0(u^*, y^*, p)$. For another identifying pair or other parameters of the identification method we usually get different coefficients of the best model, and in that sense the best model should be understood (cf. the best model notion in Bubnicki, 1974). Consequently, for a given identification method we can also formulate the problem of choosing the best set of parameters. Namely, if p_0 and p_1 are different parameter sets of the method, we can ask the question: Which of the numbers $J_{p_0}(a_0^0, a_1^0, \dots, a_n^0)$ and $J_{p_1}(a_0^1, a_1^1, \dots, a_n^1)$, obtained for the same identifying pair (u^*, y^*), is smaller? At present this question is far from being solved.

From the examples included in this paper it follows that for the identification method chosen and for various parameter sets the difference between the system output and the model output for the same input is negligible, so that in practice we can use an arbitrary parameter set. From a theoretical point of view there exist parameters for which the system (71) is 'better' or 'worse' identified. In this sense, the identification is a 'soft' notion, which suggests the possibility of using fuzzy sets theory.

Let $\mathcal{P}$ denote the parameter set for the chosen identification method of the system (71). Moreover, consider a decreasing function

$$\lambda : [0, +\infty] \longrightarrow [0, 1]$$

such that

$$\lambda(0) = 1, \quad \lim_{t \to \infty} \lambda(t) = 0.$$

Then $\mathcal{P}$ can be presented as a set of pairs

$$\mathcal{P} = \{(p, \mu_{\mathcal{P}}(p))\},$$

where

$$\mu_{\mathcal{P}}(p) := \lambda\big[J_p(a_0^0, a_1^0, \ldots, a_n^0)\big]. \tag{74}$$

In this manner the identification order of the system (71) is assigned to every element $p \in \mathcal{P}$ (with given identifying pair (u^*, y^*)). Namely, the system (71) is

- unidentified, if $\mu_{\mathcal{P}}(p) = 0$,
- partially identified, if $\mu_{\mathcal{P}}(p) \in (0,1)$, and
- totally identified, if $\mu_{\mathcal{P}}(p) = 1$.

Here (74) defines the membership function of the fuzzy set $\mathcal{P}$. In such an approach, the problem of choosing the parameters p can be replaced by the problem of constructing the set $\mathcal{P}$.

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