Int. J. Appl. Math. Comput. Sci., 2001, Vol.11, No.1, 237–269

ON NETWORK MODELS AND THE SYMBOLIC SOLUTION OF NETWORK EQUATIONS

KURT J. REINSCHKE*

This paper gives an overview of the formulation and solution of network equations, with emphasis on the historical development of this area. Networks are mathematical models. The three ingredients of network descriptions are discussed. It is shown how the network equations of one-dimensional multi-port networks can be formulated and solved symbolically. If necessary, the network graph is modified so as to obtain an admittance representation for all kinds of multi-ports. N-dimensional networks are defined as graphs with the algebraic structure of N-dimensional vectors. In civil engineering, framed structures in two and three spatial dimensions can be modeled as 3-dimensional or 6-dimensional networks. The separation of geometry from topology is a characteristic feature of such networks.

Keywords: history of network theory, network graphs, network equations, modified nodal analysis, admittance representation of multi-ports, multidimensional networks

1. Introduction

Network modeling is applicable to any real-world system that fulfills the following conditions. The signals occurring in the real-world system involve two types of variables:

(a) *flow variables* (FVs for short, also called through variables) obeying a *cut law*, i.e., the flow quantities going through any closed cutting surface sum up to zero,

(b) *difference variables* (DVs for short, also called across variables) obeying a *circuit law*, i.e., the difference quantities across adjacent points along any closed path add up to zero.

Networks are interconnections of a finite number of network elements (modeled as spatially lumped) which require interrelations between FVs and DVs defining the set of *network element relations* (NERs).

For the special case of networks consisting of electrical wires, in 1845 Kirchhoff, a 21-year-old student at that time, published what is now called the 'node and mesh' rules for electrical circuits (Kirchhoff, 1845). In the second half of the 19th century,

* Institut für Regelungs- und Steuerungstheorie, Technische Universität Dresden, Mommsenstr. 13, D–01062 Dresden, Germany, `e-mail: kr@erss11.et.tu-dresden.de`

electrical phenomena were often explained by referring to mechanical systems. The term *electromotive force* was coined to stress the similarity between a voltage source and a mechanical force: Both being driving forces in electrical and mechanical systems. Hence, the first set of analogies (mechanical force corresponding to voltage, velocity corresponding to current) was introduced and applied during the subsequent decades. In the early 1930s, Hähnle (1932) and Firestone (1933) recognized the deficiencies of this "classical" analogy and introduced a more complete type of analogy associating forces with currents, and velocities with voltages. The approach of this paper is not to discuss analogies. Rather, we focus on applying the concepts of network analysis directly to various fields of physics and engineering.

As an introductory example let us consider an electromechanical oscillating system (taken form (Reinschke and Schwarz, 1976)) whose cross-section is shown in Fig. 1(a). A permanent magnet (with mass M_1) oscillates within an electromagnet in the vertical direction. The mechanical oscillations arise due to the effect of the springs (F_1). This system is encased (mass M_2) and suspended by springs (F_3). The casing rests on rubber feet whose influence can be modeled by the springs F_2 connected to the ground. The system oscillates by applying a current through the excitor coil (Transducer 1). The relative motion between the electromagnet and the permanent magnet induces a voltage in the measurement coil (Transducer 2). The task could be to find the velocities and acceleration of the electromagnet as well as the current through the load resistor R_L. The described system can be modeled as a network as follows: Transducer 1 converts the driving current I_{w_1} into the force F_{w_1}, whereas Transducer 2 converts a relative velocity v_{w_2} into the voltage u_{w_2}. Letting the transducer constants be $\ddot{u}_1$ and $\ddot{u}_2$, we have $F_{w_1} = \ddot{u}_1 I_{w_1}$, $V_{w_1} = U_{w_1}/\ddot{u}_1$ and $U_{w_2} = \ddot{u}_2 V_{w_2}$, $I_{w_2} = F_{w_2}/\ddot{u}_2$. We denote by L_1 and L_2 the inductances of the transducer coils. These inductances are magnetically coupled with the iron parts of the permanent magnet and electromagnet (mutual inductance L_{12}). The two coils are each glued to metallic holders which can be modeled as a one-winding coil with inductance L_3 (resp. L_4). The holders are coupled to the transducer coils through mutual inductances L_{13} and L_{24}. The substantially weaker coupling between L_3 and $L_2 \,/\, L_4$, as well as between L_4 and $L_1 \,/\, L_3$, is ignored. R_{L_2} denotes the loss resistance of the measurement coil, R_i denotes the copper losses in the excitor coil together with the internal resistance of the voltage source, and R_L is the load resistance (i.e., the input resistance of the connected device). Within the mechanical part of the system each of the springs is modeled by a stiffness constant N and a friction constant H; the masses are denoted by M_1, M_2 and M_3. A pictorial representation of the network model is shown in Fig. 1(b). The one-port network elements are consumers, FV storages and DV storages. Each transducer appears as a two-port element. The inductively coupled coils appear as a 4-port network element. The *network graph* reflects the network topology. The lines along which the DVs do not change are modeled as *nodes*. The *branch* orientations can be chosen arbitrarily and are pictorially represented by arrows. The network graph shown in Fig. 1(c) is disconnected and consists of five separate subgraphs. Disconnected network graphs can be transformed into connected network graphs by identifying pairs of nodes belonging to separate subgraphs. For the example system, one possibility is depicted in Fig. 1(d).

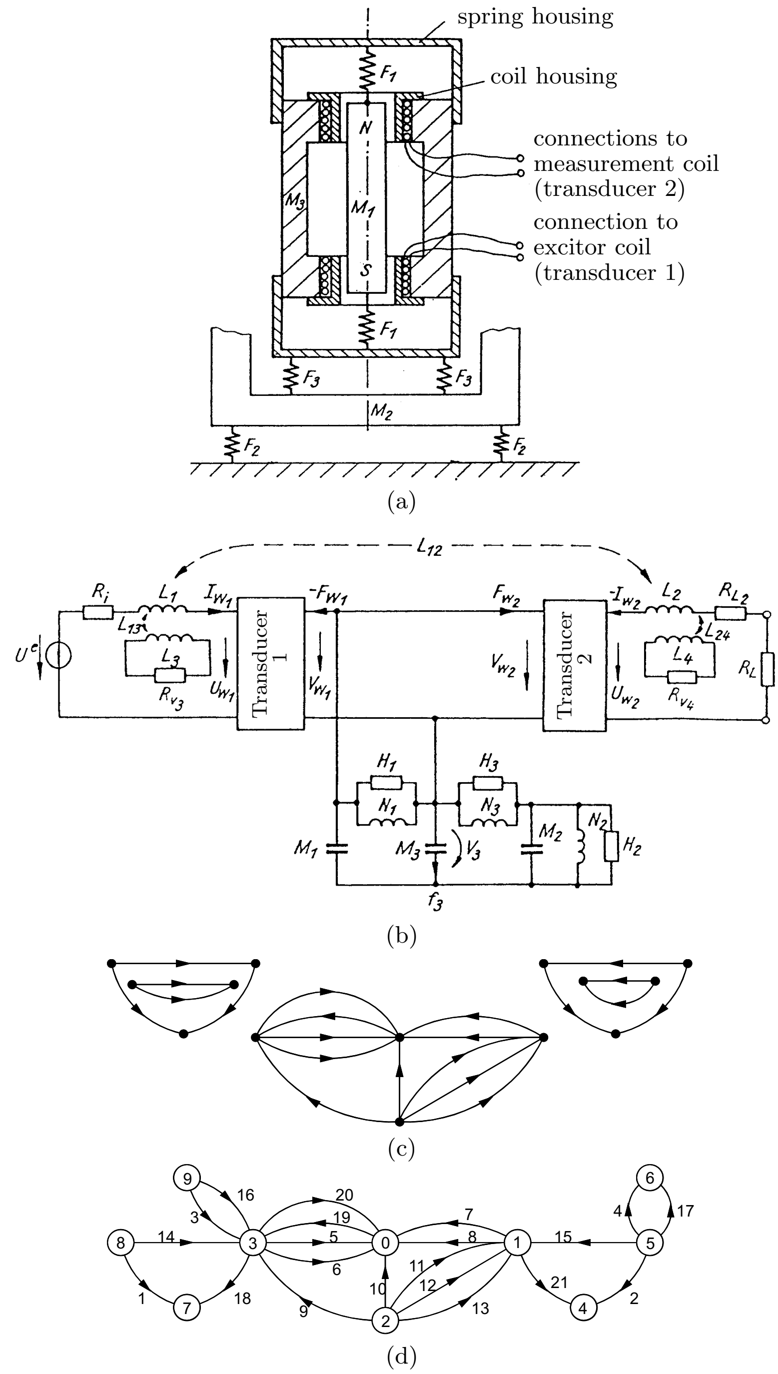


Fig. 1. Electromechanical oscillating system: (a) cross-section of physical device, (b) network model, (c) network graph consisting of five node-disjoint digraphs, (d) network graph modified to one connected digraph.

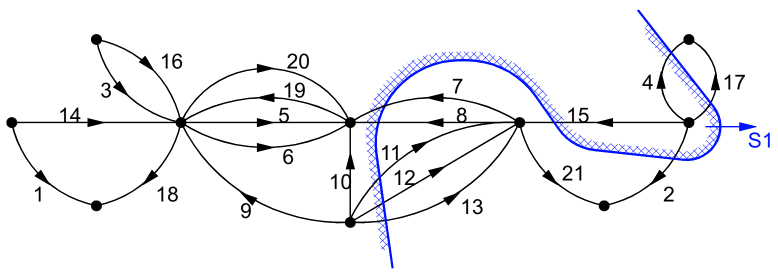
This network graph has 10 nodes denoted by $0, 1, \ldots, 9$ and 21 branches denoted by $1, 2, \ldots, 21$.

2. The Network Problem

2.1. Topological Properties of the Network Graph

We assume the network graph $\mathcal{G}$ to be a connected digraph with z oriented branches (each of them connecting two different nodes) and $k+1$ nodes (where the node 0 serves as a reference node). The connectivity properties of network graphs can be specified by means of cut surfaces or by means of circuits.

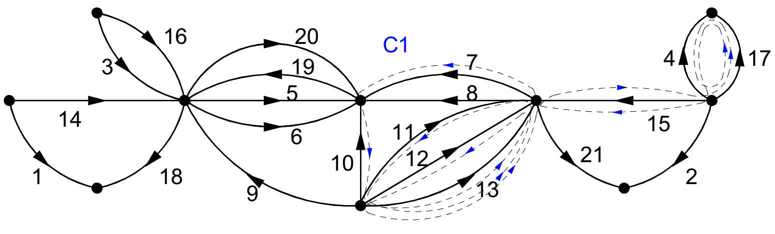
A subset of branches crossed by a cutting line forms a *cut-set* of branches if the deletion of all the crossed branches would disconnect the network graph. After associating an orientation with the cutting surface, the cut-set branch relations can be specified by means of a *cut-set branch incidence vector* whose entry ζ is defined as 1, -1, or 0 if the branch ζ belongs to the cut-set and is equally oriented, opposite oriented or does not belong to the cut-set. If we consider several cut-sets, say r altogether, all the information may gathered in an (r, z) incidence matrix $\tilde{K}$. For the introductory example, an oriented cutting line $S1$ and the corresponding row vector of the incidence matrix $\tilde{K}$ are depicted in Fig. 2.



$$
\begin{array}{c}
\zeta \cdots \\ \vdots \\ S1 \\ \vdots
\end{array}
\begin{array}{c}
\begin{array}{ccccccccccccccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 & 20 & 21 \end{array} \\
\left(\begin{array}{ccccccccccccccccccccc} \\ 0 & 1 & 0 & 1 & 0 & 0 & -1 & -1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ \\ \end{array} \right) = \tilde{K}
\end{array}
$$

Fig. 2. Oriented cut-set and cut-set branch incidence matrix.

In this paper, the term *circuit* denotes an oriented closed path on the network graph. Given any circuit, each branch is associated with an integer indicating how many times the circuit passes through the branch. One additional passing in accordance with the branch orientation increases the index by 1, one additional passing in the opposite direction decreases the index by 1. Consequently, the given circuit can be uniquely characterized by a *circuit branch indicator vector*. If we consider several circuits, say t altogether, all the information can be gathered in a $(t,\, z)$ indicator matrix $\tilde{M}$ used by Weyl (1923). For the example system, one circuit C1 and the corresponding row vector of $\tilde{M}$ are depicted in Fig. 3.



$$
\begin{array}{c}
\zeta \cdots \; 1 \; 2 \; 3 \; 4 \; 5 \; 6 \; 7 \; 8 \; 9 \; 10 \; 11 \; 12 \; 13 \; 14 \; 15 \; 16 \; 17 \; 18 \; 19 \; 20 \; 21 \\
\begin{array}{c} \vdots \\ C1 \\ \vdots \end{array}
\left(\begin{array}{ccccccccccccccccccccc}
 & & & & & & & & & & & & & & & & & & & & \\
0 & 0 & 0 & -2 & 0 & 0 & 1 & 0 & 0 & -1 & -1 & -1 & 3 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\
 & & & & & & & & & & & & & & & & & & & &
\end{array} \right) = \tilde{M}
\end{array}
$$

Fig. 3. Circuit and circuit branch indicator matrix.

The componentwise products of the row vectors of $\tilde{K}$ and of $\tilde{M}$ sum up to zero. This may be interpreted as orthogonality of the row vectors and written concisely in the form

$$\boxed{\tilde{K}\,\tilde{M}^T = 0.} \tag{1}$$

Cut sets corresponding to linearly independent row vectors of $\tilde{K}$ are called *independent* cut sets. The maximum number of independent cut sets is equal to k. One maximal collection of independent cut sets which is particularly suited to our purpose of network analysis is given by the k branch sets incident with the individual nodes $1, 2, \ldots, k$. For the example system, Fig. 4 shows this collection of independent cut sets. Then the matrix $\tilde{K}$ becomes the *node branch incidence matrix* K introduced by Poincaré (1895). K is of size $(k,\, z)$ and is row regular. The entry $(\kappa,\, \zeta)$ of K is defined as 1, -1, or 0, if the branch ζ starts at node κ, terminates at node κ, or is not

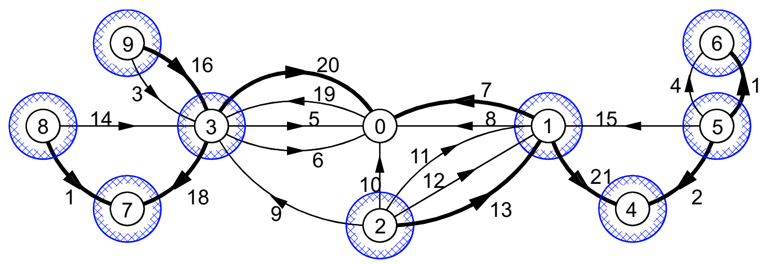


Fig. 4. Independent node cut sets and one possible set of tree branches.

incident with node κ. Any non-vanishing minor of order k of K, i.e., the determinant of any $(k,\, k)$ submatrix formed by the common entries of the row vectors $1, 2, \ldots, k$ and any pairwise different set of k column vectors $\zeta_1,\, \zeta_2, \ldots,\, \zeta_k$ of K, corresponds to a (spanning) *tree* of $\mathcal{G}$. More specifically,

$$K^{\zeta_1\, \zeta_2 \cdots \zeta_k}_{1\; 2\, \cdots k} = \begin{cases} 1 \text{ or } -1 & \text{if the branches } \zeta_1, \zeta_2, \ldots, \zeta_k \text{ form a tree,} \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

A (spanning) tree of $\mathcal{G}$ is a connected subgraph involving k branches and all the $k+1$ nodes. In Fig. 4 the branches of one particular tree of the example system are marked by thick lines.

In 1923 Weyl observed: "The integral solutions of $Kx = 0$ give the circuits of the network graph." (The term "integral solution" may be expressed differently: the components of x are integers.)

Consequently, there are at most $z-k$ independent circuits. After specification of a spanning tree, a set of $z-k$ independent circuits can be defined that is particularly suited to network analysis: the so-called *basic meshes* determined by the $z-k$ co-tree branches, i.e., by the branches belonging to the subset complementary to the tree.

The basic meshes are generated as follows: Pick a co-tree branch and add only tree branches so as to obtain a circuit of shortest length. The orientation of the co-tree branch is carried over to the orientation of the mesh. In Fig. 5, the procedure is illustrated by two basic meshes for the example system. The circuit branch indicator matrix $\tilde{M}$ appears now as the *mesh branch incidence matrix* M of size $(z-k,\, z)$.

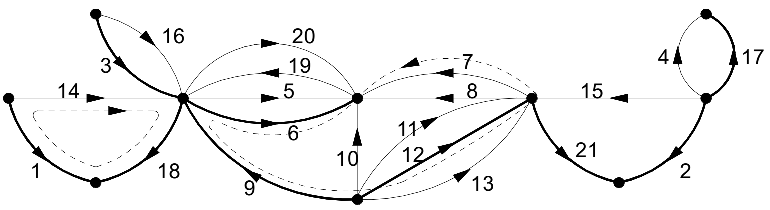


Fig. 5. A spanning tree in the network graph and the two basic meshes that are defined by the links 7 and 14.

Rank $M = z - k$ since each mesh contains one branch not contained in all the other meshes. The orthogonality relations read

$$K\, M^T = 0. \tag{3}$$

Equation (3) is a special case of (1). Fortunately, the incidence matrices K and M are sufficient to describe completely the cut and circuit laws. This is true since every cut-set and every circuit-set, respectively, can be described as a linear combination of the k independent branch sets incident with the nodes as well as the $z-k$ independent basic meshes associated with any chosen tree.

We remark that although the term "tree" was coined by Cayley (1857), Kirchhoff had made use of the concept ten years earlier (Kirchhoff, 1847). Weyl was the first to completely expound network graph topology in (Weyl, 1923), explicitly acknowledging the related works of Poincaré (1895) and Veblen (1916).

2.2. Formulation of the Network Equations

Figure 6 shows, in pictorial form, the branch representation in its most general form, including the sign conventions used in this paper. The branch is associated with an independent FV source and an independent DV source. The three dots across which

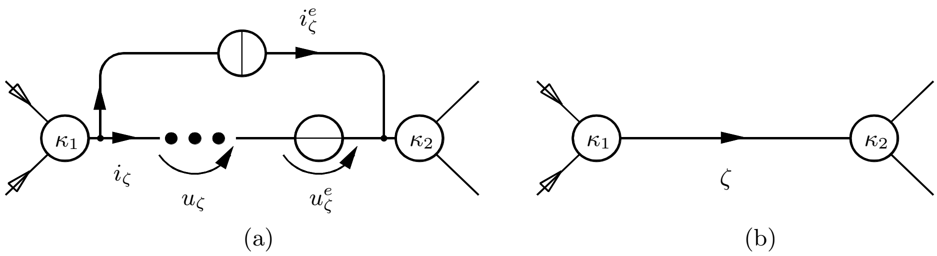


Fig. 6. General form of a branch: (a) pictorial representation of branch ζ leading from node κ_1 to node κ_2, (b) branch ζ as part of the network graph.

the DV u_ζ appears in Fig. 6 symbolize the NERs of branch ζ. The three dots may stand for the symbol of a resistor (i.e., an ohmic resistor in electrical networks), a FV storage (e.g., a capacitor) or a DV storage (e.g., an inductor). In all these cases, the NERs represent a unique mapping between the FV i_ζ and the DV u_ζ. This is typical of the so-called one-port branches. Figure 6, however, remains valid also for multi-port networks, i.e., networks with interdependencies between FVs and DVs of different branches caused by inductive couplings, controlled sources, or genuine multiport elements such as transistors, transducers, etc. A general n-port network element is depicted in Fig. 7. Examples of 2-ports are ideal transformers, gyrators, ideal amplifiers, controlled DV sources, controlled FV sources, nullors (i.e., a nullator branch combined with a norator branch). Figure 8(a) gives the general pictorial representation of a two-port network element together with its NERs in implicit form. Figure 8(b) explains the special case of a nullor.

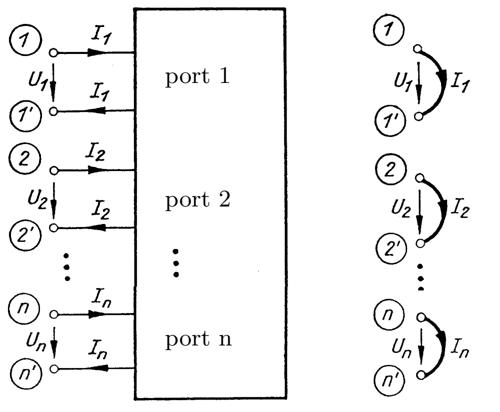
The branch DVs, the branch FVs, the DVs across the independent DV sources, the FVs through the independent FV sources, and the node DVs are considered to be components of the vectors u, i, u^e, i^e, i_0 and u_ϕ, respectively.

The cut law for the FVs may be formulated as $K\,(i + i^e) = 0$, or, taking into account (3) and expressed differently, $i + i^e \in$ image $\{M^T\}$, i.e., $M^T i_0 = i + i^e$, where the $z - k$ componenets of i_0 may be interpreted as mesh DVs. In the sequel, we can make use of both the mathematically equivalent formulations of the cut law,

$$K(i + i^e) = 0 \quad \text{or} \quad M^T i_0 = i + i^e. \tag{4}$$

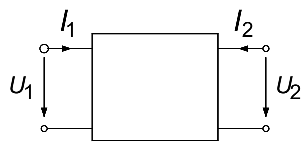
Analogously, there are two equivalent formulations of the circuit law:

$$M(u + u^e) = 0 \quad \text{or} \quad K^T u_\phi = u + u^e, \tag{5}$$



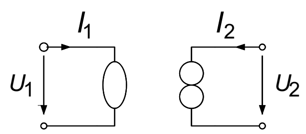
(a) (b)

Fig. 7. General n-port: (a) pictorial representation of an n-port network element, (b) representation as part of the network graph.



$$A\begin{pmatrix} I_1 \\ I_2 \end{pmatrix} + B\begin{pmatrix} U_1 \\ U_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

(a)



$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} I_1 \\ I_2 \end{pmatrix} \pm \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} U_1 \\ U_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

(b)

Fig. 8. Two-port network elements: (a) general two-port and its NERs, (b) nullor and its NERs.

where the k components of u_ϕ may be interpreted as node DVs (with respect to the reference node 0). It should be noted that the FV-vectors in (4) are entirely independent of the DV-vectors in (5). Moreover, the FVs and the DVs subject to (4) and (5) may be elements of a very loose algebraic structure (abelian group with respect to addition) since multiplication with an incidence matrix means merely addition and/or subtraction of the network variables.

If the FV-vectors and DV-vectors are interpreted as elements in a z-dimensional space equipped with an inner product, then (3) has an immediate consequence:

$$(u + u^e)^T(i + i^e) = u_\phi K M^T i_0 = 0.$$

Stated geometrically, the $(z - k)$-dimensional subspace of FV-vectors is orthogonal to the k-dimensional subspace of DV-vectors. It took the network theorists decades to understand thoroughly this fundamental fact. Related discussions are contained in (Kron, 1939; LeCorbeiller, 1950; Synge, 1951; Tellegen, 1953; Weyl, 1923).

Any complete network description contains the third principal ingredient: network element relations (NERs) between FVs and DVs. The NERs require a stronger algebraic structure of the network variables such as multiplicability with real numbers, differentiability with respect to time, etc. The NERs in their most general form may be implicitly written as

$$f(u,\, i) = 0. \tag{6}$$

All the three ingredients of a network description can be arranged in a transformation diagram (Branin, 1966), see Fig. 9.

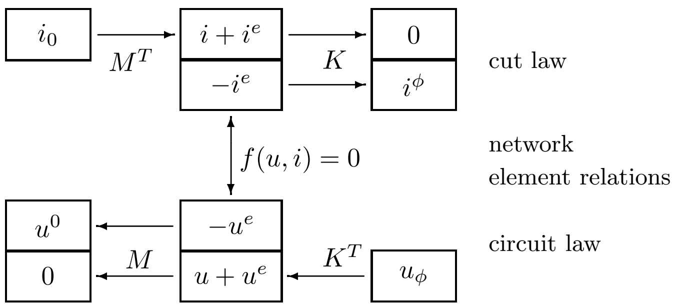


Fig. 9. The three ingredients of network models.

The network problem can be defined as follows: Given

(i) a network graph with incidence matrices K and/or M,

(ii) NERs $f(u,\, i) = 0$,

(iii) independent source vectors i^e and u^e,

find an FV-vector i and a DV-vector u such that eqs. (4)–(6) are fulfilled.

From Fig. 9 we observe that various systems of equations admit network analysis:

(i) the branch-DV branch-FV equations

$$\begin{pmatrix} M & 0 \\ 0 & K \end{pmatrix} \begin{pmatrix} u \\ i \end{pmatrix} = \begin{pmatrix} -M\,u^e \\ -Ki^e \end{pmatrix}, \quad f(u,\, i) = 0 \tag{7}$$

with $2z$ unknowns, often referred to as sparse tableau analysis (STA),

(ii) equations for node-DV branch-FV analysis (NBA)

$$f(K^T u_\phi - u^e,\, i) = 0, \quad Ki = -K\, i^e = i^\phi \tag{8}$$

with $k + z$ unknowns,

(iii) equations for mesh-FV branch-DV analysis (MBA)

$$f(u,\, M^T i_0 - i^e) = 0, \quad M u = -M\, u^e = u^0 \tag{9}$$

with $2z - k$ unknowns,

(iv) equations for node-DV mesh-FV analysis (NMA)

$$f(K^T u_\phi - u^e,\, M^T i_0 - i^e) = 0 \tag{10}$$

with z unknowns.

The duality between (8) and (9) deserves to be noted. It appears that Maxwell was the first to use the idea of duality in his study of frameworks (Maxwell, 1870). In this paper, we mainly restrict our attention to NBA. In general, little can be said about the unique solvability of the nonlinear equations (8). However, if the NERs are of the form

$$f(u,\, i) = Y u - i = 0,$$

where $u^T Y u \neq 0$ for all $u \neq 0$, the linear operator Y is termed "ohmic." Then the network problem has a unique solution (Roth, 1959).

Network models may be classified according to the algebraic structure of the branch FVs and the branch DVs. If they are real or complex numbers, i.e., they are elements of a one-dimensional vector space, then one refers to *one-dimensional networks*. If the branch FVs and branch DVs are elements of an N-dimensional vector space, then the networks are called N*-dimensional networks*.

3. One-Dimensional Networks

3.1. One-Port Networks

For *one-port networks*, the NERs are given by z FV-DV relations (for the z individual branches)

$$f_\zeta(u_\zeta,\, i_\zeta) = 0 \quad (\zeta = 1, \ldots, z). \tag{11}$$

For linear time-invariant network elements, the NERs have the well-known time and frequency domain equations:

$$\begin{aligned} &\text{resistor:} && i_\zeta(t) = G\, u_\zeta(t), && I_\zeta(s) = G\, U_\zeta(s),\\ &\text{FV storage:} && i_\zeta(t) = C\, \dot{u}_\zeta(t), && I_\zeta(s) = sC\, U_\zeta(s),\\ &\text{DV storage:} && \dot{i}_\zeta(t) = \tfrac{1}{L} u_\zeta(t), && I_\zeta(s) = \tfrac{1}{sL} U_\zeta(s). \end{aligned} \tag{12}$$

In the frequency domain, the concept of *branch admittances* defined by $y_{\zeta\zeta}(s) = U_\zeta(s)/I_\zeta(s)$ has proved to be useful. Making use of the *branch admittance matrix*

$$Y(s) = \langle y_{11}(s),\, y_{22}(s), \ldots, y_{zz}(s) \rangle\,,$$

eqn. (8) can be rewritten as $Y(K^T U_\phi - U^e) + I = 0$ and $KI = -KI^e$, which leads to the *node-DV equations*

$$KYK^T\,U_\phi = K(YU^e - I^e). \tag{13}$$

If we had used (9), similar reasoning would have led us to the *mesh-FV equations*

$$MZM^T\,I_0 = M(ZI^e - U^e), \tag{14}$$

where $Z(s) = \langle z_{11}(s),\, z_{22}(s), \ldots,\, z_{zz}(s)\rangle = (Y(s))^{-1}$ denotes the *branch impedance matrix.*

It was one of the earliest findings of network theory that node-DV equations and mesh-FV equations

$$KYK^T U_\phi = I^{e,\,\phi}_{\mathrm{aux}} \quad \text{and} \quad MZM^T I_0 = U^e_{\phi,\,\mathrm{aux}}$$

can be solved symbolically by inspecting the network graph (Kirchhoff, 1847; Maxwell, 1882). The Cauchy-Binet formula (Cauchy, 1815) gives a key to a thorough understanding of the facts announced in (Kirchhoff, 1847; Maxwell, 1882):

$$\begin{aligned}
\det(KYK^T) = \det(K(YK^T)) &= \sum_{i=1}^{\binom{z}{k}} K^{i_1\, i_2 \cdots i_k}_{\;1\;\; 2 \cdots\; k} \left(YK^T\right)^{1\;\, 2 \cdots\, k}_{i_1\, i_1 \cdots i_k} \\
&= \sum_{i=1}^{\binom{z}{k}} K^{i_1\, i_1 \cdots i_k}_{\;1\;\; 2 \cdots\; k} \left(\sum_{j=1}^{\frac{z}{k}} Y^{j_1\, j_2 \cdots j_k}_{i_1\, i_2 \cdots i_k} (K^T)^{1\;\, 2\, \cdots\, k}_{j_1\, j_2 \cdots j_k} \right) \\
&= \sum_{i,j}^{\binom{z}{k}} K^{i_1\, i_2 \cdots i_k}_{\;1\;\; 2 \cdots\; k} \quad Y^{j_1\, j_2 \cdots j_k}_{i_1\, i_2 \cdots i_k} \quad K^{j_1\, j_1 \cdots j_k}_{\;1\;\; 2 \cdots\; k}\,,
\end{aligned} \tag{15}$$

where $K^{i_1\, i_2 \cdots i_k}_{\;1\;\; 2 \cdots k}$ was explained above, see (2). Since the branch admittance matrix Y is diagonal in the case of linear 1-port networks, we have $Y^{j_1\, j_2 \cdots j_k}_{i_1\, i_2 \cdots i_k} \neq 0$ iff $\{i_1\, i_2 \cdots i_k\} = \{j_1\, j_2 \cdots j_k\}$. Then (Maxwell, 1882) $\det(KYK^T) = \sum_{(i)} Y^{i_1\, i_2 \cdots i_k}_{i_1\, i_2 \cdots i_k} = \sum_{(i)}$ (product of the admittances of all branches of tree i).

Analogously, it can be derived that (Kirchhoff, 1847) $\det(MZM^T) = \sum_{(i)}$(product of the impedances of all branches of co-tree i).

Furthermore,

$$\frac{\det(MZM^T)}{\det(KYK^T)} = \det Z = (\det Y)^{-1}.$$

Formally replacing every branch admittance by the real number 1, the total number of trees can be calculated as

$$n_T = \det(KK^T) = \det(MM^T).$$

In the case of RLC-networks we can state the following: The network determinants $\det(KYK^T)$ or $\det(MZM^T)$ can be calculated symbolically by means of enumeration of all the trees of the network graph. Every tree corresponds to one term in a sum, cf. the right-hand side of (15). All these terms have the same sign, and no cancellations of terms can thus occur. The tree enumeration method was published in the thirties (Ting, 1935; Tsai, 1939; Wang, 1934), later on algebraically substantiated (Bellert, 1962; Bott and Duffin, 1953; Duffin, 1959; Seshu and Reed, 1961; Trent, 1955), and has been implemented for CAD purposes since the sixties (Chen, 1967; Chua and Lin, 1975; Dmitrischin, 1969; Mayeda and Seshu, 1965; Trochimenko, 1972). For related papers published in the seventies and eighties, see the monographs (Gieben and Sansen, 1991; Lin, 1991). The applicability of this approach has a limitation: the number of trees may increase exponentially with the number of nodes. Indeed, for a complete network graph (i.e., a graph in which every pair of nodes is connected by exactly one branch) the number of trees equals $k^{(k-2)}$. Fortunately, complete graph structures are not typical of practical network models. But even for ladder networks we observe a growth in the number of trees depending exponentially on the number of ladder sections (see Lin, 1991, p.47). As for the actual usefulness of this method, much depends on the skills of the investigator. Frequently, it does not make sense to explicitly print out thousands of symbolic expressions corresponding to thousands of trees. The trees need not actually be determined. All information is contained in the main diagonal elements of $(KYK^T)_{\kappa\kappa}$, where the admittances of all the branches connected to the node κ are summed-up symbolically. Then the product $\prod_{\kappa=1}^{k}(KYK^T)_{\kappa\kappa}$, if evaluated according to the rules of the *Wang algebra*, $(\,x + x \overset{W}{=} 0,\; x \cdot x \overset{W}{=} 0\,)$, yields the desired network determinant in symbolic form. Of course, a complete resolution of all brackets will often appear as an inefficient way of evaluating the determinant.

As an example, consider the small electrical network depicted in Fig. 10 and find a symbolic expression for the network determinant $\det(KYK^T)$. Obviously, $k = 3$.

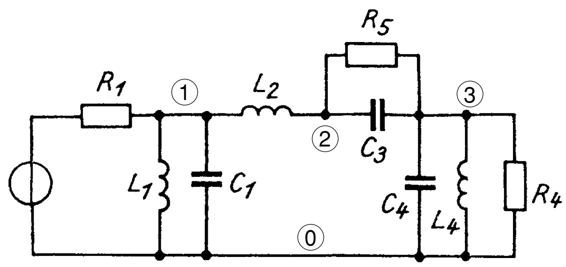


Fig. 10. Example of an RLC-network.

As for the number of branches, let us discuss two possibilities. If each passive element corresponds to one branch, then $z = 9$ and the number of trees is

$$n_T = \det\begin{pmatrix} 4 & -1 & 0 \\ -1 & 3 & -2 \\ 0 & -2 & 5 \end{pmatrix} = 39.$$

If we take $z = 4$ branches with the admittances $y_1 = G_1 + sC_1 + (sL_1)^{-1}$, $y_2 = (sL_2)^{-1}$, $y_3 = G_5 + sC_3$, $y_4 = G_4 + sC_4 + (sL_4)^{-1}$, then $n_T = \det\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} = 4$. The above-mentioned rules of the Wang-algebra give

$$\begin{aligned} \det(KYK^T) &\overset{W}{=} (y_1 + y_2)(y_2 + y_3)(y_3 + y_4) \\ &\overset{W}{=} (y_1 + y_2)(y_2(y_3 + y_4) + y_3 y_4) \\ &\overset{W}{=} y_1[y_2(y_3 + y_4) + y_3 y_4] + y_2 y_3 y_4. \end{aligned}$$

3.2. Multi-Port Networks with Admittance Representations

One-dimensional networks containing NERs with interdependencies between FVs and DVs of different branches are called *multi-port networks*. First, let us assume that all the multi-ports contained in the network have an *admittance representation*. Then the NERs of the multi-ports can be written as $I_M - Y_M U_M = 0$, and the NERs of the one-ports as before, $I - YU = 0$. To get a complete set of network equations, the cut and circuit laws are formulated as

$$\begin{pmatrix} K K_M \end{pmatrix} \begin{pmatrix} I + I^e \\ I_M + I_M^e \end{pmatrix} = 0, \qquad \begin{pmatrix} U + U^e \\ U_M + U_M^e \end{pmatrix} = \begin{pmatrix} K^T \\ K_M^T \end{pmatrix} U_\phi.$$

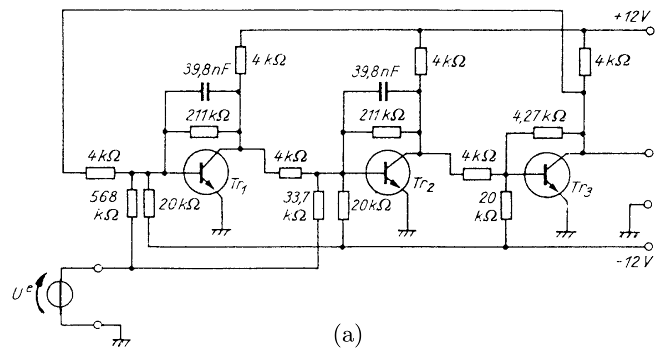
The nomenclature has been slightly changed from the one previously used so as to illustrate the role played by the multi-port network elements. To avoid confusion, an example with 8 one-port branches and three two-ports is depicted in Fig. 11. The node-DV equations appear now in the augmented form

$$(K\ K_M)\langle Y\ Y_M \rangle (K\ K_M)^T U_\phi = (K\ K_M) \begin{pmatrix} YU^e - I^e \\ Y_M U_M^e - I_M^e \end{pmatrix}.$$

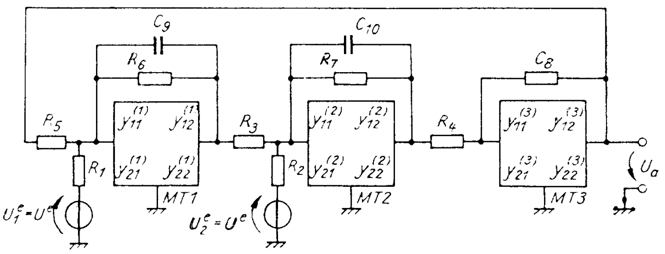
The network determinant can again be evaluated by applying the Cauchy-Binet formula twice,

$$\begin{aligned} \det[(K\ K_M)\,\langle Y\ Y_M \rangle\,(K\ K_M)^T] &= \det(\overline{K}\,\overline{Y}\,\overline{K}^T) \\ &= \sum_{(i,j)} \overline{K}^{i_1\, i_2 \cdots i_k}_{1\ 2\ \cdots k} \quad \overline{Y}^{j_1\, j_2 \cdots j_k}_{i_1\, i_2\, \cdots i_k} \quad \overline{K}^{j_1\, j_2 \cdots j_k}_{1\ 2\ \cdots k}. \end{aligned}$$

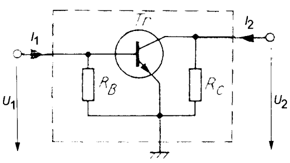
In contrast to (15), there now exist non-vanishing minors $\overline{Y}^{j_1\, j_2 \cdots j_k}_{i_1\, i_2 \cdots i_k}$ taken from the row sets $i_1, \ldots, i_k$ and the column sets $j_1, \ldots, j_k$ with $\{i_1, \ldots, i_k\} \neq \{j_1, \ldots, j_k\}$. The associated '$i$'-tree and '$j$'-tree may differ from each other. Nevertheless, it is possible to formulate topological rules for a symbolic evaluation of the network determinant, see, e.g., (Reinschke and Schwarz, 1976). Due to a lack of space, the details are omitted here. In Section 3.3, related questions will be discussed in a more general



(a)

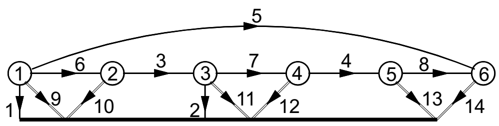


(b)



$$\begin{pmatrix} I_1 \\ I_2 \end{pmatrix} = \begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix} \begin{pmatrix} U_1 \\ U_2 \end{pmatrix}$$

(c)



$$\langle Y\ Y_M \rangle = \begin{pmatrix} y_1 &&&&&&&&&& \\ & y_2 &&&&&&&&& \\ && \ddots &&&&&&&& \\ &&& y_8 &&&&&&& \\ &&&& y_{11}^{(1)} & y_{12}^{(1)} &&&&& \\ &&&& y_{21}^{(1)} & y_{22}^{(1)} &&&&& \\ &&&&&& y_{11}^{(2)} & y_{12}^{(2)} &&& \\ &&&&&& y_{21}^{(2)} & y_{22}^{(2)} &&& \\ &&&&&&&& y_{11}^{(3)} & y_{12}^{(3)} \\ &&&&&&&& y_{21}^{(3)} & y_{22}^{(3)} \end{pmatrix}$$

(d)

Fig. 11. Active RC circuit: (a) RC circuit with three transistors, (b) small signal network model, (c) admittance representation of a transistor, (d) network graph and branch admittance matrix.

framework. As for the computer-aided network analysis, it is sometimes advantageous to use systems of network equations whose coefficients depend affinely on the complex frequency s. This is not the case if the network contains both DV and FV storages. *Gyrators* introduced in the forties (Tellegen, 1948) are non-reciprocal passive two-port elements which are able to transform a DV storage into an FV storage and vice versa. The deliberate insertion of gyrators offers a general possibility to avoid one of the two types of storages. Figure 12 also illustrates another possibility: The NERs of all

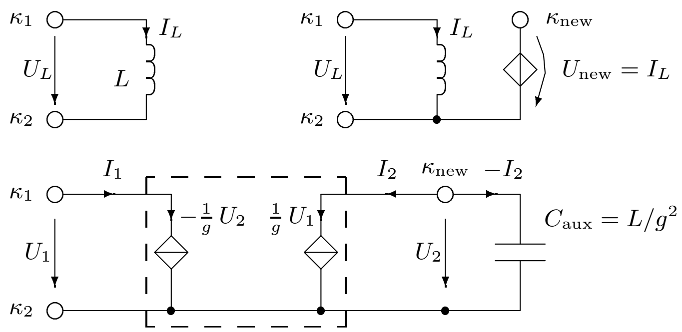


Fig. 12. Two possibilities to get modified nodal equations whose coefficients are affine functions of the complex frequency s.

inductor branches can be summarized in the equation $U_L = s < L > I_L$. We augment the network graph by as many additional branches and as many additional isolated nodes as there exist inductor branches. Each additional branch connects one terminal node of one inductor branch with one isolated new node and works as a current controlled voltage source. Denoting the node inductor-branch incidence matrix by K_L, and the node branch incidence matrix of all the other branches by $\tilde{K}$, the node equations of the modified network can be written as (a detailed proof relies on the arguments explained in Section 3.3)

$$\begin{pmatrix} sL & K_L^T \\ K_L & \tilde{K}\tilde{Y}\tilde{K}^T \end{pmatrix} \begin{pmatrix} I_L \\ U_\phi \end{pmatrix} = \cdots .$$

The price to be paid is obvious: By including one further inductor in the network, the number of graph nodes increases by one. The total number of network trees, however, remains unchanged.

A few remarks about the symbolic solution of linear network equations are in order. If the inner structure of the network equations is neglected, any method of symbolic evaluation of determinants can be applied. In particular, graph theory provides useful tools to tackle this problem. There are several possibilities of constructing digraphs that have a one-to-one correspondence with a given square matrix A and obtaining the determinant $\det A$ by inspection of the digraph. (For example, see the Appendix in (Reinschke, 1988).) The first graph-theoretic interpretation of determinants was published by Cauchy (1815), reformulated by Jacobi (1841), and re-invented by Coates (1959). Each term of $\det A$ corresponds to a *spanning cycle family* in the digraph. This method is cancellation-free, i.e., if the matrix entries are mutually independent, then no terms which cancel each other are generated. In König's monograph (1936) square matrices are represented by bipartite graphs. Each term of the determinant corresponds to a *matching*, and this method is also cancellation-free. Seen from the mathematical point of view, both the approaches are equivalent. As for computer implementations, the matching algorithms have advantages. The symbolic solution of linear algebraic equations can be traced back to the evaluation of determinants as follows: Let $Ax = a,\ y = c^T x$. Then $y = c^T A^{-1} a = \det \begin{pmatrix} A & a \\ -c^T & 0 \end{pmatrix} / \det A = D/N$ and $\det \begin{pmatrix} A & a \\ -c^T & p \end{pmatrix} = p\,N + D$. The augmented determinant is a linear polynomial in the parameter p. The coefficients are the denominator D and the numerator N. The so-called *signal-flow graphs* published by Mason became popular in the sixties (Mason, 1953). These graphs have the disadvantage that the algebraic equations must be of the form $x = Ax + b$.

Network equations, in particular those which arise in node-DV analysis, have an inner structure which leads to mutual dependencies between the matrix entries. Unfortunately, general determinant-based evaluation methods such as the Cauchy-Coates method are not capable of taking advantage of the matrix structure. Roughly speaking, the node-DV equations are of the form

$$K\,Y\,K^T = \sum_{i,j=1}^{k} y_{ij}\, K_{\bullet i}\,(K_{\bullet j})^T,$$

where $K_{\bullet j}$ denotes the j-th column of the node branch incidence matrix K. The dyadic product $K_{\bullet i}\,(K_{\bullet j})^T$ gives a structural "stamp" defined by the node branch incidence relations of the branches j and i. Stated in another way, the network coefficient matrix is a weighted sum of stamps of the same size. Each term of the sum reflects the influence of one network parameter appearing as an entry of the branch admittance matrix Y of the network. Graph-based methods for the symbolic solution of the network equations are discussed in the next section.

3.3. Generation of Admittance Representations for All Kinds of Network Elements and Topological Determination of Network Determinants

Many multi-port models such as ideal transformers, operational amplifiers, nullors, DV controlled and FV controlled DV sources or FV controlled FV sources do not have an admittance representation. Each linear n-port, however, can be specified by a linear implicit representation

$$A\,I_M + B\,U_M = 0 \tag{16}$$

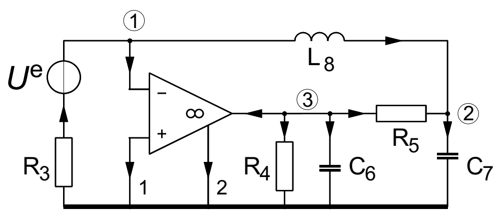
with (n, n) matrices A and B. For any linear time-invariant network, a complete set of network equations can be written as

$$\begin{pmatrix} E & -Y & 0 & 0 & 0 \\ 0 & E & 0 & 0 & -K^T \\ 0 & 0 & E & 0 & -K_M^T \\ 0 & 0 & 0 & A & BK_M^T \\ 0 & 0 & 0 & K_M & KYK^T \end{pmatrix} \begin{pmatrix} I \\ U \\ U_M \\ I_M \\ U_\phi \end{pmatrix} = \begin{pmatrix} 0 \\ -U^e \\ -U_M^e \\ BU_M^e \\ -I_\phi^e \end{pmatrix},$$

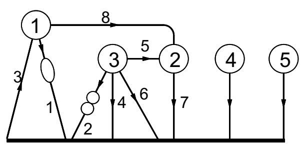
where $-I_\phi^e := K(YU^e - I^e) - K_M I_M^e$ and E denotes the identity matrix.

The essential part of those network equations consists of the last two hyper-rows. We shall show that they may be interpreted as nodal equations of a modified network model. For this purpose, the network graph is supplemented with n additional branches between n isolated nodes and the reference node, acting as FV controlled DV sources. It deserves to be noticed that the total number of (spanning) trees does not change with this modification of the network graph.

An example is shown in Fig. 13: an RLC-network with an operational amplifier modeled by a nullor (Fig. 13(a)) and the corresponding augmented network graph (Fig. 13(b)). There are two additional branches between the newly introduced isolated nodes 4, 5 and the reference node. The mathematical formulation of the circuit law and the cut law for the augmented network are presented in Fig. 14. The NERs are nothing else than the desired admittance representation of all network elements. The matrix $\overline{\overline{Y}}$ may be regarded as an augmented branch admittance matrix. The network problem is solvable iff the network determinant does not vanish. Again, the Cauchy-Binet formula is suited to symbolically determine the network determinant. Let z_M



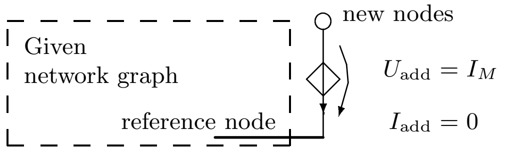
(a)



(b)

Fig. 13. Network model and augmented network graph.

Augmented network graph:



cut law: $$\begin{pmatrix} E & 0 & 0 \\ 0 & K_M & K \end{pmatrix} \begin{pmatrix} I_{\text{add}} \\ I_M \\ I \end{pmatrix} = \begin{pmatrix} 0 \\ -KI^e & -K_M I_M^e \end{pmatrix}$$

circuit law: $$\begin{pmatrix} U_{\text{add}} \\ U_M \\ U \end{pmatrix} = \begin{pmatrix} E & 0 \\ 0 & K_M^T \\ 0 & K^T \end{pmatrix} \begin{pmatrix} U_{\text{add}} \\ U_\phi \end{pmatrix} + \begin{pmatrix} 0 \\ -U_M^e \\ -U^e \end{pmatrix}$$

NERs: $$\begin{pmatrix} I_{\text{add}} \\ I_M \\ I \end{pmatrix} - \underbrace{\begin{pmatrix} A & B & 0 \\ E & 0 & 0 \\ 0 & 0 & Y \end{pmatrix}}_{=:\overline{\overline{Y}}} \begin{pmatrix} U_{\text{add}} \\ U_M \\ U \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Coefficient matrix of *modified nodal analysis* (MNA):

$$\overline{\overline{K}}\,\overline{\overline{Y}}\,\overline{\overline{K}}^T := \begin{pmatrix} E & 0 & 0 \\ 0 & K_M & K \end{pmatrix} \begin{pmatrix} A & B & 0 \\ E & 0 & 0 \\ 0 & 0 & Y \end{pmatrix} \begin{pmatrix} E & 0 \\ 0 & K_M^T \\ 0 & K^T \end{pmatrix}$$

$$= \begin{pmatrix} E & 0 & 0 \\ 0 & K_M & K \end{pmatrix} \begin{pmatrix} A & BK_M^T \\ E & 0 \\ 0 & YK^T \end{pmatrix} = \begin{pmatrix} A & BK_M^T \\ K_M & KYK^T \end{pmatrix}$$

Fig. 14. Augmentation of the network in order to allow of admittance representations for all kinds of network elements.

and z be the total number of multi-port and one-port branches, respectively. Then

$$\det \begin{pmatrix} A & BK_M^T \\ K_M & KYK^T \end{pmatrix} = \det \left(\overline{\overline{K}}\,\overline{\overline{Y}}\,\overline{\overline{K}}^T \right)$$

$$= \sum_{i,j=1}^{\binom{z+z_M}{k+z_M}} \overline{\overline{K}}{}^{i_1 \cdots i_{z_M+k}}_{1 \cdots z_M+k} \; \overline{\overline{Y}}{}^{j_1 \cdots j_{z_M} \cdots j_{z_M+k}}_{i_1 \cdots i_{z_M} \cdots i_{z_M+k}} \; \overline{\overline{K}}{}^{j_1 \cdots j_{z_M+k}}_{1 \cdots z_M+k}$$

$$= \sum_{(i,j)} \overline{\overline{K}}{}^{i_{z_M+1} \cdots i_{z_M+k}}_{z_M+1 \cdots z_M+k} \overline{\overline{Y}}{}^{j_1 \cdots j_{z_M} \; j_{z_M+1} \cdots j_{z_M+k}}_{i_1 \cdots i_{z_M} \; i_{z_M+1} \cdots i_{z_M+k}} \overline{\overline{K}}{}^{j_{z_M+1} \cdots j_{z_M+k}}_{z_M+1 \cdots z_M+k} \, .$$

The last equality results from the particular structure of the augmented incidence matrix $\overline{\overline{K}}$. The first z_M columns of $\overline{\overline{K}}$ must be used to get a regular (z_M+k)-minor of $\overline{\overline{K}}$ that corresponds to a tree in the augmented network graph. The $(z_M + b)$-minors of the square matrix $\left(\begin{smallmatrix} A & B \\ E & 0 \end{smallmatrix}\right)$, where $0 < b < z_M$, are most important for the symbolic evaluation of the network determinant. The minor

$$\begin{pmatrix} A & B \\ E & 0 \end{pmatrix}^{1\cdots z_M\, j_1\cdots j_b}_{1\cdots z_M\, i_1\cdots i_b} =: (Y_{MT})^{j_1\cdots j_b}_{i_1\cdots i_b}$$

is associated with that part of the network graph which consists of the multi-port branches $i_1, \ldots, i_b$ within the 'i'-tree and the multi-port branches $j_1, \ldots, j_b$ within the 'j'-tree. The small example introduced in Fig. 13 may help us to explain the basic ideas for the symbolic evaluation of the network determinant (see Fig. 15). The node

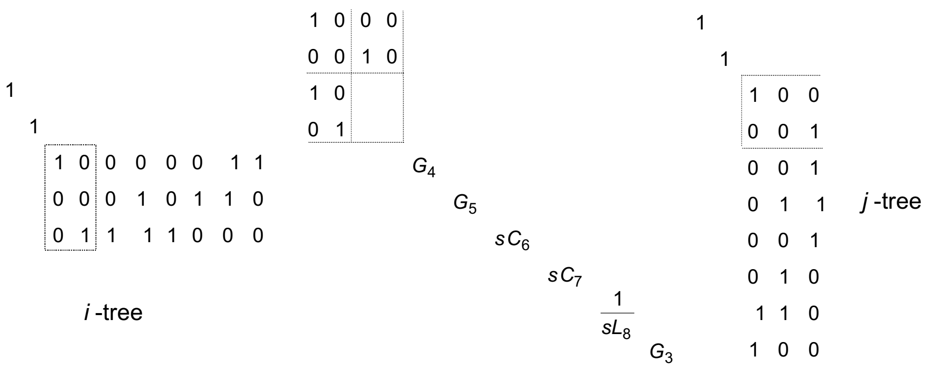


Fig. 15. Network determinant for the example of Fig. 13.

branch incidence matrix of the augmented graph has $3+2 = 5$ rows and $2+2+5 = 9$ columns. The non-vanishing terms of the network determinant are 5-minors of the augmented branch admittance matrix, i.e., determinants of (5, 5)-matrices whose rows are defined by an i-tree whereas the columns are defined by a j-tree. It is evident that each j-tree and each i-tree contain the two newly added branches. Consequently, the (2, 2)-submatrix in the left upper corner must be contained in any (5, 5)-submatrix. Continuing this discussion, we could conclude: There is exactly one non-vanishing 5-minor, and—apart from the sign—the network determinant is equal to the product of admittances of branch 5 and of branch 8. On the basis of these preliminary remarks, we are able to formulate a general evaluation rule:

$$\text{NW-Det} = \sum_{b=0}^{z_M} \sum_{i,j=1}^{\binom{z_M}{b}} (Y_{MT})^{j_1\cdots j_b}_{i_1\cdots i_b} \cdot \text{ Remainder-NW-Det.}$$

Comments on the individual terms in the double sum (see Fig. 16):

1. Let $b = 0 : (Y_{MT}) = \det A$. Removal of the z_M multi-port branches creates the remainder-NW-graph. Remainder-NW-Det $= \det(KYK^T)$.

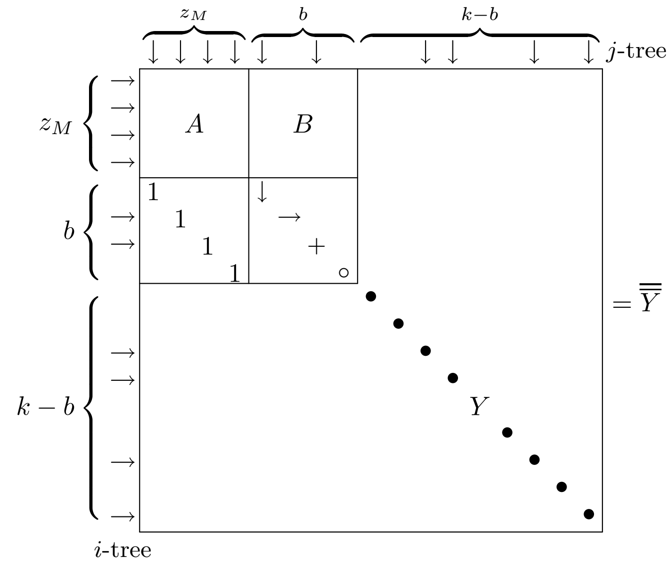


Fig. 16. Evaluation of the network determinant. Notation:
'∘' a multi-port branch to be removed,
'+' a multi-port branch to be contracted,
'↓, →' a pair of multi-port branches to be completed to branch-disjoint meshes.

2. Let $b = z_M$: $(Y_{MT})^{\cdots}_{\cdots} = \det(-B)$. The remainder-NW results from coalescing both terminal nodes of each individual multi-port branch to one node.

3. Let $0 < b < z_M$: $(Y_{MT})^{j_1 \cdots j_b}_{i_1 \cdots i_b} = (-1)^b \cdot \det((z_M,\, z_M)-$matrix generated by replacing the A-columns $i_1 \cdots i_b$ by the B-columns $j_1 \cdots j_b$).

   The remainder-NW graph is formed as follows:

   (a) Both the terminal nodes of each multi-port branch within the branch set $\{i_1, \ldots, i_b\} \cap \{j_1, \ldots, j_b\}$ coalesce to one node. The multi-port branches not contained in the branch set $\{i_1, \ldots, i_b\} \cup \{j_1, \ldots, j_b\}$ are removed.

   (b) If there remain $d(< b)$ branches in $\{i_1, \ldots, i_b\}$ and d branches in $\{j_1, \ldots, j_b\}$—neither short-circuited in step (a) nor open-circuited in step (a)—then the d branches of the j-tree correspond to d columns of B whereas the d branches of the i-tree correspond to d differently indexed rows of E. Within the network graph, the d j-tree branches and the d i-tree branches may be pairwise completed (by means of topologically suitable existing branches) to d branch-disjoint meshes. Assume that there are m different possibilities to construct d disjoint meshes. Then we denote by $y_{M,\mu}$ the μ-th mesh admittance defined as the product of the admittances of all 1-port branches occurring within the d meshes of the μ-th mesh set ($\mu = 1, \ldots, m$). We set $y_{M,\mu} = 1$ if the d meshes of the μ-th mesh set do not contain a 1-port branch. Afterwards, the μ-remainder-NW-graph is generated by coalescing each of the d meshes into one node. We have

$$\text{remainder-NW-Det} = \sum_{\mu=1}^{m} y_{M,\mu} \cdot (\mu\text{-remainder-NW-Det}).$$

If the modified network graph is connected and contains only 1-port branches, then the remainder-NW-Det may be determined according to the rules for connected 1-port networks. In the case of a disconnected modified graph, the remainder-NW-Det is equal to zero.

At first glance, the above-mentioned rules appear to be cumbersome. In applying the outlined approach, several multi-port elements can be successively taken into account, step by step. Consider a network model containing two multiports associated with matrices A_1, B_1 and A_2, B_2. Then the admittance matrix $\overline{\overline{Y}}$ can be rewritten by means of permutation of rows and columns as follows:

$$\begin{pmatrix} A & B \\ E & 0 \end{pmatrix} = \begin{pmatrix} A_1 & & B_1 & \\ & A_2 & & B_2 \\ E_1 & & & \\ & E_2 & & \end{pmatrix} \text{ equivalent } \begin{pmatrix} A_1 & B_1 & & \\ E_1 & 0 & & \\ & & A_2 & B_2 \\ & & E_2 & 0 \end{pmatrix}.$$

Let us derive more explicit symbolic expressions for networks with one two-port element in the form (16). The network determinant appears as a sum of terms which can be enumerated according to the minors of the (4, 4) matrix

$$\begin{pmatrix} a_{11} & a_{12} & b_{11} & b_{12} \\ a_{21} & a_{22} & b_{21} & b_{22} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

The terms to be summed up are:

(a) $\det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \cdot$ Remainder-NW-Det,

provided the remainder network resulting from the given network graph by removal of both two-port branches is connected;

(b) $\det \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \cdot$ Remainder-NW-Det,

provided the remainder network resulting from short-circuiting of both the two-port branches is connected;

(c) $\det \begin{pmatrix} a_{12} & b_{11} \\ a_{22} & b_{21} \end{pmatrix} \cdot$ Remainder-NW-Det,

provided the remainder network resulting from short-circuiting of the first two-port branch and removal of the second two-port branch is connected;

(d) $-\det\begin{pmatrix} a_{11} & b_{12} \\ a_{21} & b_{22} \end{pmatrix} \cdot$ Remainder-NW-Det,

provided the remainder network resulting from short-circuiting of the second two-port branch and removal of the first two-port branch is connected;

(e) $\left[\det\begin{pmatrix} a_{12} & b_{12} \\ a_{22} & b_{22} \end{pmatrix} - \det\begin{pmatrix} a_{11} & b_{11} \\ a_{21} & b_{21} \end{pmatrix}\right]$

$\times \left[\, \sum_{\mu=1}^{m} (\mu\text{-th mesh admittance}) \;\cdot\; (\mu\text{-remainder-NW-Det}) \right],$

provided there exist m meshes containing both two-port branches such that the μ-th remainder network resulting from coalescing the μ-th mesh to one node is connected.

A two-port of special interest is the "nullor" defined in Fig. 2. In this case, the terms mentioned above (a)–(d) vanish. In the term (e) we have

$$\left|\det\begin{pmatrix} a_{12} & b_{12} \\ a_{22} & b_{22} \end{pmatrix} - \det\begin{pmatrix} a_{11} & b_{11} \\ a_{21} & b_{21} \end{pmatrix}\right| = 1.$$

This implies that the symbolic evaluation of network determinants for networks with nullors should start with a search for nullor meshes, i.e., meshes containing one nullator branch and one norator branch. Two conclusions can be drawn from this:

(i) A simple criterion for the solvability of the network problem: The determinant of a network whose multi-port elements are modeled by means of n nullors may be non-zero only if the n nullator branches and the n norator branches can be completed to n branch-disjoint nullor meshes.

(ii) An efficient way to symbolically evaluate the numerators of transfer functions: Augment a given network graph with an additional nullor (whose nullator branch and norator branch, respectively, connect the nodes κ_1 and κ_2 with the reference node), and then the network determinant of the augmented network is just the $(\kappa_2,\, \kappa_1)$-cofactor of the determinant of the original network. The proof is sketched in Fig. 17.

For illustrative purposes, an example system is depicted in Fig.18. The aim is to symbolically determine the transfer function $U_{\phi 5}(s)/U^e(s)$. The denominator is equal to the network determinant. The network graph (see Fig. 18(b)) contains one nullor mesh. The remainder network coalesces into the reference node, i.e., the remainder network determinant equals 1. Hence, network determinant = nullor mesh admittance $= G_1\, G_2\, G_3\, G_4$. The numerator is given by the (1,5)-cofactor. The associated network graph results from augmenting the given network graph by a nullator branch and a norator branch as depicted in Fig. 18(c). The augmented network contains two branch-disjoint nullor meshes, each with a mesh admittance equal to 1. The remainder network graph generated through coalescing both nullor meshes contains 21 trees. Thus the numerator consists of 21 terms to be summed. The term with the highest degree in s results from the branch set $\{8, 7, 6\}$ and is equal to $s^3\, C_8\, C_7\, C_6$.

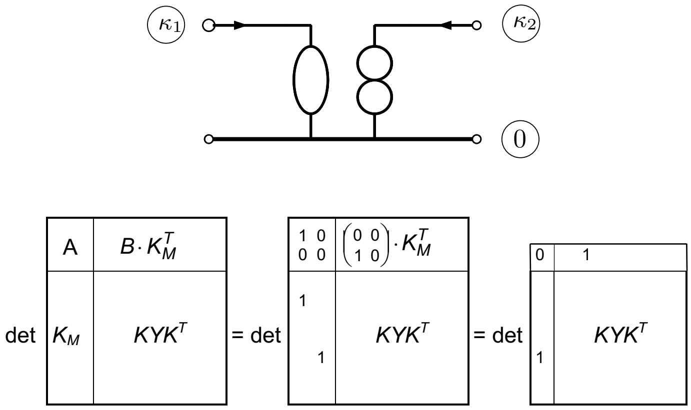


Fig. 17. The determinant of a network augmented by an added nullor as indicated is equal to the (κ_1, κ_2)-cofactor of the original network determinant.

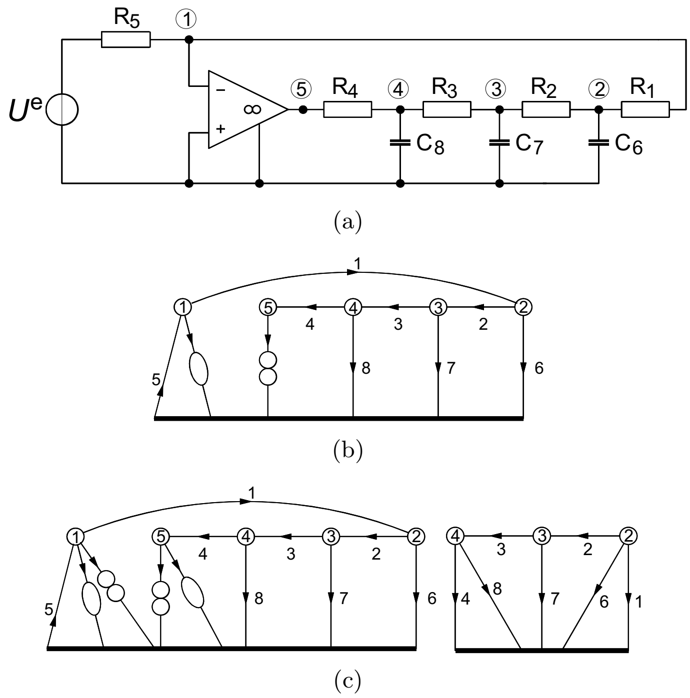


Fig. 18. (a) RC network with one operational amplifier,
(b) network graph for the network determinant,
(c) network graphs for the numerator polynomial.

3.4. Nonlinear Multi-Port Networks

Assume that the NERs are given as follows:

$$i_a = f_1(u, \dot{u}), \quad 0 = f_2(u, \dot{u}, i_b, \dot{i}_b),$$

i.e., some components of the FV-vector i are explicitly known as functions of the DV-vector u and its time derivative $\dot{u}$, whereas the other components symbolized by i_b are implicitly represented. On the basis of augmenting the network graph as depicted in Fig. 14, we are able to represent the entire FV-vector $i = (i_b^T, i_a^T)^T$ in the explicit form

$$\begin{pmatrix} i_{\text{add}} \\ i_b \\ i_a \end{pmatrix} = \begin{pmatrix} f_2(u, \dot{u}, i_b, \dot{i}_b) \\ u_{\text{add}} \\ f_1(u, \dot{u}) \end{pmatrix}. \tag{17}$$

If the aim is to carry out an MNA, the cut and circuit laws should be respectively formulated as

$$\begin{pmatrix} E & 0 & 0 \\ 0 & K_b & K_a \end{pmatrix} \begin{pmatrix} i_{\text{add}} \\ i_b + i_b^e \\ i_a + i_a^e \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{18}$$

and

$$\begin{pmatrix} u_{\text{add}} \\ u_b + u_b^e \\ u_a + u_a^e \end{pmatrix} = \begin{pmatrix} E & 0 \\ 0 & K_b^T \\ 0 & K_a^T \end{pmatrix} \begin{pmatrix} u_{\text{add}} \\ u_\phi \end{pmatrix}. \tag{19}$$

Combining (17)–(19), the network equations for the MNA are obtained as

$$\begin{aligned} f_2(K^T u_\phi - u^e, K^T \dot{u}_\phi - \dot{u}^e, i_b, \dot{i}_b) &= 0, \\ K_b i_b + K_a f_1(K^T u_\phi - u^e, K^T \dot{u}_\phi - \dot{u}^e) &= -K i^e. \end{aligned} \tag{20}$$

Next, let us consider a class of nonlinear networks which are of particular importance for computer aided design of large-scale electronic networks (see, e.g., (Günther and Feldmann, 1999) and many references cited therein). During simulation, the conservation of electric charges and magnetic fluxes should be ensured. Stated in general terms, integrated FVs, in the sequel denoted by q, and integrated DVs, in the sequel denoted by ψ, must be taken into account. For this purpose, the linear NERs $u_L = L\, \dot{i}_L$ and $i_c = C\, \dot{u}_c$ are replaced by the NERs of the form

$$u_L = \frac{\mathrm{d}}{\mathrm{d}t}\, \psi \quad \text{and} \quad i_c = \frac{\mathrm{d}}{\mathrm{d}t}\, q,$$

where ψ and q may be regarded as given nonlinear, possibly time-varying vector-valued functions

$$q = g_1(u_c, t) \quad \text{and} \quad \psi = g_2(i_L, t).$$

Then the entire set of NERs may be written as

$$u_L = \dot{\psi}, \quad i_c = \dot{q}, \quad i_a = f_1(u), \quad f_2(i_b, u) = 0,$$

where i_L is contained in i_b. On the basis of augmenting the network graph as explained above (see Fig. 14), the entire FV-vector $i = (i_c^T, i_b^T, i_a^T)^T$ can be represented in explicit form

$$i_{\text{add}} = f_2\,(i_b, u), \quad i_c = \dot{q}, \quad i_b = u_{\text{add}}, \quad i_a = f_1\,(u).$$

Taking into account both the cut and circuit laws,

$$\begin{pmatrix} E & 0 & 0 & 0 \\ 0 & K_c & K_b & K_a \end{pmatrix} \begin{pmatrix} i_{\text{add}} \\ i_c + i_c^e \\ i_b + i_b^e \\ i_a + i_a^e \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

and

$$\begin{pmatrix} u_{\text{add}} \\ u + u^e \end{pmatrix} = \begin{pmatrix} E & 0 \\ 0 & K \end{pmatrix} \begin{pmatrix} u_{\text{add}} \\ u_\phi \end{pmatrix},$$

the following equations of *charge/flux-based MNA* can be derived:

$$\begin{aligned} \dot{\psi} - K_L^T u_\phi + u_L^e &= 0, \\ K_c \dot{q} + K_b i_b + K_a f_1(K^T u_\phi - u^e) &= -K i^e, \\ f_2(i_b, K^T u_\phi - u^e) &= 0, \\ q &= g_1(u_c, t), \\ \psi &= g_2(i_L, t). \end{aligned}$$

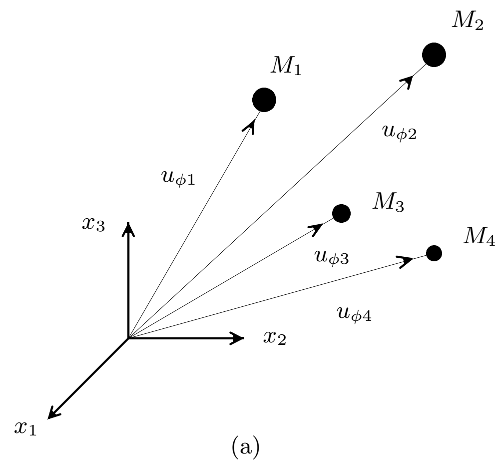
4. Multi-Dimensional Networks

By an *N-dimensional network* we understand a network model for which the FVs and DVs associated with the individual nodes and branches of the network graph are N-dimensional vectors.

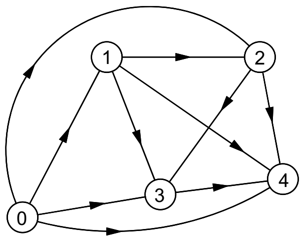
As a first example, consider the *Newtonian n-body problem* (Reibiger and Elst, 1983). The question to be answered is how n spatially lumped bodies with masses $M_1, M_2, \ldots, M_n$, driven by their mutual gravitational forces, move in the three-dimensional Euclidean space. The equations of motion are known from elementary mechanics:

$$M_j \ddot{u}_{\phi j} = \sum_{\substack{k=1 \\ k \neq j}}^{n} \gamma \, \frac{M_j \; M_k}{\| \; u_{\phi k} - u_{\phi j} \; \|^3} \, (u_{\phi k} - u_{\phi j}) \, ,$$

where γ represents the gravitational constant and $u_{\phi j}$ denotes the position of body j in a global Cartesian coordinate system. The Newtonian n-body problem can be formulated as a network problem as follows: The network graph consists of $n+1$ nodes, where the position of body j is u_{ϕ_j} and the origin of the coordinate system serves as the reference position $u_{\phi_0} = 0$, and of $n(n+1)/2$ branches connecting each node with all the others. The branch direction can be chosen such that the orientation arrow points to the higher indexed node. The case of $n = 4$ is depicted in Fig. 19.



(a)



(b)

Fig. 19. Newtonian n-body problem: (a) pictorial representation for $n = 4$, (b) network graph for $n = 4$.

The branch DVs (denoted by u_ζ) are position differences in the three-dimensional space, and the FVs (denoted by i_ζ) are three-dimensional forces. The NERs of the n branches ζ leading from reference node 0 to node κ ($\kappa = 1, 2, \ldots, n$) are given by

$$i_\zeta = M_\kappa \ddot{u}_\zeta .$$

The NERs of the remaining $n(n-1)/2$ branches ζ leading from κ_1 to κ_2, where $1 \le \kappa_1 < \kappa_2 \le n$, are given by

$$i_\zeta = \gamma \frac{M_{\kappa_1} M_{\kappa_2}}{\| u_\zeta \|^3} u_\zeta .$$

The cut and circuit laws can be respectively written as $K\,i = 0$ and $K^T u_\phi = u$. There are two main differences between this n-body network and nonlinear electrical networks: the physical dimension of $u_\zeta \cdot i_\zeta$ is energy instead of power, and the entries of the incidence matrix K are no longer $(1,1)$ but $(3,3)$ identity or zero matrices.

Next, let us enter the field of *civil engineering* and derive a network model for *structural systems* which consist of long slender members that are connected together to form a framework capable of carrying applied loads. As an example, see Fig. 20(a). In the mathematical model, specific points on the framework are indicated as the joints (which could equivalently be referred to as nodes). The segments between joints are called members (which could equivalently be referred to as branches). It is known from basic mechanics that the forces and moments may be regarded as FVs obeying a cut law, saying in mechanical terms that the equations of equilibrium are valid for any released part of the framework. On the other hand, the displacements may be regarded as DVs obeying a circuit law, saying that the equations of kinematic compatibility are valid. In contrast to one-dimensional networks, the topological connectivity properties between the joints and members are not sufficient to mathematically formulate the cut and circuit laws (see Fig. 20(b)). The geometry of the framework plays a crucial role.

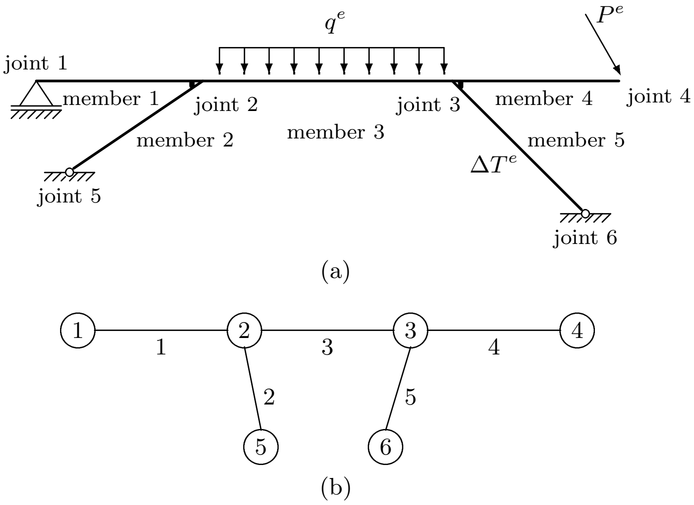


Fig. 20. Plane framework and its corresponding graph.

For the following discussion, we restrict our attention to general *frameworks in two dimensions*. The members are assumed to be prismatic (i.e., they have constant cross-section across their entire lengths) and linear elastic. First, let us derive the NERs of a single plane frame member. Figure 21 represents pictorially a released member ζ of length l_ζ. Its direction is charaterized by the angle α_ζ. The orthogonal axes x_ζ and z_ζ form a local coordinate system, where x_ζ extends from the first joint (with end loads N_1 (axial force), Q_1 (shear force), M_1 (bending moment)) to the second joint (with end loads N_2, Q_2, M_2). The angle φ indicates the rotation

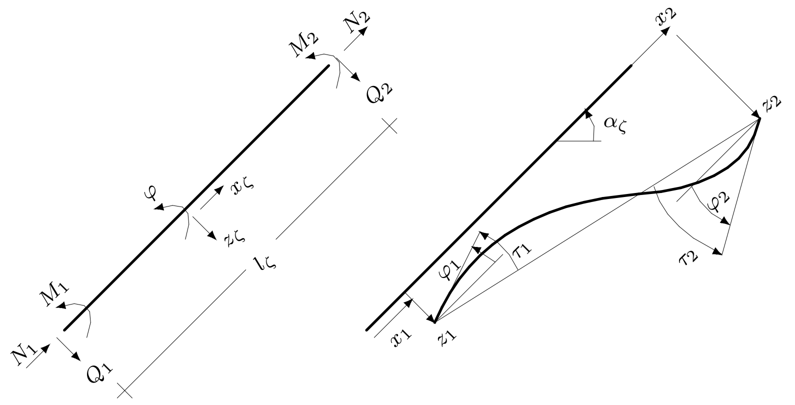


Fig. 21. Released prismatic member.

about the y-axis perpendicular to the plane. The six end loads can be arranged as a column vector $i_{\phi\zeta} = \Big(N_1\ Q_1\ M_1\ N_2\ Q_2\ M_2\Big)^T_\zeta$. Since the member ζ is thought of as being released from the framework, its end loads must obey a cut law appearing as three scalar equations of equilibrium. Therefore, only three of the six end loads are independent. Choosing $i_\zeta = \Big(N_2\ M_1\ M_2\Big)^T_\zeta$ as the three independent end loads, the cut law for the released member reads

$$i_{\phi\zeta} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1/l & -1/l \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1/l & 1/l \\ 0 & 0 & 1 \end{pmatrix}_\zeta i_\zeta =: c_\zeta\, i_\zeta. \tag{21}$$

The DVs of the absolute movements of the member ends,

$$u_{\phi\zeta} = \Big(x_1\ z_1\ \varphi_1\ x_2\ z_2\ \varphi_2\Big)^T_\zeta,$$

correspond component-wise to the FVs $i_{\phi\zeta}$. The member movement can be split into two parts: the rigid body movement which is not comprised in the load-deformation relations, and deformations which cause the member to alter its shape. Again, several choices of the three independent scalar relative movements can be realized in the local member coordinate representation. We choose $u_\zeta = \Big(\Delta x\quad \tau_1\quad \tau_2\Big)^T_\zeta$ in accordance with the chosen i_ζ. Then the kinematic compatibility conditions (circuit law for the member ζ) require

$$u_\zeta = \begin{pmatrix} -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1/l & 1 & 0 & 1/l & 0 \\ 0 & -1/l & 0 & 0 & 1/l & 1 \end{pmatrix}_\zeta u_{\phi\zeta} = c_\zeta^T u_{\phi\zeta}. \tag{22}$$

The *local member stiffness matrix* y_ζ relates the member deformations u_ζ to the member loads i_ζ,

$$i_\zeta = y_\zeta \, u_\zeta. \tag{23}$$

Neglecting shear deformations, Hooke's law yields for prismatic members the local member stiffness matrix as

$$y_\zeta = \frac{1}{l_\zeta} \begin{pmatrix} EA & 0 & 0 \\ 0 & 4EI & 2EI \\ 0 & 2EI & 4EI \end{pmatrix}_\zeta ,$$

where E, A and I denote Young's modulus of the material, the member cross-sectional area and the member cross-sectional moment with respect to the y-axis, respectively.

Together, eqns. (21)–(23) provide a complete representation of the NERs in local coordinates:

$$i_{\phi\zeta} = c_\zeta \, y_\zeta \, c_\zeta^T u_{\phi\zeta} =: y_{\phi\zeta} \, u_{\phi\zeta}. \tag{24}$$

The symmetric (6, 6) admittance matrix $y_{\phi\zeta}$ is called the *complete member stiffness matrix* in civil engineering. Note that $y_{\phi\zeta}$ is singular since rank $y_{\phi\zeta} = 3$.

The enforced member displacements, distributed member loads, and the effects of heating act as independent member DV and/or FV sources. They can be replaced by an equivalent set of concentrated loads acting on the joints at the two ends of the member (see modern texts on structural analysis, e.g., (Krätzig, 1998)). The resulting FV source vector $i_{\phi\zeta}^e$ acts "parallel" to $i_{\phi\zeta}$ $(\zeta = 1, 2, \ldots, z)$.

Up to now, each member stiffness matrix has been expressed in terms of the local member coordinates. Before combining the stiffness matrices to an overall stiffness matrix of the entire framework, it is necessary to describe all the individual member end loads and displacements, using one and the same coordinate system. As we are considering only plane frames here, the local member coordinates simply arise from the global member coordinates by rotation on the common y-axis (with angle α). Using the abbreviations $c := \cos\alpha$ and $s := \sin\alpha$, the member end loads defined by $i_{\phi\zeta}$ in local coordinates have the global coordinates

$${}_g i_{\phi\zeta} = \begin{pmatrix} c & s & 0 & & & \\ -s & c & 0 & & 0 & \\ 0 & 0 & 1 & & & \\ & & & c & s & 0 \\ & 0 & & -s & c & 0 \\ & & & 0 & 0 & 1 \end{pmatrix}_\zeta i_{\phi\zeta} =: g_\zeta \, i_{\phi\zeta}. \tag{25}$$

Note that $g_\zeta^T = (g_\zeta)^{-1}$.

Analogously, the complete end displacements can be written down in the global coordinates as ${}_g u_{\phi\zeta} = g_\zeta \, u_{\phi\zeta}$. On condition that the NERs plus independent member sources have been derived in local member coordinates in the form $i_{\phi\zeta} = y_{\phi\zeta} \, u_{\phi\zeta} + i^e_{\phi\zeta}$, the NERs can be expressed in the global coordinates as

$$ {}_g i_{\phi\zeta} = g_\zeta \, y_{\phi\zeta} \, g_\zeta^T \; {}_g u_{\phi\zeta} + g_\zeta \, i^e_{\phi\zeta} =: {}_g \, y_\zeta \;\; {}_g u_{\phi\zeta} + \; {}_g i^e_{\phi\zeta}, \tag{26} $$

where the (6, 6) matrix

$$ {}_g y_\zeta = g_\zeta \, c_\zeta \, y_\zeta \, c_\zeta^T \, g_\zeta^T \tag{27} $$

is called the *global member stiffness matrix*. Now we are able to derive the overall nodal DV equations for the entire framework by means of a suitably defined frame-node member-node incidence matrix K. To illustrate this, the $(6, 2 \times 5)$ incidence matrix K for the example framework (see Fig. 20(a)) is written out here,

$$ \begin{array}{ccc} \begin{array}{c}\text{frame}\\ \text{nodes}\end{array} & \begin{array}{cccccccccc} 1_1 & 1_2 & 2_1 & 2_2 & 3_1 & 3_2 & 4_1 & 4_2 & 5_1 & 5_2 \end{array} & \begin{array}{c}\text{member}\\ \text{nodes}\end{array} \\ \begin{array}{c} 1\\2\\3\\4\\5\\6 \end{array} & \begin{pmatrix} E & & & & & & & & & \\ & E & & E & E & & & & & \\ & & & & & E & E & & E & \\ & & & & & & & E & & \\ & & E & & & & & & & \\ & & & & & & & & & E \end{pmatrix} & = K. \end{array} \tag{28} $$

The symbols E represent $(3, 3)$ unit matrices. Let us compare the displacements $U_{\phi\kappa}$ of the frame nodes (measured in global coordinates) with the member end displacements ${}_g u_{\phi\zeta}$. The circuit law (kinematic compatibility conditions between the frame and members) requires

$$ {}_g u_\phi = K^T \, U_\phi. \tag{29} $$

The cut law (static equilibrium conditions for each node) becomes

$$ K \, {}_g i_\phi = -I^e_\phi, \tag{30} $$

where I^e_ϕ denotes the vector of independent joint loads (including support reactions) measured in the global coordinates. The combination of (26), (29) and (30) yields the desired node DV equations,

$$ K \, \langle {}_g y_1, \, {}_g y_2, \ldots, {}_g \, y_z \rangle \, K^T \, U_\phi = -K \, {}_g i^e_\phi - I^e_\phi. \tag{31} $$

Similarly to one-dimensional networks, the coefficient matrix KYK^T should not be so much regarded as a product of three matrices, but as a sum of "stamps" representing the individual members within the global admittance matrix tableau.

For the example system (Fig. 20), member 2 whose NERs are given by the (6, 6) member stiffness matrix

$$ {}_g y_2 =: \begin{pmatrix} y_2^{11} & y_2^{12} \\ y_2^{21} & y_2^{22} \end{pmatrix} $$

generates the "stamp"

$$ K \left\langle 0 \; {}_g y_2 \; 0 \; 0 \; 0 \; 0 \right\rangle K^T = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & y_2^{22} & 0 & 0 & y_2^{21} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & y_2^{12} & 0 & 0 & y_2^{11} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} . $$

The constituents of $K \, {}_g i_\phi^e$ can also be generated member-wise and then summed up. Returning to the example system, let us consider the effect of heating member 5. Heating generates axial forces $i_{\phi 5}^e = EA \, \gamma_T \Delta T^e \, (1 \; 0 \; 0 \; -1 \; 0 \; 0)^T$, whence

$$ {}_g i_{\phi 5}^e = g_5 \, i_{\phi 5}^e = EA \, \gamma_T \Delta T^e \, (c, -s, 0, -c, s, 0)_5^T =: \begin{pmatrix} {}_g i_{\phi 5,1}^e \\ {}_g i_{\phi 5,2}^e \end{pmatrix} . $$

The part of $K \, {}_g i_\phi^e$ that results from heating member 5 has the form

$$ \left(0, \; 0, \; {}_g i_{\phi 5,1}, \; 0, \; 0, \; {}_g i_{\phi 5,2}\right)^T . $$

In the foregoing discussion, we have modeled plane frameworks as three-dimensional networks. In the case of spatial frameworks we would obtain six-dimensional networks, and the entries of the incidence matrix K would be (6, 6) identity matrices or (6, 6) zero matrices. Similarly, modeling plane and spatial trusses would result in two- and three-dimensional networks, respectively.

5. Conclusion

This paper constitutes an attempt to overview the formulation and solution of network equations. In this presentation, the historic development of the concept of network modeling has been emphasized. Beyond the usual electrical network applications, network models can describe a wide variety of other real-world systems. Topological properties of the underlying network graphs provide the key to a thorough understanding of one-dimensional networks. It was shown that the network element relations can always be explicitly formulated by means of an augmented network graph. General topological rules for symbolic solution of multi-port network equations were derived. In the case of multidimensional networks, the analyst has to cope with the geometry of the systems. This has been exemplified for plane framed structures commonly occurring in civil engineering.

References

Bellert S. (1962): *Topological analysis and synthesis of linear systems.* — J. Franklin Inst., Vol.274, No.6, pp.425–443.

Branin F.H.jr. (1966): *The algebraic-topological basis for network analysis and the vector calculus.* — Proc. Symp. Generalized Networks, Brooklyn, pp.453–491.

Bott R. and Duffin R.J. (1953): *On the algebra of networks.* — Trans. AMS, Vol.74, pp.99–109.

Cauchy A.L. (1815): *Mémoire sur les fonctions qui ne peuvent obtenir que deux valeurs.* — J. l'Ecole Polytechnique, Vol.10, pp.29–97, reprinted in Oeuvres Complètes, Serie II, Tome I, pp.91–169.

Cayley A. (1857): *On the theory of the analytical forms called trees.* — Philos. Mag., Vol.4, No.13, pp.172–176.

Chen W.K. (1967): *Topological network analysis by algebraic methods.* — Proc. IEE, Vol.114, No.1, pp.86–88.

Chua L.O. and Lin P.-M. (1975): *Computer Aided Analysis of Electronic Circuits—Algorithms and Computational Techniques.* — New Jersey: Prentice-Hall.

Coates C.L. (1959): *Flow-graph solutions of algebraic equations.* — IEEE Trans. Circuit Theory, Vol.6, No.2, pp.170–187.

Dmitrischin R.W. (1969): *Rasčet linejnych cepej s metodom rasširennych strukturnych čisel.* — Isv. Vuzov SSSR, Radioelektreonnika, Vol.12, No.8, pp.806–813 (in Russian).

Duffin R.J. (1959): *An analysis of the Wang algebra of networks.* — Trans. AMS, Vol.93, pp.114–131.

Firestone B.F. (1933): *A new analogy between mechanical and electrical systems.* — J. Acoust. Soc. Am., Vol.4, pp.249–267.

Gieben G. and Sansen W. (1991): *Symbolic Analysis for Automated Design of Analog Integrated Circuits.* — Boston: Kluwer.

Günther M. and Feldmann U. (1999): *CAD-based electric-circuit modeling in industry, Part I and II.* — Surv. Math. Ind., Vol.8, pp.97–129 and 131–157.

Hähnle W. (1932): *Die Darstellung elektromechanischer Gebilde durch rein elektrische Schaltbilder.* — Wiss. Veröff. a. d. SIEMENS-Konzern, Vol.11, No.1, pp.1–23.

Jacobi C.G.J. (1841): *De formatione et proprietatibus determinantium.* — Crelle's J. f. reine und angew. Mathem., Vol.22, pp.285–318.

Kirchhoff G. (1845): *Über den Durchgang eines elektrischen Stromes durch eine Ebene, insbesondere durch eine kreisförmige.* — Poggendorfs Annalen der Physik und Chemie, Vol.64, pp.497–514.

Kirchhoff G. (1847): *Über die Auflösung der Gleichungen, auf welche man bei der Untersuchung der linearen Verteilung galvanischer Ströme geführt wird.* — Poggendorfs Annalen der Physik und Chemie, Vol.72, pp.497–508.

König D. (1936): *Theorie der Endlichen und Unendlichen Graphen.* — Leipzig: Akademische Verlagsgesellschaft.

Krätzig W.B. (1998): *Tragwerke 2. 3. Aufl.* — Berlin: Springer,

Kron G. (1939): *Tensor Analysis of Networks.* — New York: Wiley.

LeCorbeiller P. (1950): *Matrix Analysis of Networks.* — New York: Wiley.

Lin P.-M. (1991): *Symbolic Network Analysis.* — Amsterdam: Elsevier.

Mason S.J. (1953): *Feedback theory; some properties of signal-flow-graphs.* — Proc. IRE, Vol.41, No.9, pp.1144–1156.

Maxwell J.C. (1870): *On reciprocal figures, frames and diagrams of forces.* — Trans. Roy. Soc. Edinb., Vol.26, pp.1–40.

Maxwell J.C. (1882): *Electricity and magnetism I, 3rd Ed,* (Ch. VI and Appendix). — Oxford: Clarendon Press.

Mayeda W. and Seshu S. (1965): *Generation of trees without duplication.* — IEEE Trans. Circuit Theory, Vol.12, No.2, pp.181–185.

Poincaré H. (1895): *Analysis situs.* — J. de l'Ecole Polytech., Vol.2, No.1, pp.1–121.

Reibiger A. and Elst G. (1983): *Network theoretic interpretation of multibody systems.* — Acta Polytechnica Prague, Vol.8, pp.109–129.

Reinschke K.J. (1988): *Multivariable Control—A Graph-Theoretic Approach.* — Berlin: Springer-Verlag.

Reinschke K. and Schwarz P. (1976): *Verfahren zur Rechnergestützten Analyse Linearer Netzwerke.* — Berlin: Akademie-Verlag.

Roth J.P. (1959): *An application of algebraic topology: Kron's method of tearing.* — Quart. Appl. Math., Vol.17, No.1, pp.1–24.

Seshu S. and Reed M.B. (1961): *Linear Graphs and Electrical Networks.* — Reading: Addison-Wesley.

Synge J.L. (1951): *The fundamental theorem of electrical networks.* — Quart. Appl. Math., Vol.9, No.2, pp.113–127.

Tellegen B.D.H. (1948): *The gyrator, a new electric network element.* — Philips Res. Rep., Vol.3, pp.81–101.

Tellegen B.D.H. (1953): *A general network theorem with applications.* — Proc. Inst. Radio Engrs. Australia, Vol.14, pp.265–270.

Ting S.L. (1935): *On the general properties of electrical network determinants.* — Chinese J. Phys., No.1, pp.18–40.

Trent H.M. (1955): *Isomorphismus between oriented linear graphs and lumped physical systems.* — J. Acoust. Soc. Am., Vol.27, No.3, pp.500–527.

Trochimenko J.K. (1972): *Metod obobščennych čisel i analiz linejnych cepej.* — Moscow: Izd. Sov. Radio (in Russian).

Tsai C.T. (1939): *Short cut methods for expanding determinants involved in network problems.* — Chinese J. Phys., No.3, pp.148–181.

Veblen O. (1916): *Analysis situs.* — (The Cambridge Colloquium, Part II, Lectures 1916); New York: AMS, 1922.

Wang K.T. (1934): *On a new method for the analysis of electrical networks.* — Nat. Res. Inst. Eng. Acad. Sinica, Mémoir No.2, pp.1–11.

Weyl H. (1923): *Repartición de corriente en una red conductora (Introducción al análisis combinatorio).* — Revista Matematica Hispano-Americana, Vol.5, pp.153–164.

Weyl H. (1924): *Análisis situs combinatorio (continuación).* — Revista Matematica Hispano-Americana, Vol.6, pp.1–9 and pp.33-41.