DETECTION AND CONTROL PROBLEMS FOR NON-LINEAR DISTRIBUTED-PARAMETER SYSTEMS WITH DELAYS

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First, we consider non-linear discrete-time and continuous-time systems with unknown inputs. The problem of reconstructing an input using the information given by an output equation is investigated. Then we examine a control problem for non-linear discrete-time hereditary systems, i.e. the problem of finding a control which drives the state of the system from its initial value to a given desired final state. The methods used to solve these problems are based on the state-space technique and fixed-point theorems. To illustrate the outlined ideas, various numerical simulation results are presented.

Keywords: distributed parameter systems, non-linear discrete-time systems, detection, control, fixed-point theorems

1. Introduction

When considering a mathematical model for a system of physical, chemical or economic type, it is often necessary to take into account some unknown parameters that affect the system. Depending on the nature of the system, these parameters can be of different origins: errors in the approximation of the original system, some external perturbations, excitation of an unknown source, etc. The systems considered in the first part of this work are assumed to be perturbed by an unknown action and our objective is to develop methods to reconstruct this input (detection problem). This problem has been investigated in the case of linear continuous-time systems (Afifi and El Jai, 1994; 1995) and of linear discrete-time hereditary systems (Namir *et al.*, 1999). In those papers, the authors solved the detection problem using some properties of linear operators on Hilbert spaces, their inverses and adjoints.

In this work, we examine the detection problem in the case of both discrete-time non-linear distributed-parameter systems and continuous-time systems with delays. Because of the non-linearity of the systems considered, the results presented in the papers mentioned above cannot be directly extended to our case using the same techniques. Instead, we solve the detection problem by different methods that are mainly

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based on fixed-point theorems. In recent years, this tool has ben used by a large number of mathematicians to treat various problems related to the control and analysis of non-linear systems. In (Ichikawa and Pritchard, 1979) this technique was used to investigate the existence and uniqueness of solutions to evolution equations. Carmichael and Quinn (1982) used it to solve an optimal-control problem. The controllability and state-estimation problems were also treated by similar techniques (Felippe De Souza, 1984; Pritchard, 1982).

In this work, we use the fixed-point technique to study detection and control problems for non-linear systems. We formulate the detection problem as follows: For every input f that perturbs the system, $y(f)$ will denote the corresponding observation. For a given output y^d, our objective will be to characterize the set, denoted by $\mathcal{F}_{\mathrm{ad}}$, of all inputs $f^* \in \mathcal{F}$ for which y^d is the corresponding output:

$$\mathcal{F}_{\mathrm{ad}} = \left\{ f^* \in \mathcal{F} \mid y(f^*) = y^d \right\},$$

$\mathcal{F}$ being a given Hilbert space.

A more classical problem in systems analysis theory is the control problem.The objective of controllability is to find a control which steers the state of the system of interest from its initial value to a desired given final state. This problem has been intensively studied by many mathematicians and a large number of contributions can be found in the literature. However, in most of the studies, the systems considered were assumed to be linear. Only recently some authors have begun investigating control problems for non-linear continuous-time systems (Felippe De Souza, 1984; Kassara and El Jai, 1983; Magnusson *et al.*, 1981).

In this paper, we examine the control problem in the case of non-linear discrete-time hereditary systems in Hilbert spaces. This problem can be formulated as follows. Let x_0 and x^d denote respectively the initial state of the system and a given desired final state. Our objective is to characterize the set $\mathcal{U}_{\mathrm{ad}}$ of inputs u^* which steer the state of the system from x_0 to x^d:

$$\mathcal{U}_{\mathrm{ad}} = \left\{ u^* \in \mathcal{U} \mid x_f(u^*) = x^d \right\},$$

where $\mathcal{U}$ is a Hilbert space and $x_f(u^*)$ is the final state of the system corresponding to the initial state x_0 and the control u^*. The fixed-point technique proves again to be a good tool for solving this problem in the case of non-linear systems.

The paper is divided into four sections. In Sections 2 and 3, we investigate the detection problem for discrete- and continuous-time non-linear hereditary systems, respectively. In Section 4, we solve the control problem for distributed discrete-time systems with delays.

2. Detection Problem for Discrete-Time Systems

2.1. Preliminaries

In this section, we consider non-linear discrete systems of the type

$$\begin{cases} \xi_{i+1} = \sum_{j=0}^{p} A_j \xi_{i-j} + \sum_{j=0}^{q} N_j(\xi_{i-j}) + \sum_{j=0}^{m} D_j f_{i-j}, \quad 0 \le i \le n-1, \ \ n = 1, 2, \dots, \\ f_k = \phi_k \ \text{ for } \ -m \le k \le -1 \ \text{ and } \ \xi_k \ \text{ is given for } \ -\max\{p, q\} \le k \le 0, \end{cases} \tag{1}$$

with the output equation

$$y_i = \zeta(\xi_i, \xi_{i-1}, \dots, \xi_{i-r}), \quad 1 \le i \le n \ \text{ with } r \le \max\{p, q\}. \tag{2}$$

Here $A_j : \mathcal{X} \longrightarrow \mathcal{X}$ are linear operators defined on a Hilbert space $\mathcal{X}$ (the state space), $N_j : \mathcal{X} \longrightarrow \mathcal{X}$ are non-linear operators, $D_j : F \longrightarrow \mathcal{X}$ are linear operators on a Hilbert space F (the input space) and $\zeta : \mathcal{X}^{r+1} \longrightarrow Y$ is an operator (linear or not) with values in a Hilbert space Y (the output space). Note that the operators A_j, N_j, D_j and ζ are not necessarily bounded. $(\phi_{-m}, \phi_{-m+1}, \dots, \phi_{-1})$ is a given sequence in F. In the following, and without loss of generality, we assume that $p \ge q$. (If $p < q$, we can set $A_j = 0$ for $j = p+1, \dots, q$.)

The sequence $(f_i)_{0 \le i \le n-1}$ will always denote an unknown input that perturbs the system (1). For every such input, eqn. (2) gives the corresponding output $(y_i)_{1 \le i \le n} \in Y^n$. Assume now that we know a particular output $(y_i^d)_i$. We recall that our objective is to characterize the set $\mathcal{F}_{\text{ad}}$ of all inputs $(f_i)_i$ for which $(y_i^d)_i$ is the corresponding output:

$$\mathcal{F}_{\text{ad}} = \left\{ (f_i)_i \in F^n \mid (y_i)_i \equiv (y_i^d)_i \right\}.$$

Equations (1) and (2) describe the evolution of the state and output of a system with delays in the state, input and output. However, one can rewrite this system in a product space in such a way that these delays disappear from the state equation. Consider the Hilbert space $X := \mathcal{X}^{p+1} \times F^m$ and set

$$x_i = (\xi_i, \xi_{i-1}, \dots, \xi_{i-p}, f_{i-1}, \dots, f_{i-m})^T \in X \quad \text{for} \quad 0 \le i \le n.$$

It is readily verified that the sequence $(x_i)_{0 \le i \le n}$ satisfies

$$\begin{cases} x_{i+1} = A x_i + N(x_i) + D f_i, \quad 0 \le i \le n-1, \\ x_0 \in X, \end{cases} \tag{3}$$

where

$$A = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix}, \quad N = \begin{pmatrix} N_{11} & 0 \\ 0 & 0 \end{pmatrix}, \quad D = \begin{pmatrix} D_{11} \\ D_{12} \end{pmatrix},$$

with the operators $A_{11} : \mathcal{X}^{p+1} \longrightarrow \mathcal{X}^{p+1}$, $A_{12} : F^m \longrightarrow \mathcal{X}^{p+1}$, $A_{22} : F^m \longrightarrow F^m$, $N_{11} : \mathcal{X}^{p+1} \longrightarrow \mathcal{X}^{p+1}$, $D_{11} : F \longrightarrow \mathcal{X}^{p+1}$ and $D_{21} : F \longrightarrow F^m$ respectively given by

$$A_{11} = \begin{pmatrix} A_0 & A_1 & \cdots & \cdots & A_p \\ I_{\mathcal{X}} & 0 & \cdots & \cdots & 0 \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & I_{\mathcal{X}} & 0 \end{pmatrix}, \quad A_{12} = \begin{pmatrix} D_1 & \cdots & \cdots & \cdots & D_m \\ 0 & \cdots & \cdots & \cdots & 0 \\ \vdots & & & & \vdots \\ \vdots & & & & \vdots \\ 0 & \cdots & \cdots & \cdots & 0 \end{pmatrix},$$

$$A_{22} = \begin{pmatrix} 0 & \cdots & \cdots & \cdots & 0 \\ I_F & \ddots & & & \vdots \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & I_F & 0 \end{pmatrix}, \quad N_{11} = \begin{pmatrix} N_0 & \cdots & N_q & 0 & \cdots & 0 \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ \vdots & & & & & \vdots \\ \vdots & & & & & \vdots \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 \end{pmatrix},$$

$$D_{11} = \begin{pmatrix} D_0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad D_{12} = \begin{pmatrix} I_F \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Here $I_{\mathcal{X}}$ and I_F denote the identity mappings on $\mathcal{X}$ and F, respectively. Moreover, if we define

$$C : z := (z_i)_{0 \le i \le p+1+m} \in \mathcal{X}^{p+1} \times F^m \longmapsto C(z) = \zeta(z_0, z_1, \ldots, z_r) \in Y,$$

then the output equation (2) can be rewritten as

$$y_i = C(x_i), \quad 1 \le i \le n. \tag{4}$$

2.2. Problem Solution

Clearly, the solution to (3) is given by

$$x_i = A^i x_0 + \sum_{j=0}^{i-1} A^{i-j-1} N(x_j) + \sum_{j=0}^{i-1} A^{i-j-1} D f_j, \quad 1 \le i \le n. \tag{5}$$

Introduce the operators

$$G : x \in X^n \longmapsto Gx \in X^n \text{ with } (Gx)_i = \sum_{j=0}^{i-1} A^{i-j-1} N(x_j), \quad 1 \le i \le n,$$

and

$$L : f \in F^n \longmapsto Lf \in X^n \text{ with } (Lf)_i = \sum_{j=0}^{i-1} A^{i-j-1} D f_j, \quad 1 \le i \le n.$$

Equation (5) can then be written as

$$x = A^{\cdot} x_0 + Gx + Lf,$$

where $A^{\cdot} x_0 = (A^i x_0)_{1 \le i \le n}$.

In the following, we shall need to inverse the operator L. But L is not invertible in a general case. Consider then

$$L_1 : x \in \ker(L)^{\perp} \longmapsto L_1 x = Lx \in \text{range}\,(L).$$

This operator is invertible and its inverse, which is defined on range(L), can be extended to range(L) $\oplus$ range$(L)^{\perp}$ as follows:

$$L^{\dagger} : x + y \in \text{range}(L) \oplus \text{range}(L)^{\perp} \longmapsto L_1^{-1} x \in F^n.$$

The operator $L^{\dagger}$ is known as the 'generalized inverse' or 'pseudo-inverse' of L. If range(L) is closed, then $X^n = \text{range}(L) \oplus \text{range}(L)^{\perp}$ and $L^{\dagger}$ is defined on the whole space X^n. In particular, the mapping $L^{\dagger}$ satisfies

$$\begin{cases} LL^{\dagger} x = x, & \forall x \in \text{range}(L), \\ L^{\dagger} L f = f, & \forall f \in \ker(L)^{\perp}. \end{cases}$$

2.2.1. Method I

Let $P : X^n \longrightarrow \text{range}(L)$ be any projection on range(L), $\bar{x}$ be any fixed element of range(L) $\setminus \{0\}$ and $\xi : X^n \longrightarrow X^n$ be the mapping

$$\xi(x) = x - A^{\cdot} x_0 - Gx.$$

Consider the operator $H : X^n \longrightarrow X^n$ defined by

$$Hx = A^{\cdot} x_0 + Gx + P\xi(x) + \|y - y^d\|_{Y^n} \bar{x}, \tag{6}$$

where $\| \cdot \|_{Y^n}$ is the norm on Y^n.

Proposition 1. *We have the following properties:*

1. If x^* is a fixed-point of H, then $\xi(x^*) \in \text{range}(L)$ and

$$L^{\dagger} \xi(x^*) + \ker(L) \subset \mathcal{F}_{\text{ad}}.$$

2. If $f^* \in \mathcal{F}_{\text{ad}}$, then x^* (i.e. the corresponding state) is a fixed-point of H and

$$f^* \in L^{\dagger} \xi(x^*) + \ker(L).$$

Proof. (Part 1) Let x^* be a fixed-point of H. We have

$$x^* = A^{\cdot}x_0 + Gx^* + P\xi(x^*) + \|y^* - y^d\|_{Y^n}\bar{x}, \tag{7}$$

where $y^* = C(x^*)$, and hence

$$\xi(x^*) = P\xi(x^*) + \|y^* - y^d\|_{Y^n}\bar{x} \in \text{range}(L).$$

Therefore $P\xi(x^*) = \xi(x^*)$ and $\|y^* - y^d\|_{Y^n}\bar{x} = 0$. But since $\bar{x} \neq 0$, we deduce that $y^* = y^d$.

Equation (7) becomes

$$x^* = A^{\cdot}x_0 + Gx^* + \xi(x^*).$$

Set $v^* = L^\dagger\xi(x^*) + \phi^*$ with $\phi^* \in \ker(L)$. Since $Lv^* = \xi(x^*)$, we can write

$$\begin{cases} x^* = A^{\cdot}x_0 + Gx^* + Lv^*, \\ y^* = y^d, \end{cases}$$

which means that $v^* \in \mathcal{F}_{\text{ad}}$. We conclude that

$$L^\dagger\xi(x^*) + \ker(L) \subset \mathcal{F}_{\text{ad}}.$$

(Part 2) If $f^* \in \mathcal{F}_{\text{ad}}$ and x^* (resp. y^*) is the corresponding state (resp. output), then

$$\begin{cases} x^* = A^{\cdot}x_0 + Gx^* + Lf^*, \\ y^* = y^d. \end{cases}$$

Hence

$$\begin{cases} \xi(x^*) = Lf^* \in \text{range}(L), \\ \|y^* - y^d\|_{Y^n} = 0 \end{cases}$$

and therefore

$$H(x^*) = A^{\cdot}x_0 + Gx^* + \xi(x^*) = x^*.$$

We have $f^* = L^\dagger\xi(x^*) + \left(f^* - L^\dagger\xi(x^*)\right)$ and $L\left(f^* - L^\dagger\xi(x^*)\right) = Lf^* - \xi(x^*) = 0$. We conclude that $f^* \in L^\dagger\xi(x^*) + \ker(L)$. ■

A characterization of the set $\mathcal{F}_{\text{ad}}$ is given by the next assertion.

Proposition 2. *Let $\mathcal{P}_H$ denote the set of fixed-points of H. Then $\mathcal{F}_{\text{ad}}$ is given by*

$$\mathcal{F}_{\text{ad}} = \left\{L^\dagger\xi(x^*) \mid x^* \in \mathcal{P}_H\right\} + \ker(L).$$

Proof. Clearly, by Parts 1 and 2 of Proposition 1, we have

$$\mathcal{F}_{\text{ad}} = \bigcup_{x^* \in \mathcal{P}_H} \left(L^\dagger\xi(x^*) + \ker(L)\right) = \left\{L^\dagger\xi(x^*) \mid x^* \in \mathcal{P}_H\right\} + \ker(L).$$

■

Remarks:

- The fixed-points of H depend on neither the choice of the projection P nor the element $\bar{x}$. For example, let P_1 and P_2 be two projections on range(L) and let $\bar{x}_1$ and $\bar{x}_2$ stand for two elements in range(L). Consider the mappings

$$H_1 : x \in X \longmapsto A^{\cdot}x_0 + Gx + P_1\xi(x) + \|y - y^d\|_{Y^n}\bar{x}_1 \in X,$$

and

$$H_2 : x \in X \longmapsto A^{\cdot}x_0 + Gx + P_2\xi(x) + \|y - y^d\|_{Y^n}\bar{x}_2 \in X.$$

All we have to show is that if x^* is a fixed-point of H_1, then it is also a fixed-point of H_2. By Proposition 1, if $x^* \in \mathcal{P}_{H_1}$, then $y^* = y^d$ and $\xi(x^*) \in$ range(L). It follows that $P_2\xi(x^*) = \xi(x^*)$ and

$$H_2(x^*) = A^{\cdot}x_0 + Gx^* + \xi(x^*) = x^*.$$

- Clearly, Proposition 1 is still true if the expression $\|y - y^d\|_{Y^n}$ in (6) is substituted by the term $q(y)$, where $q : Y^n \longrightarrow \mathbb{C}$ is any function which satisfies $q(y) = 0$ if and only if $y = y^d$.

By Proposition 2, the detection problem has been transformed into that of the existence of fixed-points for H. Moreover, the set $\mathcal{F}_{\text{ad}}$ will be completely characterized if the fixed-points of H are known. This is illustrated with the following example.

Example 1. Consider the system

$$\begin{cases} x_{i+1} = Ax_i + N(x_i) + Df_i, & 0 \le i \le n-1, \quad n = 1, 2, \dots, \\ x_0 = 0, \end{cases} \tag{8}$$

$$y_i = C(x_i), \quad 1 \le i \le n, \tag{9}$$

with $X = L^2(0,1;\mathbb{R})$, $F = \mathbb{R}^2$, $Y = \mathbb{R}$. The operators A, N, D and C are respectively defined by

$$A : x \in X \longmapsto \sum_{k=1}^{\infty} e^{-k^2\pi^2\delta}\langle x, e_k\rangle_X e_k \in X,$$

$$N : x \in X \longmapsto \sum_{k=1}^{m}\langle x, e_k\rangle_X^2 e_k \in X, \quad m = 1, 2, \dots,$$

$$D : \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \in \mathbb{R}^2 \longmapsto \alpha e_1 \in X, \quad C : x \in X \longmapsto \langle x, e_1\rangle_X^3 \in \mathbb{R},$$

where $\delta \in \mathbb{R}_+ \setminus \{0\}$, $e_k(\cdot) = \sqrt{2}\sin(k\pi.)$ and $\langle \cdot,\cdot \rangle_X$ denotes the inner product on X. One can easily check that the operators L, G and ξ, the sets $\ker(L)$, $\ker(L)^\perp$ and $\text{range}(L)$ are, in this case, given by

$$\begin{cases} L : f := \begin{pmatrix} f_{i1} \\ f_{i2} \end{pmatrix}_{0 \le i \le n-1} \in \left(\mathbb{R}^2\right)^n \longmapsto Lf \in X^n, \\ (Lf)_i = \displaystyle\sum_{j=0}^{i-1} e^{-\pi^2\delta(i-j-1)} f_{j1} e_1, \quad 1 \le i \le n, \end{cases}$$

$$\begin{cases} G : x \in X^n \longmapsto Gx \in X^n, \\ (Gx)_i = \displaystyle\sum_{k=1}^{m}\sum_{j=0}^{i-1} e^{-k^2\pi^2\delta(i-j-1)} \langle x_j, e_k\rangle_X^2 e_k, \quad 1 \le i \le n, \end{cases}$$

$$\begin{cases} \xi : x \in X^n \longmapsto \xi(x) \in X^n, \\ \left(\xi(x)\right)_i = x_i - \displaystyle\sum_{k=1}^{m}\sum_{j=0}^{i-1} e^{-k^2\pi^2\delta(i-j-1)} \langle x_j, e_k\rangle_X^2 e_k, \quad 1 \le i \le n, \end{cases}$$

$$\ker(L) = \left\{ f \in \left(\mathbb{R}^2\right)^n \mid f_{i1} = 0, \quad 0 \le i \le n-1 \right\},$$

$$\ker(L)^\perp = \left\{ f \in \left(\mathbb{R}^2\right)^n \mid f_{i2} = 0, \quad 0 \le i \le n-1 \right\},$$

$$\text{range}(L) = \left\{ (\alpha_i e_1)_{1 \le i \le n} \in X^n \mid \alpha_i \in \mathbb{R} \right\}.$$

Introduce the operator $L_1 : f \in \ker(L)^\perp \longmapsto Lf \in \text{range}(L)$. L_1 is invertible and its inverse is given by

$$L_1^{-1} : (\alpha_i e_1)_{1 \le i \le n} \in \text{range}(L) \longmapsto f \in \ker(L)^\perp,$$

where

$$\begin{cases} f_{01} = \alpha_1, & \\ f_{i1} = \alpha_{i+1} - e^{-\pi^2\delta}\alpha_i, & 1 \le i \le n-1, \\ f_{i2} = 0, & 0 \le i \le n-1. \end{cases}$$

In fact, the operator L_1^{-1} is the restriction of $L^\dagger$ to $\text{range}(L)$. Set $\bar{x} = (e_1, 0, \ldots, 0)^T$ and let P be the projection

$$P : x \in X^n \longmapsto \left(\langle x_i, e_1\rangle_X e_1\right)_{1 \le i \le n} \in \text{range}(L).$$

It follows that the mapping $H : x \in X^n \longmapsto Hx \in X^n$ is then given by

$$\begin{aligned} (Hx)_i &= \langle x_i, e_1\rangle_X e_1 + \sum_{k=2}^{m}\sum_{j=0}^{i-1} e^{-k^2\pi^2\delta(i-j-1)} \langle x_j, e_k\rangle_X^2 e_k \\ &\quad + \left\| y - y^d \right\|_{Y^n} \bar{x}_i, \quad 1 \le i \le n. \end{aligned}$$

Let x^* be a fixed-point of H. We have

$$x_i^* = \langle x_i^*, e_1\rangle_X e_1 + \sum_{k=2}^{m}\sum_{j=0}^{i-1} e^{-k^2\pi^2\delta(i-j-1)}\langle x_j^*, e_k\rangle_X^2 e_k + \left\|y^* - y^d\right\|_{Y^n}\bar{x}_i. \qquad (10)$$

Hence

$$\langle x_1^*, e_1\rangle_X = \langle x_1^*, e_1\rangle_X + \|y^* - y^d\|_{Y^n}.$$

Thus $y^d = y^*$ and, consequently, $\left(y_i^d\right)^{1/3} = \langle x_i^*, e_1\rangle_X$, $1 \le i \le n$.

For $i = 1$, eqn. (10) becomes $x_1^* = \left(y_1^d\right)^{1/3} e_1$ and by induction we can show that

$$x_i^* = \left(y_i^d\right)^{1/3} e_1, \quad 1 \le i \le n. \qquad (11)$$

Respectively, we can easily check that x^* given by (11) is a fixed-point for H. Thus we have shown that x^* given by (11) is the unique fixed-point of H.

We have

$$\left(\xi\left(x^*\right)\right)_i = \left(\left(y_i^d\right)^{1/3} - \sum_{j=0}^{i-1} e^{-\pi^2\delta(i-j-1)}\left(y_j^d\right)^{2/3}\right) e_1.$$

Let $y_0^d = 0$. It follows that

$$L^\dagger\xi(x^*) = \begin{pmatrix} \left(y_{i+1}^d\right)^{1/3} - e^{-\pi^2\delta}\left(y_i^d\right)^{1/3} - \left(y_i^d\right)^{2/3} \\ 0 \end{pmatrix}_{0\le i\le n-1},$$

and therefore

$$\begin{aligned}\mathcal{F}_{\text{ad}} &= L^\dagger\xi(x^*) + \ker(L) \\ &= \left\{\begin{pmatrix} \left(y_{i+1}^d\right)^{1/3} - e^{-\pi^2\delta}\left(y_i^d\right)^{1/3} - \left(y_i^d\right)^{2/3} \\ \alpha_i \end{pmatrix}_{0\le i\le n-1} \mid \alpha_i \in \mathbb{R}\right\}.\end{aligned}$$

♦

2.2.2. Method II

Now we present the other method to characterize the set $\mathcal{F}_{\text{ad}}$ that can be used only in case the output operator is linear. Assume then that the observation is given by

$$y_i = Cx_i, \quad 1 \le i \le n,$$

where $C : X \longrightarrow Y$ is a linear operator. Consider again the operators G and L as defined in the previous section. Let $\mathcal{M} = X^n \times Y^n$ and introduce the operators

$$\mathcal{A} : x_0 \in X \longmapsto \begin{pmatrix} Ax_0 \\ CAx_0 \end{pmatrix} \in \mathcal{M}, \quad \mathcal{G} : x \in X^n \longmapsto \begin{pmatrix} Gx \\ CGx \end{pmatrix} \in \mathcal{M},$$

$$\mathcal{L} : f \in F^n \longmapsto \begin{pmatrix} Lf \\ CLf \end{pmatrix} \in \mathcal{M},$$

$$\widetilde{\xi} : \begin{pmatrix} x \\ z \end{pmatrix} \in \mathcal{M} \longmapsto \begin{pmatrix} x \\ z \end{pmatrix} - \mathcal{A}x_0 - \mathcal{G}x \in \mathcal{M}.$$

Let $\mathcal{P} : \mathcal{M} \longmapsto \text{range}(\mathcal{L})$ be any projection onto $\text{range}(\mathcal{L})$ and consider the mapping

$$\mathcal{H} : \begin{pmatrix} x \\ z \end{pmatrix} \in \mathcal{M} \longmapsto \mathcal{A}x_0 + \mathcal{G}x + \mathcal{P}\widetilde{\xi} \begin{pmatrix} x \\ y^d \end{pmatrix} + \begin{pmatrix} 0 \\ z - y^d \end{pmatrix} \in \mathcal{M}.$$

Proposition 3. *We have the following characterizations:*

1. If $\binom{x^*}{z^*} \in \mathcal{M}$ *is a fixed-point of* $\mathcal{H}$, *then* $\widetilde{\xi}\binom{x^*}{y^d} \in \text{range}(\mathcal{L})$ *and*

$$\mathcal{L}^\dagger \widetilde{\xi} \begin{pmatrix} x^* \\ y^d \end{pmatrix} + \ker(\mathcal{L}) \subset \mathcal{F}_{\text{ad}}.$$

2. If $f^* \in \mathcal{F}_{\text{ad}}$, *then* $\binom{x^*}{y^d}$ *is a fixed-point of* $\mathcal{H}$ *and*

$$f^* \in \mathcal{L}^\dagger \widetilde{\xi} \begin{pmatrix} x^* \\ y^d \end{pmatrix} + \ker(\mathcal{L}),$$

where x^* *is the state corresponding to* f^*.

Proof. (Part 1) If $\binom{x^*}{z^*}$ is a fixed-point of $\mathcal{H}$, then

$$\begin{pmatrix} x^* \\ z^* \end{pmatrix} = \mathcal{A}x_0 + \mathcal{G}x^* + \mathcal{P}\widetilde{\xi} \begin{pmatrix} x^* \\ y^d \end{pmatrix} + \begin{pmatrix} 0 \\ z^* - y^d \end{pmatrix}.$$

Hence

$$\begin{aligned} \mathcal{P}\widetilde{\xi} \begin{pmatrix} x^* \\ y^d \end{pmatrix} &= \widetilde{\xi} \begin{pmatrix} x^* \\ z^* \end{pmatrix} - \begin{pmatrix} 0 \\ z^* - y^d \end{pmatrix} \\ &= \begin{pmatrix} x^* - Ax_0 - Gx^* \\ z^* - CAx_0 - CGx^* \end{pmatrix} - \begin{pmatrix} 0 \\ z^* - y^d \end{pmatrix} \\ &= \widetilde{\xi} \begin{pmatrix} x^* \\ y^d \end{pmatrix} \in \text{range}(\mathcal{L}). \end{aligned}$$

We have

$$\begin{pmatrix} x^* \\ y^d \end{pmatrix} = \begin{pmatrix} x^* \\ z^* \end{pmatrix} - \begin{pmatrix} 0 \\ z^* - y^d \end{pmatrix} = \mathcal{A}x_0 + \mathcal{G}x^* + \widetilde{\xi} \begin{pmatrix} x^* \\ y^d \end{pmatrix}.$$

Set

$$v^* = \mathcal{L}^\dagger \widetilde{\xi} \begin{pmatrix} x^* \\ y^d \end{pmatrix} + \phi^*,$$

where $\phi^* \in \ker(\mathcal{L})$. Since

$$\mathcal{L}v^* = \widetilde{\xi} \begin{pmatrix} x^* \\ y^d \end{pmatrix},$$

we can write

$$\begin{pmatrix} x^* \\ y^d \end{pmatrix} = \mathcal{A}x_0 + \mathcal{G}x^* + \mathcal{L}v^*,$$

i.e.

$$\begin{cases} x^* = A^{\cdot}x_0 + Gx^* + Lv^*, \\ y^d = CA^{\cdot}x_0 + CGx^* + CLv^* = Cx^*, \end{cases}$$

which means that $v^* \in \mathcal{F}_{\text{ad}}$. We conclude that

$$\mathcal{L}^\dagger \widetilde{\xi} \begin{pmatrix} x^* \\ y^d \end{pmatrix} + \ker(\mathcal{L}) \subset \mathcal{F}_{\text{ad}}.$$

(Part 2) If $f^* \in \mathcal{F}_{\text{ad}}$ and x^* is the corresponding state, then we have

$$\begin{cases} x^* = A^{\cdot}x_0 + Gx^* + Lf^*, \\ y^d = Cx^*, \end{cases}$$

or, equivalently,

$$\begin{pmatrix} x^* \\ y^d \end{pmatrix} = \mathcal{A}x_0 + \mathcal{G}x^* + \mathcal{L}f^*.$$

Hence $\widetilde{\xi}\begin{pmatrix} x^* \\ y^d \end{pmatrix} = \mathcal{L}f^* \in \text{range}(\mathcal{L})$ and therefore

$$\mathcal{H}\begin{pmatrix} x^* \\ y^d \end{pmatrix} = \mathcal{A}x_0 + \mathcal{G}x^* + \widetilde{\xi}\begin{pmatrix} x^* \\ y^d \end{pmatrix} = \begin{pmatrix} x^* \\ y^d \end{pmatrix}.$$

Set

$$v^* = \mathcal{L}^\dagger \widetilde{\xi} \begin{pmatrix} x^* \\ y^d \end{pmatrix}.$$

We have $f^* = v^* + (f^* - v^*)$ and $\mathcal{L}(f^* - v^*) = 0$. Consequently, we conclude that

$$f^* \in \mathcal{L}^\dagger \widetilde{\xi} \begin{pmatrix} x^* \\ y^d \end{pmatrix} + \ker(\mathcal{L}).$$

∎

Let $\mathcal{P}_{\mathcal{H}}$ denote the set of fixed-points of $\mathcal{H}$ and set

$$\mathcal{P}'_{\mathcal{H}} := \left\{ x^* \in X^n \mid \exists z^* \in Y^n \text{ such that } \begin{pmatrix} x^* \\ z^* \end{pmatrix} \in \mathcal{P}_{\mathcal{H}} \right\}.$$

Proposition 4. *The set of admissible inputs is given by*

$$\mathcal{F}_{\text{ad}} = \{L^\dagger \xi(x^*) \mid x^* \in \mathcal{P}'_{\mathcal{H}}\} + \ker(L).$$

Proof. By Proposition 3, we have

$$\mathcal{F}_{\text{ad}} = \bigcup_{x^* \in \mathcal{P}'_{\mathcal{H}}} \left(\mathcal{L}^\dagger \widetilde{\xi} \begin{pmatrix} x^* \\ y^d \end{pmatrix} + \ker(\mathcal{L}) \right) = \left\{ \mathcal{L}^\dagger \widetilde{\xi} \begin{pmatrix} x^* \\ y^d \end{pmatrix} \mid x^* \in \mathcal{P}'_{\mathcal{H}} \right\} + \ker(\mathcal{L}).$$

Note that $\ker(\mathcal{L}) = \ker(L)$,

$$\text{range}(\mathcal{L}) = \left\{ \begin{pmatrix} x \\ Cx \end{pmatrix} \mid x \in \text{range}(L) \right\}$$

and

$$\widetilde{\xi} \begin{pmatrix} x^* \\ y^d \end{pmatrix} = \begin{pmatrix} \xi(x^*) \\ y^d - CA^{\cdot} x_0 - CGx^* \end{pmatrix}.$$

Introduce the operator

$$\mathcal{L}_1 : f \in \ker(\mathcal{L})^\perp \longmapsto \mathcal{L}f \in \text{range}(\mathcal{L}).$$

We can check easily that its inverse is given by

$$\mathcal{L}_1^{-1} : \begin{pmatrix} x \\ Cx \end{pmatrix} \in \text{range}(\mathcal{L}) \longmapsto L^\dagger x \in F^n, \quad (x \in \text{range}(L)).$$

The inverse of $\mathcal{L}_1$ is the restriction of $\mathcal{L}^\dagger$ to $\text{range}(\mathcal{L})$. Hence, for all $x^* \in \mathcal{P}'_{\mathcal{H}}$ we have

$$\mathcal{L}^\dagger \widetilde{\xi} \begin{pmatrix} x^* \\ y^d \end{pmatrix} = \mathcal{L}_1^{-1} \begin{pmatrix} \xi(x^*) \\ y^d - CA^{\cdot} x_0 - CGx^* \end{pmatrix} = L^\dagger \xi(x^*).$$

This completes the proof. ■

Example 2. Consider again the system (8) and (9) given in Example 1. Now we assume that the output operator is linear and given by

$$C : x \in X \longmapsto \langle x, e_1 \rangle_X \in \mathbb{R}.$$

From the knowledge of the operators G and ξ, we deduce easily the form of operators $\mathcal{G}$ and $\widetilde{\xi}$:

$$\mathcal{G} : x \in X^n \longmapsto \begin{pmatrix} \left(\sum_{k=1}^{m} \sum_{j=0}^{i-1} e^{-k^2\pi^2\delta(i-j-1)} \langle x_j, e_k \rangle_X^2 e_k \right)_{1 \le i \le n} \\ \left(\sum_{j=0}^{i-1} e^{-\pi^2\delta(i-j-1)} \langle x_j, e_1 \rangle_X^2 \right)_{1 \le i \le n} \end{pmatrix} \in \mathcal{M},$$

$$\widetilde{\xi} : \begin{pmatrix} x \\ z \end{pmatrix} \in \mathcal{M} \longmapsto \begin{pmatrix} \left(x_i - \sum_{k=1}^{m} \sum_{j=0}^{i-1} e^{-k^2\pi^2\delta(i-j-1)} \langle x_j, e_k \rangle_X^2 e_k \right)_{1 \le i \le n} \\ \left(z_i - \sum_{j=0}^{i-1} e^{-\pi^2\delta(i-j-1)} \langle x_j, e_1 \rangle_X^2 \right)_{1 \le i \le n} \end{pmatrix} \in \mathcal{M}.$$

Introduce the projection

$$\mathcal{P} : \begin{pmatrix} x \\ z \end{pmatrix} \in \mathcal{M} \longmapsto \begin{pmatrix} (\langle x_i, e_1 \rangle_X e_1)_{1 \le i \le n} \\ (\langle x_i, e_1 \rangle_X)_{1 \le i \le n} \end{pmatrix} \in \text{range}(\mathcal{L}).$$

The mapping $\mathcal{H}$ is then given by

$$\begin{aligned} \mathcal{H}\begin{pmatrix} x \\ z \end{pmatrix} &= \mathcal{G}x + P\widetilde{\xi}\begin{pmatrix} x \\ y^d \end{pmatrix} + \begin{pmatrix} 0 \\ z - y^d \end{pmatrix} \\ &= \begin{pmatrix} \left(\langle x_i, e_1 \rangle_X e_1 + \sum_{k=2}^{m} \sum_{j=0}^{i-1} e^{-k^2\pi^2\delta(i-j-1)} \langle x_j, e_k \rangle_X^2 e_k \right)_{1 \le i \le n} \\ (\langle x_i, e_1 \rangle_X)_{1 \le i \le n} + z - y^d \end{pmatrix}. \end{aligned}$$

One can easily check that $\binom{x^*}{z^*} \in \mathcal{M}$ is a solution to

$$\mathcal{H}\begin{pmatrix} x \\ z \end{pmatrix} = \begin{pmatrix} x \\ z \end{pmatrix}$$

if and only if $x^* = \left(y_i^d e_1\right)_{1 \le i \le n}$. Therefore

$$\begin{aligned} \mathcal{F}_{\text{ad}} &= L^{\dagger}\xi(x^*) + \ker(L) \\ &= \left\{ \begin{pmatrix} y_{i+1}^d - e^{-\pi^2\delta} y_i^d - \left(y_i^d\right)^2 \\ \alpha_i \end{pmatrix}_{0 \le i \le n-1} \mid \alpha_i \in \mathbb{R} \right\}, \quad y_0^d = 0. \end{aligned}$$

♦

2.2.3. Special Case

In order to characterize the set $\mathcal{F}_{\text{ad}}$, in the two previous methods we have transformed the detection problem into that of finding fixed points of some mapping (H or $\mathcal{H}$). However, under appropriate conditions on C and D, the set $\mathcal{F}_{\text{ad}}$ can be characterized by a simpler method.

Proposition 5. *If the operators C and D are linear and bounded and the operator CD is one-to-one, then there exists a unique element $f^* \in F^n$ such that $y^* = y^d$. It is given by*

$$f_0^* = (D^*C^*CD)^{-1}\left(y_1^d - CAx_0 - CN(x_0)\right),$$

$$f_i^* = (D^*C^*CD)^{-1}D^*C^*\left[y_{i+1}^d - C\left(A^{i+1}x_0 - \sum_{j=0}^{i} A^{i-j}N\left(x_j^*\right) - \sum_{j=0}^{i-1} A^{i-j}Df_j^*\right)\right], \quad 1 \le i \le n-1.$$

Proof. We have

$$y_i^d = CA^ix_0 + C\sum_{j=0}^{i-1} A^{i-j-1}N(x_j^*) + C\sum_{j=0}^{i-1} A^{i-j-1}Df_j^*, \quad 0 \le i \le n.$$

From $y_1^d = CAx_0 + CNx_0 + CDf_0^*$, we deduce that

$$f_0^* = (D^*C^*CD)^{-1}\left(y_1^d - CAx_0 - CN(x_0)\right). \tag{12}$$

The operator D^*C^*CD is invertible because it is self-adjoint and positive definite.

Assume that $f_0^*, \dots, f_{i-1}^*$, $1 \le i \le n-2$ have been calculated. Since

$$y_{i+1}^d = CA^{i+1}x_0 + C\sum_{j=0}^{i} A^{i-j}N(x_j^*) + C\sum_{j=0}^{i-1} A^{i-j}Df_j^* + CDf_i^*,$$

it follows that

$$f_i^* = (D^*C^*CD)^{-1}D^*C^*\left[y_{i+1}^d - CA^{i+1}x_0 - C\sum_{j=0}^{i} A^{i-j}N(x_j^*) - C\sum_{j=0}^{i-1} A^{i-j}Df_j^*\right]. \tag{13}$$

This completes the proof. ∎

Proposition 5 suggests the following algorithm for determining the sequence f^*:

Step 1: Calculate f_0^* using (12).

Step 2: Set $i = 1$.

Step 3: Calculate $x_i^* = Ax_{i-1}^* + N\left(x_{i-1}^*\right) + Df_{i-1}^*$.

Step 4: Calculate f_i^* using (13).

Step 5: Replace i by $i + 1$.

Step 6: If $i = n$ then stop, else return to Step 3.

Example 3. Consider the following non-linear continuous-time system:

$$\begin{cases} \dot{x}(t) = \Delta x(t) + M\big(x(t)\big) + Ef(t), & 0 < t < 1, \\ x(0) = 0, \end{cases} \tag{14}$$

with the output equation

$$y(t) = \langle x(t), e_1 \rangle_X \in \mathbb{R}, \tag{15}$$

where $x(t) \in X = L^2(0,1;\mathbb{R})$, $f(t) \in F = \mathbb{R}$, $E : \alpha \in \mathbb{R} \longmapsto \alpha e_1 \in X$, and M is a non-linear operator defined on X by

$$M : x \in X \longmapsto \sum_{k=1}^{m} \sin\big(\langle x, e_k \rangle_X\big) e_k \in X, \quad m \geq 1.$$

The Laplacian Δ is the infinitesimal generator of the strongly continuous semigroup $(S(t))_{t\geq 0}$ defined by

$$S(t)x = \sum_{k=1}^{\infty} e^{-k^2\pi^2 t} \langle x, e_k \rangle_X e_k, \quad x \in X.$$

It is easy to check that the operator M is uniformly Lipschitzian with

$$\|M(x_1) - M(x_2)\|_X \leq \|x_1 - x_2\|_X, \quad \forall x_1, x_2 \in X.$$

Hence the system (14) has a unique mild solution in $L^2(0,T;X)$ (Balakrishnan, 1976). It is given by

$$x(t) = \int_0^t S(t-r) M\big(x(r)\big)\,\mathrm{d}r + \int_0^t S(t-r) E f(r)\,\mathrm{d}r.$$

In order to sample the system (14) and (15), let $\delta = 1/n$, $t_i = i\delta$, $x_i = x(t_i)$, $f_i = f(t_i)$, $y_i = y(t_i)$, $n = 1, 2, \ldots$. For a small δ we make the approximations

$$x(t) = x_i, \quad f(t) = f_i \quad \text{for all} \quad t \in [t_i, t_{i+1}[.$$

We have

$$\begin{aligned} x_{i+1} &= x(t_i+\delta) = S(\delta)\left(\int_0^{t_i} S(t_i-r)M\big(x(r)\big)\,\mathrm{d}r + \int_0^{t_i} S(t_i-r)Ef(r)\,\mathrm{d}r\right) \\ &\quad + \left(\int_{t_i}^{t_i+\delta} S(t_i+\delta-r)M\,\mathrm{d}r\right)(x_i) + \left(\int_{t_i}^{t_i+\delta} S(t_i+\delta-r)E\,\mathrm{d}r\right) f_i \\ &= Ax_i + N(x_i) + Df_i, \end{aligned}$$

where

$$A = S(\delta), \quad N = \int_0^{\delta} S(r)M\,\mathrm{d}r, \quad D = \int_0^{\delta} S(r)E\,\mathrm{d}r.$$

For all $\alpha \in \mathbb{R}$, we get

$$D\alpha = \int_0^{\delta} S(r)E\alpha\,\mathrm{d}r = \alpha\left(\int_0^{\delta} e^{-\pi^2 r}\,\mathrm{d}r\right) e_1 = \frac{\alpha}{\pi^2}\left(1-e^{-\pi^2\delta}\right) e_1.$$

Hence

$$CD : \alpha \in \mathbb{R} \longmapsto \frac{\alpha}{\pi^2}\left(1-e^{-\pi^2\delta}\right) \in \mathbb{R}.$$

Thus the operators C and D are linear and bounded, and CD is injective (in fact, it is invertible). The assumptions of Proposition 5 are satisfied and, for all $n \in \mathbb{N}\setminus\{0\}$, we can use the foregoing algorithm to reconstruct the corresponding sequence $(f_i^*)_{0\le i\le n-1}$. In the numerical simulation presented below, we have tested this algorithm for the input (to be reconstructed) $f(t) = e^{-t}, \quad 0 \le t \le 1$. The second column of Table 1 contains the exact values of f for some points in $[0,1]$, and in the other columns the corresponding approximations are given. Figure 1 constitutes an alternative representation of the numerical results. ♦

Table 1. Some approximation of f.

		$n=2$	$n=4$	$n=6$	$n=8$	$n=10$
$f(0.0)$	1	1	1	1	1	1
$f(0.1)$	0.904837	1	1	1	1	0.904837
$f(0.2)$	0.818731	1	1	0.846482	0.882497	0.818731
$f(0.3)$	0.740818	1	0.778801	0.846482	0.778801	0.740818
$f(0.4)$	0.670320	1	0.778801	0.716531	0.687289	0.670320
$f(0.5)$	0.606531	0.606531	0.606531	0.606531	0.606531	0.606531
$f(0.6)$	0.548812	0.606531	0.606531	0.606531	0.606531	0.548812
$f(0.7)$	0.496585	0.606531	0.606531	0.513417	0.535261	0.496585
$f(0.8)$	0.449329	0.606531	0.472367	0.513417	0.472367	0.449329
$f(0.9)$	0.406570	0.606531	0.472367	0.434598	0.416862	0.406570

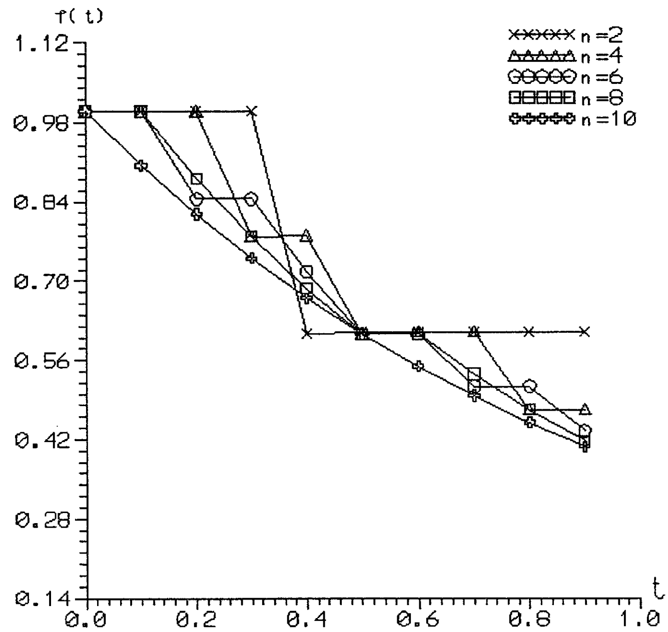

Fig. 1. Approximation of f.

3. Detection Problem for Continuous-Time Systems

In this section, we examine the detection problem in the case of systems whose state evolution is described by a non-linear, continuous-time, distributed-parameter system with delays in the input. More precisely, we consider the systems which can be written down after a transformation as

$$\begin{cases} \dot{x}(t) = Ax(t) + N\big(x(t)\big) + \sum_{i=0}^{n} D_i f(t-h_i) \\ \qquad + \int_{-h}^{0} D(r) f(t+r)\,\mathrm{d}r, \quad 0 < t < T, \\ x(0) = x_0 \in X, \quad f(r) = \phi(r), \quad r \in [-h, 0[, \\ \quad 0 = h_0 < h_1 < \dots < h_n = h \le T < \infty. \end{cases} \tag{16}$$

The non-linear operator $N : X \longrightarrow X$ is defined on a Hilbert space X. The linear operator A is the infinitesimal generator of a strongly continuous semigroup $(S(t))_{t\geq 0}$ on X. Moreover, ϕ is a given function in $L^2(-h, 0; F)$, $D_i \in \mathcal{L}(F, X)$ and $D(\cdot) \in L^2(-h, 0; \mathcal{L}(F, X))$ where F is a Hilbert space. To the system (16) we associate the output equation

$$y(t) = C\big(x(t)\big), \quad 0 \le t \le T, \tag{17}$$

where $C : X \longrightarrow Y$ is an operator (possibly non-linear) that takes its values in a Hilbert space Y. We assume that $y(\cdot) \in \mathcal{Y}$, where $\mathcal{Y}$ is a space of functions from the interval $[0, T]$ to the output space Y. In what follows, we write $x := x(\cdot) \in L^2(0, T; X)$, $f := f(\cdot) \in L^2(0, T; F)$ and $y := y(\cdot) \in \mathcal{Y}$.

We recall that our objective is to characterize the set

$$\mathcal{F}_{\text{ad}} = \left\{ f \in L^2(0, T; F) \mid y = y^d \right\},$$

where $y^d \in \mathcal{Y}$ is a given observation and y is the output corresponding to f. Sufficient conditions for the existence and uniqueness of a solution for system (16) are given by the following result:

Lemma 1. *Assume that the operator N is uniformly Lipschitzian, i.e. there exists $0 < k < \infty$ such that*

$$\|N(x_1) - N(x_2)\|_X \leq k \|x_1 - x_2\|_X, \quad \forall x_1, x_2 \in X.$$

Then the mild solution to (16) exists in $L^2(0, T; X)$ and is unique. It is given by

$$\begin{aligned} x(t) &= S(t)x_0 + \int_0^t S(t-r) N\big(x(r)\big) \, \mathrm{d}r + \sum_{i=0}^{n} \int_0^t S(t-r) D_i f(r - h_i) \, \mathrm{d}r \\ &\quad + \int_0^t S(t-r) \left(\int_{-h}^{0} D(s) f(r+s) \, \mathrm{d}s \right) \mathrm{d}r. \end{aligned} \tag{18}$$

Proof. See (Balakrishnan, 1976). ■

In the sequel, we assume that the operator N is uniformly Lipschitzian. Moreover, we choose $S(\theta) = 0$ for $\theta < 0$. We can then easily show that (18) can be written as

$$x = a + Gx + Lf,$$

where

$$\begin{aligned} a(t) &= S(t)x_0 + \sum_{i=0}^{n} \int_{-h_i}^{0} S(t - s - h_i) D_i \phi(s) \, \mathrm{d}s \\ &\quad + \int_0^h S(t-s) \left(\int_{s-h}^{0} D(\theta - s) \phi(\theta) \, \mathrm{d}\theta \right) \mathrm{d}s, \end{aligned}$$

$$\begin{cases} G : x \in L^2(0, T; X) \longmapsto Gx \in L^2(0, T; X), \\ (Gx)(t) = \displaystyle\int_0^t S(t-s) N\big(x(s)\big) \, \mathrm{d}s, \end{cases}$$

and

$$\begin{cases} L : f \in L^2(0,T;F) \longmapsto Lf \in L^2(0,T;X), \\ (Lf)(t) = \displaystyle\sum_{i=0}^{n} \int_0^t S(t-s-h_i) D_i f(s)\,\mathrm{d}s + \int_0^h S(t-s)\Big(\int_0^s D(\theta - s) f(\theta)\,\mathrm{d}\theta\Big)\,\mathrm{d}s \\ \qquad + \begin{cases} 0, & 0 \le t \le h, \\ \displaystyle\int_h^t S(t-s)\Big(\int_{s-h}^s D(\theta - s) f(\theta)\,\mathrm{d}\theta\Big)\,\mathrm{d}s, & h \le t \le T. \end{cases} \end{cases}$$

As for the discrete case, we now present two methods to characterize the set $\mathcal{F}_{\mathrm{ad}}$. Both the methods are based on the fixed-point technique and their proofs are similar to the ones presented in the case of discrete systems.

3.1. Method I

Let $q : \mathcal{Y} \to \mathbb{C}$ be any function which satisfies $q(y) = 0$ if and only if $y = y^d$, $\bar{x}$ be any fixed element of $\mathrm{range}(L) \setminus \{0\}$, $P : L^2(0,T;X) \longrightarrow \mathrm{range}(L)$ be any projection onto $\mathrm{range}(L)$ and $\xi : L^2(0,T;X) \longrightarrow L^2(0,T;X)$ be the mapping

$$\xi(x) = x - a - Gx.$$

Introduce the mapping $H : L^2(0,T;X) \longrightarrow L^2(0,T;X)$ with

$$Hx = a + Gx + P\xi(x) + q(y)\bar{x}.$$

Proposition 6. *Let $\mathcal{P}_H$ denote the set of fixed-points of H. Then $\mathcal{F}_{\mathrm{ad}}$ is given by*

$$\mathcal{F}_{\mathrm{ad}} = \big\{L^\dagger \xi(x^*) \mid x^* \in \mathcal{P}_H\big\} + \ker(L).$$

Proof. It is similar to that of Proposition 2. ■

Example 4. Consider the system

$$\begin{cases} \dot{x}(t) = \Delta x(t) + N\big(x(t)\big) + Df(t-h), & 0 < t < T, \\ x(0) = 0; \;\; f(r) = 0, \quad r \in [-h, 0[, \\ 0 < h < T < \infty, \end{cases} \tag{19}$$

with the output equation

$$y(t) = \langle x(t), e_1 \rangle_X \in \mathbb{R}, \tag{20}$$

where $x(t) \in X = L^2(0,1;\mathbb{R})$, $f(t) \in F = \mathbb{R}$, $D : \alpha \in \mathbb{R} \longmapsto \alpha e_1 \in X$. As is well-known, the Laplacian Δ is the infinitesimal generator of the strongly continuous semigroup $(S(t))_{t \ge 0}$ defined by

$$S(t)x_0 = \sum_{k=1}^{\infty} e^{-k^2\pi^2 t} \langle x_0, e_k \rangle_X e_k, \quad \forall x_0 \in X,$$

with $e_k(\cdot) = \sqrt{2}\sin(k\pi\cdot)$. Moreover, N is a non-linear operator defined on X by

$$N(x_0) = \sum_{k=1}^{m} \sin\left(\langle x_0, e_k\rangle_X\right) e_k \in X, \quad m \geq 1.$$

The operator N is uniformly Lipschitzian (cf. Example 3) and hence, by Lemma 1, eqn. (19) has a unique solution in $L^2(0,T;X)$. Consequently, $y \in L^2(0,T;\mathbb{R})$. We assume that $y^d \in L^2(0,T;\mathbb{R})$ and that it satisfies $y^d(t) = 0$ for $0 \leq t \leq h$ and $y^d \in H^1(h,T;\mathbb{R})$. Clearly, the operators G and L and the sets $\ker(L)$ and $\ker(L)^\perp$ are given by

$$\begin{cases} G : x \in L^2(0,T;X) \longmapsto Gx \in L^2(0,T;X), \\ (Gx)(t) = \displaystyle\sum_{k=1}^{m} \left(\int_0^t e^{-k^2\pi^2(t-s)} \sin\left(\langle x(s), e_k\rangle_X\right) \mathrm{d}s \right) e_k, \end{cases}$$

$$\begin{cases} L : f \in L^2\left(0,T;\mathbb{R}\right) \longmapsto Lf \in L^2\left(0,T;X\right), \\ (Lf)(t) = \begin{cases} 0, & 0 \leq t \leq h, \\ \left(\displaystyle\int_0^{t-h} e^{-\pi^2(t-r-h)} f(r)\,\mathrm{d}r \right) e_1, & h \leq t \leq T, \end{cases} \end{cases}$$

$$\ker(L) = \left\{ f \in L^2(0,T;\mathbb{R}) \mid f(t) = 0 \ \text{ a.e. } t \in [0, T-h] \right\},$$

$$\ker(L)^\perp = \left\{ f \in L^2(0,T;\mathbb{R}) \mid f(t) = 0 \ \text{ a.e. } t \in [T-h, T] \right\}.$$

Let

$$\beta := \left\{ x \in L^2(0,T;X) \;\middle|\; \begin{array}{ll} x(t) = 0, & 0 \leq t \leq h, \\ x(t) = g(t)e_1, & h \leq t \leq T, \end{array} \quad \text{with } g \in H^1\left(h,T;\mathbb{R}\right) \right\}.$$

It is readily verified that $\mathrm{range}(L) \subset \beta$. Let $x \in \beta$ and define

$$f(t) = \begin{cases} \dot{g}(t+h) + \pi^2 g(t+h), & 0 \leq t < T-h, \\ 0, & T-h \leq t \leq T. \end{cases}$$

It follows that $f \in L^2(0,T;\mathbb{R})$ and $Lf = x$. Thus $\mathrm{range}(L) = \beta$.

Note that $f \in \ker(L)^\perp$. The restriction of $L^\dagger$ to $\mathrm{range}(L)$ is hence given by

$$\begin{cases} L^\dagger : x \in \mathrm{range}(L) \longmapsto L^\dagger x \in \ker(L)^\perp, \\ \left(L^\dagger x\right)(t) = \begin{cases} \langle \dot{x}(t+h), e_1\rangle + \pi^2 \langle x(t+h, e_1\rangle, & 0 \leq t < T-h, \\ 0, & T-h \leq t \leq T. \end{cases} \end{cases}$$

Introduce the projection

$$P : x \in L^2(0,T;X) \longmapsto Px \in \mathrm{range}(L),$$

where we have

- if $\langle x, e_1\rangle_X \in H^1(h,T;\mathbb{R})$, then

$$(Px)(t) = \begin{cases} 0, & 0 \le t \le h, \\ \langle x(t), e_1\rangle_X e_1, & h < t \le T, \end{cases}$$

- otherwise, $(Px)(t) = 0$ for $0 \le t \le T$.

Let $\bar{x}$ be any fixed element in $\text{range}(L) \setminus \{0\}$ and consider the mapping

$$Hx = Gx + P\xi(x) + \|y - y^d\|_{L^2(0,T;Y)}\bar{x}.$$

We will prove that H has a unique fixed-point given by

$$x^*(t) = \begin{cases} 0, & 0 \le t \le h, \\ y^d(t)e_1, & h \le t \le T. \end{cases} \tag{21}$$

We have

$$\begin{aligned} (\xi(x^*))(t) &= x^*(t) - (Gx^*)(t) \\ &= x^*(t) - \sum_{k=1}^{m} \left(\int_0^t e^{-k^2\pi^2(t-s)} \sin\left(\langle x^*(s), e_k\rangle_X\right) \mathrm{d}s \right) e_k. \end{aligned}$$

Clearly, $(\xi(x^*))(t) = 0$ for $t \in [0,h]$.

Let $t \in [h,T]$. We get

$$\begin{aligned} (\xi(x^*))(t) &= y^d(t)e_1 - \sum_{k=1}^{m} \left(\int_h^t e^{-k^2\pi^2(t-s)} \sin\left(\langle y^d(s)e_1, e_k\rangle_X\right) \mathrm{d}s \right) e_k, \\ &= \left(y^d(t) - \int_h^t e^{-\pi^2(t-s)} \sin\left(y^d(s)\right) \mathrm{d}s \right) e_1. \end{aligned}$$

Hence, by the assumptions on y^d, $\xi(x^*) \in \text{range}(L)$. We have

$$y^*(t) = Cx^*(t) = \begin{cases} 0, & 0 \le t \le h, \\ y^d(t), & h \le t \le T. \end{cases}$$

Therefore

$$Hx^* = Gx^* + \xi(x^*) = Gx^* + x^* - Gx^* = x^*.$$

Furthermore, let z^* be a fixed-point of H. Then $y^* := Cz^* = y^d$ and $\xi(z^*) \in \text{range}(L)$. Hence

$$y^d(t) = y^*(t) = \langle z^*(t), e_1\rangle_X, \quad 0 \le t \le T.$$

On the other hand, since $\xi(z^*) \in \text{range}(L)$, we have $\xi(z^*)(t) = 0$ for $0 \le t \le h$. Thus

$$z^*(t) = (Gz^*)(t) = \sum_{k=1}^{m} \left(\int_0^t e^{-k^2\pi^2(t-s)} \sin\left(\langle z^*(s), e_k\rangle_X\right) \mathrm{d}s \right) e_k.$$

Let $1 \le k \le m$ and $g_k(t) := |\langle z^*(t), e_k\rangle_X|$. We have

$$g_k(t) = \left| \int_0^t e^{-k^2\pi^2(t-s)} \sin\left(\langle z^*(s), e_k\rangle_X\right) \mathrm{d}s \right| \le \int_0^t |\langle z^*(t), e_k\rangle_X| \, \mathrm{d}s \le \int_0^t g_k(s)\, \mathrm{d}s.$$

By the Gronwall Lemma we have

$$0 \le g_k(t) \le g_k(0)e^t = 0.$$

It follows that $x(t) = 0$ for $0 \le t \le h$.

Let $h < t \le T$. We have

$$\begin{aligned} z^*(t) &= (Gz^*)(t) + (P\xi(z^*))(t) \\ &= \sum_{k=1}^{m} \left(\int_0^t e^{-k^2\pi^2(t-s)} \sin\left(\langle z^*(s), e_k\rangle_X\right) \mathrm{d}s \right) e_k + \langle z^*(t), e_1\rangle_X e_k \\ &\quad - \left(\int_0^t e^{-\pi^2(t-s)} \sin\left(\langle z^*(s), e_1\rangle_X\right) \mathrm{d}s \right) e_1 \\ &= y^d(t)e_1 + \sum_{k=2}^{m} \left(\int_0^t e^{-k^2\pi^2(t-s)} \sin\left(\langle z^*(s), e_k\rangle_X\right) \mathrm{d}s \right) e_k. \end{aligned}$$

For $2 \le k \le m$, we have

$$f_k(t) := |\langle z^*(t), e_k\rangle_X| = \left| \int_0^t e^{-k^2\pi^2(t-s)} \sin\left(\langle z^*(s), e_k\rangle_X\right) \mathrm{d}s \right| \le \int_0^t f_k(s)\, \mathrm{d}s,$$

and, again by applying the Gronwall Lemma, we deduce that $\langle z^*(t), e_k\rangle_X = 0$ for $h < t \le T$ and $2 \le k \le m$. It follows that

$$z^*(t) = \begin{cases} 0, & 0 \le t \le h, \\ y^d(t)e_1, & h \le t \le T. \end{cases}$$

We conclude that H has a unique fixed-point given by (21).

We have

$$(\xi(x^*))(t) = \begin{cases} 0, & 0 \le t \le h, \\ \left(y^d(t) - \int_h^t e^{-\pi^2(t-s)} \sin\left(y^d(s)\right) \mathrm{d}s \right) e_1, & h \le t \le T. \end{cases}$$

Hence

$$\left(L^{\dagger}\xi(x^*)\right)(t) = \begin{cases} \langle\dot{\xi}(x^*)(t+h), e_1\rangle_X + \pi^2\langle\xi(x^*)(t+h), e_1\rangle_X, & 0 \le t < T-h \\ 0, & T-h \le t \le T, \end{cases}$$

$$= \begin{cases} \dot{y}^d(t+h) - \sin\left(y^d(t+h)\right) + \pi^2 y^d(t+h), & 0 \le t < T-h, \\ 0, & T-h \le t \le T. \end{cases}$$

Consequently,

$$\mathcal{F}_{\text{ad}} = L^{\dagger}\xi(x^*) + \ker(L),$$

$$= \left\{ \begin{array}{l} f \in L^2(0,T;\mathbb{R}), \\ f(t) = \begin{cases} \dot{y}^d(t+h) - \sin\left(y^d(t+h)\right) + \pi^2 y^d(t+h), & 0 \le t < T-h, \\ \alpha(t), & T-h \le t \le T, \end{cases} \end{array} \right.$$

where $\alpha \in L^2(T-h,T;\mathbb{R})$. ♦

3.2. Method II

In this section, we assume that the observation is given by

$$y(t) = Cx(t), \quad 0 \le t \le T,$$

where $C : X \longrightarrow Y$ is a bounded linear operator, so that $y \in L^2(0,T;Y)$.

Consider again the operators G and L as defined in the previous section. Let $\mathcal{M}$ be the Hilbert space $\mathcal{M} = L^2(0,T;X) \times L^2(0,T;Y)$,

$$\mathcal{A} = \begin{pmatrix} a \\ Ca \end{pmatrix}.$$

and introduce the operators

$$\mathcal{G} : x \in L^2(0,T;X) \longmapsto \begin{pmatrix} Gx \\ CGx \end{pmatrix} \in \mathcal{M},$$

$$\mathcal{L} : f \in L^2(0,T;F) \longmapsto \begin{pmatrix} Lf \\ CLf \end{pmatrix} \in \mathcal{M}.$$

$$\widetilde{\xi} : \begin{pmatrix} x \\ z \end{pmatrix} \in \mathcal{M} \longmapsto \begin{pmatrix} x \\ z \end{pmatrix} - \mathcal{A} - \mathcal{G}x \in \mathcal{M}.$$

Let $\mathcal{P} : \mathcal{M} \longrightarrow \text{range}(\mathcal{L})$ be any projection onto $\text{range}(\mathcal{L})$ and consider the mapping

$$\mathcal{H} : \begin{pmatrix} x \\ z \end{pmatrix} \in \mathcal{M} \longmapsto \mathcal{A} + \mathcal{G}x + \mathcal{P}\widetilde{\xi}\begin{pmatrix} x \\ y^d \end{pmatrix} + \begin{pmatrix} 0 \\ z - y^d \end{pmatrix} \in \mathcal{M}.$$

Let $\mathcal{P}_{\mathcal{H}}$ denote the set of fixed-points of $\mathcal{H}$. Set

$$\mathcal{P}'_{\mathcal{H}} := \left\{ x^* \in L^2(0,T;X) \mid \exists z^* \in L^2(0,T;Y) \text{ such that } \begin{pmatrix} x^* \\ z^* \end{pmatrix} \in \mathcal{P}_{\mathcal{H}} \right\}.$$

Proposition 7. *The set of admissible inputs is given by*

$$\mathcal{F}_{\text{ad}} = \left\{ L^{\dagger}\xi(x^*) \mid x^* \in \mathcal{P}'_{\mathcal{H}} \right\} + \ker(L).$$

Proof. It is similar to that of Proposition 4. ■

4. Control Problem

Consider the non-linear discrete-time system

$$\begin{cases} x_{i+1} = Ax_i + N(x_i) + Bu_i, & 0 \leq i \leq n-1, \\ x_0 \in X. \end{cases} \tag{22}$$

As usual, $N : X \longrightarrow X$ denotes a non-linear operator defined on the Hilbert space X, $A : X \longrightarrow X$ and $B : U \longrightarrow X$ are linear operators with U being a Hilbert space (the control space). We write $x = (x_i)_{1 \leq i \leq n}$ and $u = (u_i)_{0 \leq i \leq n-1}$.

Let $x^d \in X$ be a given state. We recall that our objective is to characterize the set of controls $u^* \in U^n$ which drive the state of system (22) from x_0 at $i = 0$ to x^d at $i = n$, i.e.

$$\mathcal{F}_{\text{ad}} = \left\{ u^* \in U^n \mid x_n^* = x^d \right\},$$

where x_n^* is the state of system (22) at $i = n$ corresponding to the initial state x_0 and the input u^*. It turns out that this problem can be solved by the same methods that we have used to treat the detection problem. So introduce the operators

$$G : x \in X^n \longmapsto Gx \in X^n \text{ with } (Gx)_i = \sum_{j=0}^{i-1} A^{i-j-1} N(x_j), \quad 1 \leq i \leq n,$$

and

$$L : u \in U^n \longmapsto Lu \in X^n, \text{ with } (Lu)_i = \sum_{j=0}^{i-1} A^{i-j-1} Bu_j, \quad 1 \leq i \leq n.$$

The solution to system (22) can be written as

$$x = A^{\cdot} x_0 + Gx + Lu.$$

4.1. Method I

Let $P : X^n \longrightarrow \text{range}(L)$ be any projection on $\text{range}(L)$, $\bar{x}$ be any fixed element of $\text{range}(L) \setminus \{0\}$ and $\xi : X^n \longrightarrow X^n$ be the mapping

$$\xi(x) = x - A^{\cdot}x_0 - Gx.$$

Consider the operator $H : X^n \longrightarrow X^n$ defined by

$$Hx = A^{\cdot}x_0 + Gx + P\xi(x) + \|x_n - x^d\|_X \bar{x}.$$

Proposition 8. *The set $\mathcal{U}_{\text{ad}}$ is given by*

$$\mathcal{U}_{\text{ad}} = \left\{L^{\dagger}\xi\left(x^*\right) \mid x^* \in \mathcal{P}_H\right\} + \ker(L).$$

Proof. It is similar to that of Proposition 2. ∎

Example 5. Consider the system

$$\begin{cases} x_{i+1} = Ax_i + N(x_i) + Bu_i, & 0 \le i \le n-1, \quad n = 1, 2, \ldots, \\ x_0 = 0, \end{cases} \tag{23}$$

with $X = L^2(0,1;\mathbb{R})$, $U = \mathbb{R}$. The operators A and N are as in Example 1, and the operator B is defined by

$$B : \alpha \in \mathbb{R} \longmapsto \alpha e_1 \in X.$$

The final desired state x^d is assumed to be of the form $x^d = \langle x^d, e_1\rangle_X e_1$. This condition is necessary for x^d to be reachable.

The operator L and the set $\text{range}(L)$ are then given by

$$\begin{cases} L : u \in \mathbb{R}^n \longmapsto Lu \in X^n, \\ (Lu)_i = \displaystyle\sum_{j=0}^{i-1} e^{-\pi^2\delta(i-j-1)} u_j e_1, \quad 1 \le i \le n, \end{cases}$$

$$\text{range}(L) = \left\{(\alpha_i e_1)_{1 \le i \le n} \in X^n \mid \alpha_i \in \mathbb{R}\right\}.$$

Clearly, the operator L is one-to-one and $L^{\dagger}$ is given by

$$L^{\dagger} : (\alpha_i e_1)_{1 \le i \le n} \in \text{range}(L) \longmapsto u \in \mathbb{R}^n,$$

where

$$\begin{cases} u_0 = \alpha_1, \\ u_i = \alpha_{i+1} - e^{-\pi^2\delta}\alpha_i, \quad 1 \le i \le n-1. \end{cases}$$

Consider the same projection P and element $\bar{x}$ as in Example 1. The operator $H : X \longrightarrow X$ is then given by

$$(Hx)_i = \langle x_i, e_1 \rangle_X e_1 + \sum_{k=2}^{m} \sum_{j=0}^{i-1} e^{-k^2\pi^2\delta(i-j-1)} \langle x_j, e_k \rangle_X^2 e_k$$
$$+ \|x_n - x^d\|_X \bar{x}_i, \quad 1 \le i \le n.$$

Proceeding as in Example 1, we can easily verify that the fixed-points of H are given by

$$\mathcal{P}_H = \left\{ (\alpha_i e_1)_{1 \le i \le n-1} \in X^n \mid \alpha_i \in \mathbb{R} \text{ and } \alpha_n = \langle x^d, e_1 \rangle_X \right\}.$$

Therefore

$$\begin{aligned} \mathcal{U}_{\text{ad}} &= \{ L^\dagger \xi(x^*) \mid x^* \in \mathcal{P}_H \} \\ &= \left\{ \left(\alpha_{i+1} - e^{-\pi^2\delta}\alpha_i - \alpha_i^2 \right)_{0 \le i \le n-1} \mid \alpha_i \in \mathbb{R}, \text{ and } \alpha_n = \langle x^d, e_1 \rangle_X \right\}. \end{aligned}$$

♦

4.2. Method II

Set $\mathcal{M} = X^n \times X$ and introduce the operators

$$\mathcal{A} : x_0 \in X \longmapsto \begin{pmatrix} Ax_0 \\ A^n x_0 \end{pmatrix} \in \mathcal{M}, \quad \mathcal{G} : x \in X^n \longmapsto \begin{pmatrix} Gx \\ (Gx)_n \end{pmatrix} \in \mathcal{M},$$

$$\mathcal{L} : u \in U^n \longmapsto \begin{pmatrix} Lu \\ (Lu)_n \end{pmatrix} \in \mathcal{M},$$

$$\widetilde{\xi} : \begin{pmatrix} x \\ z \end{pmatrix} \in \mathcal{M} \longmapsto \begin{pmatrix} x \\ z \end{pmatrix} - \mathcal{A}x_0 - \mathcal{G}x \in \mathcal{M}.$$

Let $\mathcal{P} : \mathcal{M} \longmapsto \text{range}(\mathcal{L})$ be any projection onto $\text{range}(\mathcal{L})$. Consider the mapping

$$\mathcal{H} : \begin{pmatrix} x \\ z \end{pmatrix} \in \mathcal{M} \longmapsto \mathcal{A}x_0 + \mathcal{G}x + \mathcal{P}\widetilde{\xi} \begin{pmatrix} x \\ x^d \end{pmatrix} + \begin{pmatrix} 0 \\ z - x^d \end{pmatrix} \in \mathcal{M}.$$

Proposition 9. *The set* $\mathcal{U}_{\text{ad}}$ *is given by*

$$\mathcal{U}_{\text{ad}} = \{ L^\dagger \xi(x^*) \mid x^* \in \mathcal{P}'_{\mathcal{H}} \} + \ker(L).$$

Proof. It is similar to that of Proposition 4. ■

Example 6. Consider again the system (23) with the desired final state as in Example 5. The operators $\mathcal{G}$ and $\widetilde{\xi}$ are then given by

$$\mathcal{G} : x \in X^n \longmapsto \begin{pmatrix} \left(\sum_{k=1}^{m} \sum_{j=0}^{i-1} e^{-k^2\pi^2\delta(i-j-1)} \langle x_j, e_k \rangle_X^2 e_k \right)_{1 \le i \le n} \\ \sum_{k=1}^{m} \sum_{j=0}^{n-1} e^{-k^2\pi^2\delta(i-j-1)} \langle x_j, e_k \rangle_X^2 e_k \end{pmatrix} \in \mathcal{M}$$

and

$$\widetilde{\xi} : \begin{pmatrix} x \\ z \end{pmatrix} \in \mathcal{M} \longmapsto \begin{pmatrix} \left(x_i - \sum_{k=1}^{m} \sum_{j=0}^{i-1} e^{-k^2\pi^2\delta(i-j-1)} \langle x_j, e_k \rangle_X^2 e_k \right)_{1 \le i \le n} \\ z_i - \sum_{k=1}^{m} \sum_{j=0}^{n-1} e^{-k^2\pi^2\delta(i-j-1)} \langle x_j, e_k \rangle_X^2 e_k \end{pmatrix} \in \mathcal{M}.$$

Introduce the projection

$$\mathcal{P} : \begin{pmatrix} x \\ z \end{pmatrix} \in \mathcal{M} \longmapsto \begin{pmatrix} (\langle x_i, e_1 \rangle_X e_1)_{1 \le i \le n} \\ \langle x^n, e_1 \rangle_X e_1 \end{pmatrix} \in \operatorname{range}(\mathcal{L}).$$

The mapping $\mathcal{H}$ is then given by

$$\begin{aligned} \mathcal{H}\begin{pmatrix} x \\ z \end{pmatrix} &= \mathcal{G}x + P\widetilde{\xi}\begin{pmatrix} x \\ x^d \end{pmatrix} + \begin{pmatrix} 0 \\ z - x^d \end{pmatrix} \\ &= \begin{pmatrix} \left(\langle x_i, e_1 \rangle_X e_1 + \sum_{k=2}^{m} \sum_{j=0}^{i-1} e^{-k^2\pi^2\delta(i-j-1)} \langle x_j, e_k \rangle_X^2 e_k \right)_{1 \le i \le n} \\ z + \langle x_n^*, e_1 \rangle_X e_1 - x^d + \sum_{k=2}^{m} \sum_{j=0}^{n-1} e^{-k^2\pi^2\delta(i-j-1)} \langle x_j^*, e_k \rangle_X^2 e_k \end{pmatrix}. \end{aligned}$$

One can easily check that $\binom{x^*}{z^*} \in \mathcal{M}$ is a solution to

$$\mathcal{H}\begin{pmatrix} x \\ z \end{pmatrix} = \begin{pmatrix} x \\ z \end{pmatrix}$$

if and only if $x^* = (\alpha_i e_1)_{1 \le i \le n}$, $\alpha_i \in \mathbb{R}$ and $\alpha_n = \langle x^d, e_1 \rangle_X$. Therefore

$$\begin{aligned} \mathcal{U}_{\text{ad}} &= \left\{ L^\dagger \xi(x^*) \mid x^* \in \mathcal{P}_{\mathcal{H}} \right\} \\ &= \left\{ \left(\alpha_{i+1} - e^{-\pi^2\delta}\alpha_i - \alpha_i^2 \right)_{0 \le i \le n-1} \mid \alpha_i \in \mathbb{R} \text{ and } \alpha_n = \langle x^d, e_1 \rangle_X \right\}. \end{aligned}$$

♦

5. Conclusion

In this work, we have investigated the problems of detection and control for non-linear distributed-parameter systems with delays. In our opinion, it will be also worthwile to study the regulation, compensator, observer, stability, stabilizability and observability problems for non-linear, discrete and distributed systems with delays. It seems to us that the techniques used in this paper can be adopted to solve these problems (fixed-point theorems, state-space techniques, spectral decompositions, pseudo-inverses, etc.). These possibilities are now under investigation.

References

Afifi L. and El Jai A. (1994): *Strategic sensors and spy sensors.* — Appl. Math. Comp. Sci., Vol.4, No.4, pp.553–573.

Afifi L. and El Jai A. (1995): *Spy sensors and detection.* — Int. J. Syst. Sci., Vol.26, No.8, pp.1447–1463.

Balakrishnan A.V. (1976): *Applied Functional Analysis.* — New York: Springer, pp.264–265.

Carmichael N. and Quinn M.D. (1982): *Notes on optimal control and estimation results of nonlinear systems.* — Proc. 3rd IFAC Symp. *Control of Distributed Parameter Systems*, Toulouse, France.

Felippe De Souza J.A.M. (1984): *Nonlinear control and estimation using fixed point theorems.* In: Methods and Applications of Measurements and Control (S. Tzafestas and Hamza, Ed.). — Anaheim: Acta Press, Vol.1, pp.138–141.

Ichikawa A. and Pritchard A.J. (1979): *Existence, uniqueness and stability of nonlinear evolution equations.* — Control Theory Centre Report No.65, University of Warwick.

Kassara K. and El Jai A. (1983): *Algorithmes pour la commande d'une classe de systèmes à paramètres repartis non linéaires.* — Revue Marocaine d'Automatique d'Informatique et de Traitment du Signal, Vol.1, No.1, pp.3–24.

Magnusson K., Pritchard A.J. and Quinn M.D. (1981): *The application of fixed point theorems to global nonlinear controllability problems.* — Control Theory Centre Report No.87, University of Warwick.

Namir A., Lahmidi F. and Alif O. (1999): *Detection and regulation problems for discrete-time distributed systems with delays.* — Int. J. Appl. Math. Comp. Sci., Vol.9, No.2, pp.101–123.

Pritchard A.J. (1982) *Nonlinear infinite dimensional systems theory.* — Proc. 3rd IFAC Symp. *Control of Distributed Parameter Systems*, Toulouse, France.

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