Int. J. Appl. Math. and Comp. Sci., 2000, Vol.10, No.2, 333–344

SOLVING PARABOLIC EQUATIONS BY USING THE METHOD OF FAST CONVERGENT ITERATIONS

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The paper describes an approach to solving parabolic partial differential equations that generalizes the well-known parametrix method. The iteration technique proposed exhibits faster convergence than the classical parametrix approach. A solution is constructed on a manifold with the application of the Laplace-Beltrami operator. A theorem is formulated and proved to provide a basis for finding a unique solution. Simulation results illustrate the superiority of the proposed approach in comparison with the classical parametrix method.

Keywords: parabolic equations, fundamental solution, Riemannian manifold, rate of convergence, iteration technique, numerical simulation

1. Introduction

There is a number of processes and systems that are described by partial differential equations of parabolic type. As an example, heat and mass transfer, and diffusion can be mentioned. The general form of the equation is as follows:

$$\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial}{\partial x^k}\left(g^{jk}(x)\frac{\partial u}{\partial x^j}\right) + b^k(x)\frac{\partial u}{\partial x^k} \equiv Lu. \tag{1}$$

Here $x = [x^1, \dots, x^n]^T \in \mathbb{R}^n$, $g^{jk}(x)$ and $b^k(x)$ are coefficients of diffusion and transfer, respectively. The right-hand side of (1) contains summation with respect to indices j and k (the summation symbol is omitted as is customary in tensor algebra).

As it was mentioned above, physical processes of type (1) are related to phenomena of diffusion and heat transfer, what is common in automatic control systems. In this case, the controlled process is usually characterized by a medium concentration or a temperature, and the solution $u(t, x)$ represents these quantities. Control system design and implementation require solving the corresponding equation in real time when the calculation speed is highly important. So it is much desirable to have a fast convergent method of solution. Parabolic equations are also used in ecological monitoring to determine excessive contaminant concentrations (possibly very small), which also requires fast convergence for an effective response.

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2. Existing Approaches to Solving Parabolic Equations

Equation (1) is supplemented by an initial condition $u(0, x) = \varphi(x)$ and, if $x \in D \subset \mathbb{R}^n$, boundary conditions are added in a region D. This paper is devoted to the Cauchy problem for (1) in the whole space with a general solution form

$$u(t,x) = \int_{\mathbb{R}^n} \varphi(y) p(t,x,y)\, \sigma(\mathrm{d}y),$$

where $p(t, x, y)$ denotes a fundamental solution to (1) and $\sigma(\mathrm{d}y)$ is an arbitrary unit in the space $\mathbb{R}^n$. In the case of Cartesian coordinates, we have $\sigma(\mathrm{d}y) = \Pi\, \mathrm{d}y^k$, $k = 1, \ldots, n$.

An explicit form of fundamental solution is known only for a limited group of equations, i.e. equations with constant coefficients $g^{jk}(x)$ and $b^k(x)$, for which the fundamental solution is

$$p(t,x,y) = \frac{1}{(2\pi t)^{n/2}\sqrt{\det g^{jk}}} \exp\left\{-\frac{1}{2t} g_{jk}\left(y^j - x^j - tb^j\right)\left(y^k - x^k - tb^k\right)\right\},$$

where g_{jk} is the matrix inverse to the diffusion matrix. A classical iteration technique for constructing fundamental solutions for equations with time-varying coefficients is the parametrix method (Friedman, 1964) which provides the solution in the form:

$$p(t,x,y) = p_0(t,x,y) + \int_0^t \mathrm{d}\tau \int_{\mathbb{R}^n} p_0(t-\tau, x, z) r(\tau, z, y)\, \sigma(\mathrm{d}z),$$

where $p_0(t, x, y)$ satisfies (1) with coefficients 'frozen' at the point y, and $r(t, x, y)$ is constructed via the iteration technique. An initial approximation p_0 can be chosen as

$$p(t,x,y) = \frac{1}{(2\pi t)^{n/2}\sqrt{\det g^{jk}(y)}} \exp\left\{-\frac{1}{2t} g_{jk}(y)\left(y^j - x^j\right)\left(y^k - x^k\right)\right\}. \tag{2}$$

If coefficients of (1) satisfy Lipschitz conditions, the residual can be assessed using the estimate

$$\left| Lp_0 - \frac{\partial p_0}{\partial t} \right| < \frac{c}{\sqrt{t}} \left(1 + \frac{\|x-y\|^2}{t}\right) p_0,$$

i.e. it has an integrable singularity as time vanishes. Actually, this inequality guarantees convergence of the iteration process while constructing the function $r(t, x, y)$.

At the beginning of the 1970's parabolic equations (1) were extended to Riemannian manifolds. Foremost, the works (Cheng and Yau, 1975; 1981) should be mentioned whose authors were awarded Field's medal.

The equations of type (1) are most often solved by making use of numerical techniques presented in abundant relevant literature. A thorough description of numerical techniques, such as finite differences and finite elements, collocations, non-linear two-point boundary problems, can be found in (Na, 1979; Shih, 1984). Details and numerous examples regarding the Galerkin method are presented in (Fletcher, 1984). The analytical methods discussed in the next section proved to be too slowly convergent. The references (Bakry *et al.*, 1997; Bondarenko, 1997; Grigoryan, 1998) contain rather good estimates of solutions to (1), but they do not provide a procedure for obtaining the solutions themselves.

In this paper we propose an iteration method for obtaining solutions, which generalizes the parametrix method.

3. Problem Statement

As an illustration of the influence of the initial approximation on the convergence rate, the following one-dimensional equation is considered:

$$\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial}{\partial x}\left(\sigma^2(x)\frac{\partial u}{\partial x}\right) + b(x)\frac{\partial u}{\partial x}, \qquad u(0,x) = \varphi(x), \tag{3}$$

where $\sigma(x) > \varepsilon$ and φ is continuous.

Let

$$m(t,x,y) = \frac{1}{\sqrt{2\pi t\sigma(y)\sigma(x)}} \exp\left\{ -\frac{1}{2t}\left(\int_x^y \frac{\mathrm{d}z}{\sigma(z)}\right)^2 + \int_x^y \frac{b(z)}{\sigma^2(z)}\,\mathrm{d}z \right\}.$$

The following relation is true:

$$\lim_{t\downarrow 0} \int_{-\infty}^{\infty} \varphi(y) m(t,x,y)\,\mathrm{d}y = \varphi(x),$$

which is proved via the substitution

$$y \to v, \qquad v = \frac{1}{\sqrt{t}}\int_x^y \frac{\mathrm{d}s}{\sigma(s)}, \qquad y = f_x(\sqrt{t}v),$$

where the transformation $f_x(u)$, $f_x(0) = x$ is inverse with respect to

$$\int_x^y \frac{\mathrm{d}s}{\sigma(s)} = u.$$

The inverse transformation exists because of the monotonicity of u as a function of y. The discrepancy for eqn. (3) is

$$h(t,x,y) = \frac{\partial m}{\partial t} - \frac{1}{2}\frac{\partial}{\partial x}\left(\sigma^2(x)\frac{\partial m}{\partial x}\right) - b(x)\frac{\partial m}{\partial x} = \lambda(x) m,$$

where

$$\lambda(x) = \frac{\left(\sigma'(x)\right)^2}{8} + \frac{\sigma''(x)\sigma(x)}{4} + \frac{b'(x)}{2} + \frac{b^2(x)}{2\sigma^2(x)}.$$

If $c_1 \le \lambda(x) \le c_2$ then, according to the maximum principle for the fundamental solution, we have

$$\exp(-c_2 t) \le \frac{p(t,x,y)}{m(t,x,y)} \le \exp(-c_1 t).$$

The fundamental solution can be found in the form:

$$p(t,x,y) = m(t,x,y) + \int_0^t \mathrm{d}\tau \int_{-\infty}^{\infty} m(t-\tau,x,z) r(\tau,z,y)\,\mathrm{d}\tau,$$

where the sought function is

$$r(t,x,y) = \sum_{n=0}^{\infty} r_n(t,x,y), \quad r_0 = h(t,x,y),$$

$$r_n(t,x,y) = \int_0^t \mathrm{d}\tau \int_{-\infty}^{\infty} h(t-\tau,x,z) r_{n-1}(\tau,z,y)\,\mathrm{d}z.$$

From the estimate

$$\bigl|h(t,x,y)\bigr| < cm(t,x,y)$$

it follows that

$$\bigl|r_1(t,x,y)\bigr| < c^2 \int_0^t \mathrm{d}\tau \int_{-\infty}^{\infty} m(t-\tau,x,z) m(\tau,z,y)\,\mathrm{d}z < c^2 t m(t,x,y),$$

which is proved by making the following substitutions:

$$z \to u, \quad u = \sqrt{\frac{t}{\tau(t-\tau)}} \int_x^z \frac{\mathrm{d}s}{\sigma(s)} - \sqrt{\frac{t-\tau}{t\tau}} \int_x^y \frac{\mathrm{d}s}{\sigma(s)},$$

$$z = f_x \left(\sqrt{\frac{\tau(t-\tau)}{t}} u + \frac{t-\tau}{t} \int_x^y \frac{\mathrm{d}s}{\sigma(s)} \right).$$

The estimate for the n-th iteration

$$\bigl|r_n(t,x,y)\bigr| < \frac{c^{n+1}}{n!} t^n m(t,x,y),$$

produces

$$\bigl|r(t,x,y)\bigr| < c_1 \exp(ct) m(t,x,y).$$

Consequently, for $t \in (0, T]$ we have

$$p(t, x, y) < m(t, x, y)\big(1 + c_1 t \exp(t)\big).$$

If a heat transfer kernel is to be built for a multidimensional case, an approximation different from (2) is used. The new approximation is based on the fact that the diffusion matrix induces in the space $\mathbb{R}^n$ a new Riemannian distance defined by

$$\rho^2(x, y) = \min_{\gamma} T \int_0^T g_{jk}\big(\gamma(s)\big)\dot{\gamma}^j(s)\dot{\gamma}^k(s)\,\mathrm{d}s, \tag{4}$$

where the minimum is taken among different curves connecting the points $x, y \in M$. Thus it is obvious to consider (1) on the Riemannian manifold M formed by inducing the distance (4) on the initial space $\mathbb{R}^n$.

4. Notation and Conditions

Let M be a Riemannian manifold. The geodesic line connecting points x and y in M (i.e. the solution to the variational problem (4)) will be denoted by $\gamma(s)$, where s is a natural parameter, $\gamma(0) = y$, $\gamma(\rho(x, y)) = x$. The matrix $g_{1k}(x)$ is inverse to the diffusion matrix and forms the metrics tensor of the manifold, which is used to determine the connection coefficients (Cromol *et al.*, 1968)

$$\Gamma^k_{ij} = \frac{1}{2} g^{kr} \left(\frac{\partial g_{jr}}{\partial x^i} + \frac{\partial g_{ir}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^r} \right),$$

and the curvature tensor

$$R_{ijkl} = \frac{1}{2} \left(\frac{\partial^2 g_{jk}}{\partial x^i \partial x^l} - \frac{\partial^2 g_{jl}}{\partial x^i \partial x^k} - \frac{\partial^2 g_{ik}}{\partial x^j \partial x^l} + \frac{\partial^2 g_{il}}{\partial x^j \partial x^k} \right) + g_{pq}\big(\Gamma^p_{jk}\Gamma^q_{jl} - \Gamma^p_{ik}\Gamma^q_{jl}\big).$$

The sectional curvature of the manifold in directions u and v is determined by

$$r = -R_{ijkl} u^i v^j u^k v^l = -\Big(\big(R(x)u, v\big)u, v\Big),$$

up to a positive multiplier. Denote by $\sigma(\mathrm{d}y)$ an arbitrary unit of the manifold. The following assumption will be needed throughout the paper.

Assumption 1. The manifold is complete and one-connected. Moreover, its sectional curvature is non-positive at every point and decreases fast enough as $\|x\| \to \infty$ (for details, see Cromol *et al.*, 1968).

One of the manifold characteristics is a Jacobi field $Z(s)$ along a geodesic $\gamma(s)$, which is defined as a solution to the Jacobi equation

$$Z''(s) = R\big(\gamma(s)\big)\big(\dot{\gamma}(s), Z(s)\big)\dot{\gamma}(s).$$

In a linear space, $Z(s)$ is a linear function of the parameter s. For a manifold of a nonpositive curvature, $\|Z(s)\|$ is a function convex downward, i.e. it increases faster than s. This occurence allows us to determine basic Jacobi fields $Z_k(s)$, $k = 1, \dots, n$ which form an orthogonal basis at the point x. Introducing such fields allows us to find a solution to the problem of choosing basis vectors in the manifold. The problem is non-trivial while a tangent displacement of the vector along different curves in the manifold leads to different results.

If the condition holds, then for every couple of points $x, y \in M$ a nonnegative function $a(x,y)$ can be introduced, which is defined in terms of the basic Jacobi fields (Bondarenko, 1997) and depends on the manifold curvature. Let

$$\varphi(x,y) = \int_0^{\rho(x,y)} \frac{a\big(\gamma(s), y\big)}{s}\, \mathrm{d}s,$$

where

$$a(x,y) = \sum_{k=2}^{n} \Big(\rho Z_k'(\rho) - Z_k(\rho), Z_k(\rho)\Big),$$

$Z_k(\rho)$ are basic Jacobi fields at the point $x = \gamma(\rho)$, and $a(x,y)$ satisfies

$$0 \le a(x,y) \le \int_0^{\rho} \tau \mathrm{Ric}\big(\gamma(\tau)\big)\big(\dot{\gamma}(\tau), \dot{\gamma}(\tau)\big)\, \mathrm{d}\tau,$$

where $\mathrm{Ric}(x)$ is a convolution of the curvature tensor. The condition imposed on the manifold guarantees that the function $a(x,y)$ is bounded.

5. Constructing the Equation on the Manifold

Consider again (1) while assuming that the transfer coefficients are defined in the following way:

$$b^k(x) = \frac{1}{2}\Gamma^j_{jr}(x) g^{rk}(x).$$

In this case (1) takes the self-adjoint form

$$\frac{\partial u}{\partial t} = \frac{1}{2}\Delta u, \tag{5}$$

where Δ is the Laplace-Beltrami operator. Therefore $p(t,x,y) = p(t,y,x)$.

Introduce the function

$$m(t,x,y) = (2\pi t)^{-n/2} \exp\left\{-\frac{\rho^2(x,y)}{2t} - \frac{\varphi(x,y)}{2}\right\}.$$

Lemma 1. *If the condition given above is fulfilled for the residual*

$$h(t,x,y) = \frac{1}{2}\Delta m - \frac{\partial m}{\partial t},$$

then

$$\left|h(t,x,y)\right| < cm(t,x,y), \quad t > 0, \quad x,y \in M.$$

Proof. Let Z_k is a basis of the Jacobi fields along the geodesic γ, $\gamma(0) = y$, $\gamma(\rho(x,y)) = x$. Calculate the derivatives of the function $m(t,x,y)$:

$$\operatorname{grad} m(t,x,y) = -m(t,x,y)\left(\frac{\rho(x,y)}{t}\dot{\gamma}(x) + \frac{1}{2}\operatorname{grad}\varphi(x,y)\right),$$

$$\begin{aligned}\Delta m(t,x,y) = m(t,x,y)\Bigg(&\frac{\rho^2(x,y)}{t^2} + \frac{\rho(x,y)}{t}(\operatorname{grad}\varphi(x,y),\dot{\gamma}(x)) \\ &+ \frac{1}{4}\left\|\operatorname{grad}\varphi(x,y)\right\|^2 + \frac{1}{2}\Delta\varphi(x,y) \\ &- \frac{\rho(x,y)}{t}\sum\left(Z_k'(\rho), Z_k(\rho)\right)\Bigg),\end{aligned}$$

$$\frac{\partial m(t,x,y)}{\partial t} = m(t,x,y)\left(\frac{\rho^2(x,y)}{2t^2} - \frac{n}{2t}\right).$$

From

$$\sum\left(Z_k'(\rho), Z_k(\rho)\right) = n + a(x,y)$$

it follows that the residual is

$$h(t,x,y) = m(t,x,y)\left(\frac{1}{8}\left\|\operatorname{grad}\varphi(x,y)\right\|^2 + \frac{1}{2}\Delta\varphi(x,y)\right).$$

As far as $\|\operatorname{grad}\varphi\|$ and $\Delta\varphi$ are bounded, the above statement is proved. ■

The fundamental solution for (5) is written as

$$p(t,x,y) = m(t,x,y) + \int_0^t d\tau \int_M m(t-\tau,x,z)r(\tau,z,y)\,\sigma(dz), \tag{6}$$

where the sought function r satisfies the integral Volterra equation

$$r(t,x,y) = h(t,x,y) + \int_0^t \mathrm{d}\tau \int_M h(t-\tau,x,z) r(t,z,y)\,\sigma(\mathrm{d}z). \tag{7}$$

Theorem 1. *Under the conditions of Lemma 1, eqn. (6) has a unique solution satisfying the estimate*

$$\big|r(t,x,y)\big| < c\exp(ct)\exp\left\{-\frac{\rho^2(x,y)}{2t}\right\},$$

where c is a constant.

Proof. Using the ordinary iteration technique to solve (7), we get

$$r(t,x,y) = h(t,x,y) + \sum_{n=1}^{\infty} r_n(t,x,y), \tag{8}$$

where

$$r_n(t,x,y) = \int_0^t \mathrm{d}\tau \int_M h(t-\tau,x,z) r_{n-1}(\tau,z,y)\,\sigma(\mathrm{d}z).$$

In much the same way as in the one-dimensional case, the estimate of the first iteration

$$r_1(t,x,y) = \int_0^t \mathrm{d}\tau \int_M h(t-\tau,x,z) h(\tau,z,y)\,\sigma(\mathrm{d}z)$$

is determined via substitution of the integration variable $z \to u \in T_xM$, where T_xM is a tangent to the space

$$Z = \exp_x\left(\sqrt{\frac{\tau(t-\tau)}{t}}u - \frac{t-\tau}{t}\rho(x,y)\dot{\gamma}(x)\right), \tag{9}$$

the exponential mapping $\exp_x$ being a 'projection' of the vector tangent at the point x onto the manifold. Equality (9) allows us to find the estimate

$$\big|r_1(t,x,y)\big| < ct(2\pi t)^{-n/2}\exp\left\{-\frac{\rho^2(x,y)}{2t}\right\}.$$

Then it is easy to establish the estimate for the n-th iteration

$$\big|r_n(t,x,y)\big| < \frac{c^n t^n}{n!}(2\pi t)^{-n/2}\exp\left\{-\frac{\rho^2(x,y)}{2t}\right\}$$

which is our claim. ∎

Remark 1. In the classical parametrix method, the r_n asymptotic behaviour as $t^{n/2}/\Gamma(n/2)$ takes place with respect to n, where $\Gamma(x)$ is the gamma function. Thus the series (8) exhibits slower convergence. The acceleration of the convergence proposed in this paper is achieved via an appropriate selection of the initial approximation $m(t,x,y)$ and a small (not containing singularities with respect to t) innovation $h(t,x,y)$.

Corollary 1. *The representation (6) for the fundamental solution takes place together with the following estimate:*

$$\left| \int_0^t \mathrm{d}\tau \int_M h(t-\tau,x,z) r(\tau,z,y)\, \sigma(\mathrm{d}z) \right| < ct \exp(ct) \exp\left\{ -\frac{\rho^2(x,y)}{2t} \right\}.$$

To find the fundamental solution the Riemannian distance $\rho(x,y)$ and the function $\varphi(x,y)$ must be calculated, which is a non-trivial problem. The Riemannian distance $\rho(x,y)$ is a solution to the variational problem (4) transformed into a set of Euler equations (in this case, they are called geodesic equations), which allows us to determine the geodesic γ:

$$\ddot{\gamma}^k(s) = g^{kr}(\gamma(s)) \left(\frac{1}{2} \frac{\partial g_{ij}}{\partial x^r}(\gamma(s)) - \frac{\partial g_{ir}}{\partial x^j}(\gamma(s)) \right) \dot{\gamma}^i(s) \dot{\gamma}^j, \tag{10}$$

$$\gamma(0) = y, \qquad \gamma(T) = x,$$

and, even in the two-dimensional case, assumes a rather sophisticated form

$$\begin{aligned} \ddot{\gamma}^1 = & \left(\frac{1}{2} g^{12} \frac{\partial g_{11}}{\partial x^2} - \frac{1}{2} g^{11} \frac{\partial g_{11}}{\partial x^1} - g^{12} \frac{\partial g_{12}}{\partial x^1} \right) (\dot{\gamma}^1)^2 \\ & - \left(g^{11} \frac{\partial g_{11}}{\partial x^2} + g^{12} \frac{\partial g_{22}}{\partial x^1} \right) \dot{\gamma}^1 \dot{\gamma}^2 \\ & + \left(\frac{1}{2} g^{11} \frac{\partial g_{12}}{\partial x^1} - \frac{1}{2} g^{12} \frac{\partial g_{22}}{\partial x^2} - g^{11} \frac{\partial g_{12}}{\partial x^2} \right) (\dot{\gamma}^2)^2. \end{aligned}$$

The solution to the geodesic equation depends on the derivatives of the matrix g_{jk}.

The function $\varphi(x,y)$ allows us to reduce the influence of the terms in the residual $h(t,x,y)$ with singularities at $t \downarrow 0$. Its computation is based on the formula

$$\varphi(x,y) = \int_0^{\rho(x,y)} \left(\Delta\rho' - \frac{n-1}{\rho'} \right) \mathrm{d}\rho'. \tag{11}$$

Thus, constructing the fundamental solution for (1) by using the method proposed here requires some extra calculations in comparison with the known approach, but it does provide faster convergence.

6. Numerical Simulation

As mentioned above, the algorithm of constructing the fundamental solution requires extra (with respect to the classic parametrix method) calculation of the Riemannian distance $\rho(x, y)$ and the function $\varphi(x, y)$ for computing the initial approximation $m(t, x, y)$. The Riemannian distance $\rho(x, y)$ is determined as a solution to the boundary problem for the geodesic equations (10), which is numerically calculated by Newton's method. The function $\varphi(x, y)$ is calculated based on the formula (11) and using numerical integration and finite-difference approximation for determining $\Delta\rho'$.

In order to demonstrate that the initial approximation $m(t, x, y)$ provides faster convergence than the classical parametrix one, their residuals are compared. The smaller residual is obtained, the better approximation is chosen, and the faster convergence takes place.

Numerical simulation of both the initial approximations was performed on the parabolic hyperboloid defined by the formula $z = xy$ in the Cartesian coordinates, which satisfies Assumption 1. Results of the simulation are presented in Fig. 1 ($t = 0.5$, $x = (1, 0.1)$, $y = (-1, 0.5)$) and Table 1 ($t = 0.01$, $x = (0.1, 0.1)$, $y = (-0.5, 0.3)$).

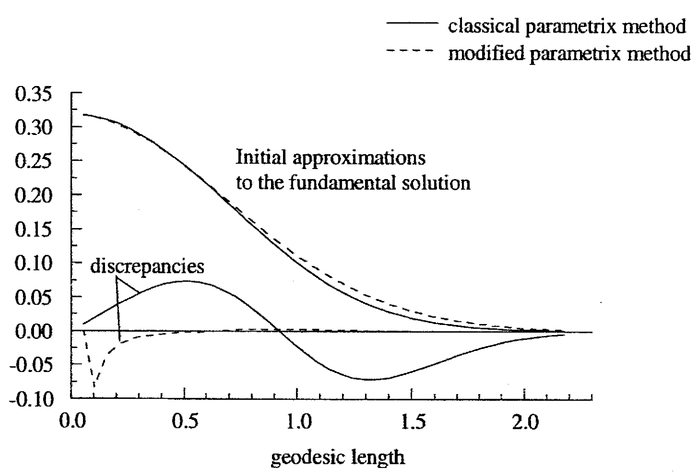

Fig. 1. Initial approximations to the fundamental solution and the corresponding discrepancies on the manifold of a parabolic hyperboloid with $t = 0.5$.

The advantages of the proposed modification of the parametrix method become obvious when the parameter t vanishes. In contrast to the classical parametrix approximation whose residual has a singularity while $t \downarrow 0$ (the maximum value reaches 22.5), the modified parametrix approximation produces a discrepancy of a substantially smaller order. We may assert that this order is less than 0.0001, despite the values in Fig. 1 and Tab. 1, because the peak near $s = 0.1$ is caused by the errors of numerical simulation of the residual itself. Calculating the modified parametrix approximation and its residual on a sphere, for which an analytic formula for the residual exists, shows exactly this order (Table 2).

Table 1. Initial approximations of the fundamental solution and the corresponding discrepancies on the manifold of a parabolic hyperboloid with $t = 0.01$.

Length s	Initial approximation of the classical parametrix method		Initial approximation $m(t, x, y)$ of the modified parametrix method	
	approximation	discrepancy	approximation	discrepancy
0.066	12.8032645	22. 511593	12.7504588	0.0250228
0.1319	6.5952518	1.1187486	6.6185392	−1.9982935
0.1979	2.1409309	−18.4730502	2.2241829	−0.2676849
0.2638	0.4311178	−12.3559834	0.4839477	−0.031716
0.3297	0.053061	−3.3780994	0.068209	−0.0027715
0.3956	0.0039395	−0.4590479	0.0062316	−0.0001695
0.4614	0.0001746	−0.0329347	0.0003694	−7.21E − 06

Table 2. Initial approximations of the fundamental solution and the corresponding residuals on the manifold of a sphere with $t = 0.01$.

Length s	Initial approximation of the classical parametrix method		Initial approximation $m(t, x, y)$ of the modified parametrix method	
	approximation	discrepancy	approximation	discrepancy
0.014	15.76013	−0.46079	15.76064	4.29E−05
0.028	15.30318	−1.75403	15.30462	0.000117
0.042	14.57123	−3.63054	14.57367	0.000199
0.056	13.60567	−5.73129	13.60865	0.000282
0.07	12.4588	−7.65817	12.46131	0.00036
0.084	11.1891	−9.04923	11.18968	0.000427
0.098	9.856279	−9.64257	9.853239	0.000478
0.112	8.516741	−9.31629	8.50845	0.00051
0.126	7.219816	−8.09865	7.20501	0.000522
0.14	6.005198	−6.14962	5.983206	0.000515
0.154	4.901622	−3.72082	4.872491	0.000491
0.168	3.926746	−1.10477	3.891245	0.000454
0.182	3.088033	1.415375	3.047538	0.000407
0.196	2.384344	3.606327	2.340643	0.000354
0.21	1.807932	5.309668	1.76299	0.000301
0.2239	1.346529	6.450102	1.302252	0.000248

7. Conclusions

In this paper we have proposed a modified parametrix method with an initial approximation that is much closer to the fundamental solution to a parabolic partial differential equation than the classical parametrix one. The estimate obtained in a one-dimensional space and on a Riemannian manifold proves faster convergence of the iteration process. This result has been confirmed by numerical simulation on the manifold of a parabolic hyperboloid.

References

Bakry D., Concordet D., Ledoux M. (1997): *Optimal heat kernel bounds under logarithmic Sobolev inequalities.* — ESAIM Prob. Stat., Vol.1, pp.391–407.

Bondarenko V.G. (1997): *Diffusion sur variété de courbure non positive.* — Comptes Rendus A.S., Paris, Vol.324, No.10, pp.1099–1103.

Cheng S.Y. and Yau S.T. (1975): *Differential equations on Riemannian manifolds and their geometric applications.* — Comm. Pure Appl. Math., Vol.28, No.3, pp.333–354.

Cheng S.Y., Yau S.T. (1981): *On the upper estimate of the heat kernel of a complete Riemannian manifold.* — Amer. J. Math., Vol.103, No.5, pp.1021–1063.

Cromol D., Klingenberg W., Meyer W. (1968): *Riemannishe Geometrie im Groben.* — Lecture Notes on Math., Vol.55, Berlin, Heidelberg, New York: Springer Verlag.

Fletcher C.A.J. (1984): *Computational Galerkin Methods.* — Berlin: Springer Verlag.

Friedman A. (1964): *Partial Differential Equations of Parabolic Type.* — New York: Prentice-Hall.

Grigoryan A. (1998): *Estimates of heat kernels on Riemannian manifolds.* — Proc. Int. Conf. *Special Theory and Geometry*, Edinburgh, pp.3–53.

Na T.Y. (1979): *Computational Methods in Engineering — Boundary Value Problem.* — New York: Academic Press.

Shih T.-M. (1984): *Numerical Heat Transfer.* — New York: Hemisphere Corporation.

Received: 28 May 1999
Revised: 12 January 2000