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KNOWLEDGE, VAGUENESS AND LOGIC

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Dedicated to
Professor Zdzisław Pawlak

The aim of the paper is to outline an idea of solving the problem of the vagueness of concepts. The starting point is a definition of the concept of vague knowledge. One of the primary goals is a formal justification of the classical viewpoint on the controversy about the truth and object reference of expressions including vague terms. It is proved that grasping the vagueness in the language aspect is possible through the extension of classical logic to the logic of sentences which may contain vague terms. The theoretical framework of the conception refers to the theory of Pawlak's rough sets and is connected with Zadeh's fuzzy set theory as well as bag (or multiset) theory. In the considerations formal logic means and the concept system of set theory have been used. The paper can be regarded as an outline of the logical theory of vague concepts.

Keywords: vague knowledge, fuzzy sets, rough sets, vague sets, formal logic

1. About Vagueness

The problem of non-precise or vague knowledge, its representation and conceptualization already has a long tradition and is still regarded as one of the most worthy of discussion. It grew out of the philosophical reflection on vague concepts of colloquial language and on the value of this language for philosophy and science. In philosophy the problem found expression mainly in arguments on the value of cultivating science in a precise way, free from vague concepts, or in a way that is not always clear and sharp. The former tendency in approaching vagueness, started by Russell (1929), dominated till the 1960s, while the latter began to be widespread owing to philosophers of language such as Wittgenstein (1953) and Austin (1961), and philosophers of science such as Kuhn (1962) and Polanyi (1962).

The presence of vague terms in natural language has led to reflections of logicians and linguists as well. In logic the subject of vagueness has appeared for a long time in discussions and arguments on meaningfulness, truth and object reference of expressions like:

$$\text{John is YOUNG.} \tag{1}$$

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The question is asked: Can this expression be used correctly if John is 25 years old? The meaning of the term 'YOUNG' used in (1) is such that we are not able to give a correct answer to the question, since we cannot decide whether or not a 25-year-old person is young. And although there are positive examples of the use of this expression in relation to persons under 18, and so there exist negative examples of the use of the expression in relation to persons over, let us say 35, next to these non-controversial examples of the use of the expression we encounter situations when we do not know whether the expression is true or false. So perhaps sentence (1) has no logical value as such? And what about the principle of excluded middle? Moreover, what does the expression refer to? Does the word 'YOUNG', which is included in it and which will be functioning throughout the paper as an abbreviation for the name 'A YOUNG MAN', denote a definite set of persons? Does a 25-year-old John belong to it? Can we determine this set?

The problems touched upon belong to logical semantics and are connected with the understanding of vagueness as the existence of bordering and non-bordering cases, called the *boundary* cases. The incapability of giving answers to the above questions does not result from the state of our knowledge but from the meaning of expression (1). The solution to the problems discussed has not led to satisfying results, although the problems connected with the issue have also become current because of the logical analysis of the language of empirical sciences (Przełęcki, 1964) and the reasearch of computer scientists interested in artificial intelligence. The aim of this research, begun in the work of Zadeh (1965), the originator of fuzzy set theory, is to apply computers to make use of vague information in much the same way as it is done by our brain; it has to do with the possibility of using the computer to draw conclusions employing vague concepts. Zadeh's theory allows the statement of the belonging of an object of a given reality (e.g. our John) to a set (in the example, the set of YOUNG people) only to a certain degree, which is described by a number from the open interval $(0, 1)$. The gradation of the belonging of an element to a set is here a characteristic feature of the corresponding vague concept. This element (in our example, John) can be called a *quasi-designatum* of the vague term determining this concept (in the example, the term 'YOUNG'), and the term itself—a *quasi-name*. The extension of this name is different from the situation when we speak of an ordinary name extension or a sharp concept extension. It is a *fuzzy set* and at the same time the *extention* of a vague concept corresponding to this quasi-name. Fuzzy set theory leads to non-classical logic, in which classical values of truth and falseness are substituted with values from the closed interval $[0, 1]$.

Thus the 1960s brought along the acceptance of the phenomenon of vagueness in language and science on the one hand, and new semantics or computer-science-oriented solutions to the problem of vagueness itself on the other. The results of these solutions, however, proved to be less than satisfying. Sufficient evidence of this are the new attempts and research in this area conducted in different centres, including Poland (Muszyński, 1988). The precursor of the formal approach to the problem of vagueness and, generally, research into vagueness in Poland is Kubiński (1958), a logician who built a non-classical vague names calculus.

It should be stressed here that in 1982, a formal theory complementary to Zadeh's theory was created under the name of the rough set theory. It was built by the Polish computer scientist Z. Pawlak (Pawlak, 1982; 1991). Pawlak's theory takes a non-numerical approach to the issue of vagueness, as opposed to the quantitative characteristics of the vagueness phenomenon by Zadeh. It touches on qualitative aspects and is based on the idea of a set approximation by a pair of sets called the *lower* and the *upper approximations of the set*. These approximations define the *positive* and *negative extension* of a vague concept, respectively. Pawlak's approach is connected with a reference to the concept of a cognitive agent's knowledge about the objects of the investigated reality (Pawlak, 1992). So at the same time it has an epistemic character. This knowledge is determined by a system of concepts, which is determined by a system of their extensions. When an extension of a concept is vague, it is determined by the *rough set* understood as the family of sets with the same lower and upper approximations.

Pawlak's ideas had a profound influence on the author's building of a logical theory of vague concepts, the outline of which is presented in the further part of this paper. It was sketched at the Sixth Polish Philosophical Convention in Toruń in 1995 (Wybraniec-Skardowska, 1996). The creation of this theory was also inspired by the ideas or opinions of all the researchers mentioned in this part of the paper.

2. Characteristics of the Approach to the Problem of Vagueness

Neither Zadeh's nor Pawlak's conceptions solve numerous problems connected with the conceptualization of vague knowledge, understood here as the formation of a system of concepts relevant to objects from the reality recognized by an agent (a man in general), and more precisely—as the assignment of a system for the denotation of these concepts. The source seems to be in the lack of a scientific description of logical bases for a theory of vagueness, working out a certain formalism that could be provided only by logic together with set theory. The extension of applying formal-logical means must be based, however, on some general assumptions regarding the way of approaching the phenomenon of vagueness itself.

The way of describing a logical theory of vagueness proposed here is not, of course, aimed at relating the mechanisms of its origination. A theory of vagueness describes what vagueness is, and not only as a language phenomenon. Such a theory cannot, obviously, reflect all approaches to the phenomenon of vagueness. It can be attempted, however, to provide the possible general characteristics of the concept of vagueness, providing also answers to basic questions rankling philosophers and language logicians for a long time, and asked in the first part of this paper.

The suggested conception, which is at the same time an attempt to theoretically describe a model for the conceptualization of vague concepts, is based on the fundamental assumption that *vagueness is a subjective feature* depending on the agent recognizing reality or using the language describing this reality. It deals with the problem of vagueness in four aspects, and it considers the following indicators taking into account these aspects:

- vague knowledge — epistemic aspect,
- vague object — ontological aspect,

- vague name — semiotic aspect (semantic and pragmatic), and
- vague concept — logical aspect.

They are correlated with one another—each previously mentioned factor influences the following one, and vice versa.

The starting point, determining the direction of consideration, is the concept of the agent's knowledge about the objects of reality, and specifically, a certain fragment of it. Figuratively speaking: knowledge shapes the language, and the language influences knowledge; the language, however, being secondary to knowledge, cannot reflect all our knowledge.

That is why is seems justified to refrain, in the foreground, from the linguistic and especially the semantic level of consideration and to set the formal theory on the cognitive level. Such an approach allows the existence of non-verbal knowledge and refers to the views of the philosophers previously mentioned, such as Wittgenstein, Austin, Polanyi, Kuhn, and others (cf. Marciszewski, 1994). The phenomenon of vagueness is treated here as an inseparable feature of the method of recognizing reality.

3. The Concept of Knowledge; Vague Knowledge

The character of the paper does not allow for a precise definition of the concept of knowledge in general, and especially the concept of vague knowledge. Thus for the needs of this paper we will limit ourselves only to not very accurate definitions, reflecting, however, the essential intuitions connected with these concepts (Wybraniec-Skardowska, 1994a; 1994b; 1997; 1998).

Each recognized fragment of reality, relatively stable in some respects (established aspects) which can be understood here as 'attributes' in information systems (Pawlak, 1983), will be presented as the ordered system

$$\Re = \langle U, R_1, R_2, \ldots, R_n \rangle,$$

consisting of a non-empty universe U of the objects of reality and all one- or multi-argument relations R_i $(i = 1, 2, \ldots, n)$, in connection with which these objects constitute a unified whole which is of interest to us for some reasons.

Reality $\Re$ *is objective in relation to cognition.* One-argument relations, understood as properties of the objects of universe U, are formally identified with non-empty subsets U and regarded as sets of the universe objects possessing these properties. They do not have to be here something single. They can be properties consisting of single properties. In Pawlak's information systems (1983; 1991) each property is characterized by a single value of an attribute, or by a finite set of the values of this attribute. Each such value defines uniquely a one-argument relation as a set of universe objects possessing the same attribute value, or possessing at least one value from the finite set of values of this attribute. Multi-argument, k-argument relations $(k > 1)$, are understood as relationships among objects of the universe and are formally identified with subsets of the Cartesian product U^k.

In what follows, we will frequently refer to the following example of reality:

Example 1. Assume that I is the group of people born in Warsaw after 1970 who emigrated to the United States. Let us distinguish a fragment of reality $\Im$, whose universe is this group of people, according to the two following aspects: AGE and LIVING TOGETHER IN A GIVEN STATE. Reality $\Im$ consists then of the group of people I, of all the properties that might be applied to these people with respect to their AGE (e.g. being under 18, being exactly 25, not being under 20, and not being over 25, etc.), and of all two-argument relations in which these persons could appear due to LIVING TOGETHER IN A DEFINITE STATE (e.g. living together in the State of New York, living together in the State of Illinois, living together in the State of California, etc.). ♦

The *agent's knowledge* about universe U of reality $\Re$ is specified by the *unit knowledge (information)* about the particular objects of this universe, in relation to all of its particular relations R_i $(i = 1, 2, \ldots, n)$ of $\Re$.

Unit information (knowledge) about object u of the universe U with respect to a definite one-argument relation R_i (property R_i) distinguished by aspect a characterizing reality $\Re$ assigns object u possible, from the point of view of the agent, relations (properties) or sets of relations (sets of properties) of reality $\Re$ distinguished in it by the aspect a.

Unit information (knowledge) about object u of the universe U with respect to a definite k-argument relation R_i $(k > 1)$ distinguished by aspect b characterizing reality $\Re$ assigns object u the possible, from the point of view of the agent, $(k-1)$-element ordered systems of objects from U or sets of such object systems, with which u could remain in any relation of reality $\Re$ distinguished by aspect b.

The knowledge of the agent is *exact*

1. in the case of a one-argument relation (property) R_i:
   when the agent assigns this property, i.e. R_i (the set of all the objecs of universe U possessing this property) to object u, if it possesses this property $(u \in R_i)$, and the empty set, if u does not possess this property, and

2. in the case of a k-argument relation R_i $(k > 1)$:
   when the agent assigns object u a set of all the ordered systems of objects from U, with which it remains in relation R_i.

The *exact unit information* about the objects of the universe is defined as follows:

1. when we consider it with respect to a one-argument relation (property) R_i:
   it is a function $\vec{R}_i : U \to \mathcal{P}(U)$ such that

$$\vec{R}_i(u) = \begin{cases} R_i & \text{if } u \in R_i, \\ \emptyset & \text{otherwise,} \end{cases} \tag{2}$$

2. when it is considered with respect to a k-argument relation R_i $(k > 1)$:
it is a function $\vec{R}_i : U \to \mathcal{P}(U^{k-1})$ such that

$$\vec{R}_i(u) = \{(u_1, u_2, \ldots, u_{k-1}) \in U^{k-1} : (u_1, u_2, \ldots, u, \ldots, u_{k-1}) \in R_i\} \tag{3}$$

for any object $u \in U$; $\mathcal{P}(U)$ is the family of all subsets of U, and $\mathcal{P}(U^{k-1})$ is the family of all subsets of the Cartesian product of $k-1$ sets of U.

Let us observe that the pieces of exact unit information R_i are simultaneously objective components of the real knowledge about objects of the universe of reality $\Re$ with respect to relation R_i functioning in it. The sets $\vec{R}_i(u)$, for $u \in U$, will be called the *images* of u with respect to relation R_i. They are determined uniquely when our knowledge is exact.

Example 2. When $\Im$ is a reality and John belongs to its universe I, then the exact unit information about John with respect to the one-argument relation (property) of being exactly 25 assigns this relation (property) to John if John is actually 25, or the empty set if John's age is different. The exact information about John with respect to the two-argument relation of living in the state of New York assigs him the class of all the persons from I who live in this state together with him if John lives in this state, and the empty set if John lives in a different state or none of the persons born in Warsaw after 1970 lives in the State of New York now. ⧫

Unit information about object u of reality $\Re$ with respect to a one-argument relation R_i (property) is *empty* when an agent recognizing reality $\Re$ does not assign object u either the R_i property or any other property regarding the same aspect that characterizes reality $\Re$; in the case when we consider a k-argument relation R_i $(k > 1)$, the information is *empty* when an agent does not assign object u any of such ordered $(k-1)$-element systems of objects from U, which are in relation R_i with it or in any other relation considering the same aspect that characterizes reality $\Re$, or object u is simply not in any relation determined by the aspect which allowed us to distinguish relation R_i.

Unit knowledge about object u is empty when the agent knows nothing about the value of function $\vec{R}_i$ for u. The value of this function is then an unknown quantity for the agent, with an empty extension. The equation

$$\vec{R}_i(u) = x, \tag{4}$$

where x is an unknown quantity, does not have, from the agent's point of view, a solution. Such an equation will be called the *equation of the agent's ignorance.*

Example 3. Information about John, an emigrant with respect to the property of being 25, is empty for us when we are not able to say anything about John's age. Knowledge about John with respect to the relation of living together in the state of New York is empty for us when we do not know at all whether John emigrated to the States. ⧫

Unit knowledge of the agent about object u with respect to relation R_i can be non-empty but *indefinite*. Then the dependence (2) or (3) is non-discoverable for the agent. This is possible when the agent assigns object u—non-uniquely possible from the agent's point of view, in different situations different, images $\vec{R}_i(u)$ of this object with regard to relation R_i. Equation (4) has at least two solutions for him, but there can be a lot of possibilities of indicating presumable solutions. Such a situation, for a one-argument relation R_i, is illustrated by the diagram of Fig. 1. All the relations (properties) indicated in this diagram are the properties possible from the point of view of the agent recognizing reality $\Re$ which he can assign to object u. The tree branches illustrate situations in which the agent considers individual possible properties of object u.

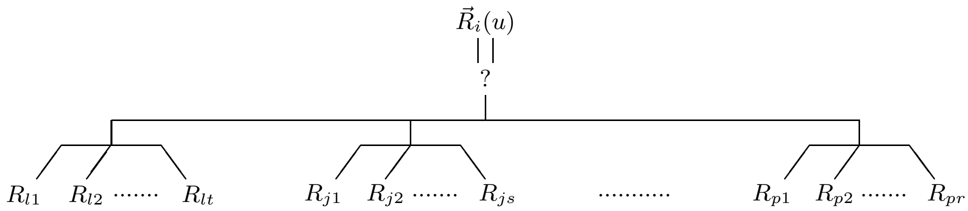

Fig. 1. Many possibilities indicating presumable solutions.

In the case when relation R_i is multi-argument, the situation of knowledge indefiniteness can be illustrated by the diagram in which the tops of the trees are substituted by the symbols of presumable images of relation R_i.

When our knowledge about object u is *indefinite*, eqn. (4), characterizing the agent's ignorance, allows for many different solutions. More formally, unit information about object u is *indefinite* for the agent with regard to relation R_i when the agent assigns object u a family consisting of families of the images of element u, with respect to relation R_i, which are possible from the agent's point of view.

In order to simplify further consideration, we will limit ourselves to the situation when relation R_i is a one-argument relation (property). Then the family in *indefiniteness* mentioned above can be described by (see Fig. 1)

$$\mathcal{F}(R_i, u) = \big\{\{R_l\}_{l\in L},\ \{R_j\}_{j\in J}, \dots, \{R_p\}_{p\in P}\big\}.$$

Families $\mathcal{V}_s = \{R_l\}_{l\in M}$, where $s < \xi$ and ξ is an integer greater than 1, are here non-empty sets of relations (properties) of reality $\Re$, possible from the point of view of the agent. These relations are chosen from among the ones characterizing objects of the universe of $\Re$ in a definite aspect, which are assigned to object u by the agent because of this very aspect. Let us note that if we treat this aspect as an attribute in an information system, a family consisting of all possible, from the point of view of the agent, families of elementary classes or families of unions of such classes corresponds to indefinite information.

Thus the *indefinite information* about object u with respect to relation R_i is determined by the following relationship: There exists $s < \xi$ $(\xi > 1)$ such that

$$\vec{R}_i(u) \in \mathcal{V}_s \in \mathcal{F}(R_i, u) = \{\mathcal{V}_s\}_{s<\xi}.$$

The indefinite information about u is thus the assignment to object u of any of the sets of relations (properties) allowed by it. These sets of relations (properties) constitute the possible solution of (4) in definite situations.

As for family $\mathcal{F}(R_i, u)$, we assume that relation (property) R_i is included in at least one relation (is part of at least one property) of each of the families $\mathcal{V}_s$, and we call it the *field of indefiniteness of knowledge of the agent with respect to relation* R_i.

If the field of indefiniteness of such knowledge consists of at least two non-single-element families of relations, we call the agent's knowledge about object u the *completely vague knowledge* for the agent.

Families of relations (properties) $\mathcal{V}_s$ $(s < \xi,\ \xi > 1)$ determining the field of indefiniteness of the agent's knowledge with respect to relation (property) R_i are called the *sets approximating this field* with respect to relation (property) R_i. We assume that each approximating set $\mathcal{V}_s$ is the covering of universe U. The sum of all the relations (properties) of such a set gives U (a complete property), i.e.

$$\bigcup \mathcal{V}_s = U \quad \text{for any} \quad s < \xi, \ \ \xi > 1.$$

Each approximating set $\mathcal{V}_s$ is the set of possible solutions of the equation of the agent's ignorance (4).

If the approximating set $\mathcal{V}_s$ is at least two-element, we call it a *vague set* adequate to the agent's knowlege about object u with regard to relation (property) R_i. Object u itself can be called the *vague object* for the agent with regard to relation (property) R_i, and the knowledge about it the *relatively vague knowledge with respect to this relation.*

Therefore the relatively vague knowledge about object u with respect to R_i is determined by the relationship

$$\vec{R}_i(u) \in \mathcal{V}_s = \{R_m\}_{m\in M},$$

where $\mathcal{V}_s$ is an established vague set and at the same time the only set of solutions to (4).

Unit knowledge about object u with respect to relation R_i is *relatively vague* for the agent when the agent assigns object u a vague set $\mathcal{V}_s$ as a set of the possible, from the agent's point of view, images of object u with respect to R_i. Such knowledge is then the assigning of any of the relations (properties) of the set $\mathcal{V}_s$ to object u.

We can call one-element approximating sets (these are sets consisting of a single relation, thus single sets) *sharp sets* for the agent and identify each such set family with its element (relation-set), thus with a set in the ordinary sense, called the *exact set.*

Each approximating set $\mathcal{V}_s$ has the greatest lower bound and the least upper bound in the family of all subsets $\mathcal{P}(U)$ of the universe U, with respect to inclusion. So, for each set $\mathcal{V}_s$ there exists the greatest subset of universe U (property in reality $\Re$), which is included in each set-relation (is part of each property) determining this set. In particular, each vague set (a family of at least two-set properties) has the greatest lower bound in $\mathcal{P}(U)$. Similarily, for each approximating set $\mathcal{V}_s$ there exists the smallest subset of universe U (property of reality $\Re$) including all set-relations determining this set. In particular, each vague set has the least upper bound. When an approximating set is a sharp set, its bounds are equal and identical with the relation (property) that determines it.

The difference between the greatest lower bound and the least upper bound of a given approximating set is called the *boundary* of this set. An approximating set is a sharp set when its boundary is the empty set; otherwise, this set is obviously a vague set.

The greatest lower bound and the least upper bound of an approximating set are called the *lower limit* and the *upper limit*, or the *lower approximation* and the *upper approximation* of the rough set, respectively. These limits determine respectively the *positive* and the *negative* region of the agent's knowledge about the object.

Example 4. When, as before, $\Im$ is the recognized reality, then the completely vague knowlege about John, the 25-year-old emigrant, with respect to the property of being 25 years old, i.e. formally, with regard to the set Y_{25} of all people living in the States in 2000 who were born in Warsaw after 1970, may assign John any of the following example families of sets (properties):

$$\mathcal{Y}_1 = \{Y_{<19}, Y_{<25}, Y_{\leq 30}\}, \qquad (Y_{25} \subseteq Y_{\leq 30}),$$

$$\mathcal{Y}_2 = \{Y_{<19}, Y_{<20}, Y_{<21}, \ldots, Y_{<29}, Y_{\leq 30}\}, \quad (Y_{25} \subseteq Y_{<26} \text{ and } \ldots\ Y_{25} \subseteq Y_{\leq 30}),$$

$$\mathcal{Y}_3 = \{Y_{16}^{20}, Y_{18}^{25}, Y_{20}^{30}\}, \qquad (Y_{25} \subseteq Y_{18}^{25} \text{ and } Y_{25} \subseteq Y_{20}^{30}),$$

$$\mathcal{Y}_4 = \{Y_{18}^{20}, Y_{19}^{25}, Y_{20}, Y_{20}^{25}\}, \qquad (Y_{25} \subseteq Y_{19}^{25} \text{ and } Y_{25} \subseteq Y_{20}^{25}),$$

which define the field of the vagueness of our knowledge as family $\mathcal{F}(Y_{25}, John) = \{\mathcal{Y}_1, \mathcal{Y}_2, \mathcal{Y}_3, \mathcal{Y}_4\}$ consisting of vague sets $\mathcal{Y}_t$ $(t = 1, 2, 3, 4)$. Each vague set $\mathcal{Y}_t$ $(t = 1, 2, 3, 4)$ is composed of some sets given in the form: Y_k, $Y_{<k}$, $Y_{\leq k}$, Y_k^l, being non-empty sets, consisting of all the people born in Warsaw after 1970 who emigrated to the USA, and who in 2000 were respectively k years old, less than k years old, not more than k years old, or were from k to l years old. Note that in each vague set $\mathcal{Y}_t$ $(t = 1,\ 2,\ 3,\ 4)$ we can distinguish at least one set (property) of which the set Y_{25} (property of being 25) is a subset (is a part). The lower limits $\mathcal{Y}_t$ are respectively sets $Y_{<19}$, $Y_{<19}$, Y_{20}, Y_{20}, and the upper limits are respectively sets $Y_{\leq 30}$, $Y_{\leq 30}$, Y_{16}^{30}, Y_{18}^{25}, whose coverings are the subsequent sets $\mathcal{Y}_t$. ♦

As presented in the example, the field of indefiniteness shows to some extent the 'situational' diversity of completely vague knowledge. Situations in which the agent assigns John the vague sets $\mathcal{Y}_1$ and $\mathcal{Y}_2$ differ considerably from the situation when the agent assigns him the vague sets $\mathcal{Y}_3$ and $\mathcal{Y}_4$. In the first case, however, it is as if the degree of vagueness were smaller, because the vague set's limits are the same. This does not mean, however, that relatively vague knowledge of the agent about John is the same in both the situations—different vague sets correspond to it in different situations.

The above example shows that it is as if there could exist a smaller degree of vagueness of the agent's knowledge, in similar situations, when the limits of vagueness are the same. Completely vague knowledge can then simply be called the *vague knowledge*.

4. Logical Conceptualization of Vague Knowledge

The concepts of knowledge or unit information about an object of reality introduced in the preceding part of the paper were made relative to an agent treated as a psychological or a sociological entity—to a man or a group of people. Such an agent was therefore in a way isolated from the functioning of a language or another system of representing knowledge, in which the agent expresses his knowledge in the same way as its average user. The knowledge of such an agent is then made objective by pragmatic and semantic rules of the language or any other system of representing knowledge. It influences the knowledge of every potential user of this system, and it is being influenced by the knowledge of any other user of this kind (Pelc, 1971).

Without defining precisely the previously considered difference between an actual agent and a potential one, the latter will be acknowledged as a *logical subject* and we shall refer only to it in our further consideration, speaking of an agent.

We are going to speak further about the knowledge of an agent, represented in a system of signs, e.g. in a natural language or in an information system, treated here as a system of representing knowledge (Pawlak, 1983; 1991). Such knowledge, respecting the knowledge coded in a given system of signs, consists of the 'resultant' knowledge of information (knowledge) of particular users of this system. It can be characterized by relationships similar to those given earlier in the paper, without relativizing the concepts introduced there to a definite agent.

Completely vague unit knowledge or vague knowledge is then represented in a language by an atomic sentence containing a vague term as a predicate (a *completely vague term* or a *vague term*) to which a logical concept corresponds, here called a *completely vague concept* or a *vague concept*, respectively. To this concept (term), depending on the degree of vagueness of knowledge, corresponds a non-one-element indefiniteness field called the *denotation of a completely vague concept (term)* or the *denotation of a vague concept (term)*, respectively.

When unit knowledge is relatively vague, it is represented in a language by an atomic sentence with a vague term (a *relatively vague term*) to which a logical concept corresponds, here called *relatively vague*. To this concept (term) corresponds a vague set with appropriate limits. We shall call this set the *denotation of a relatively vague concept (term)*.

Example 5. Let us consider, like before, reality $\Im$ whose universe contains 25-year-old John. The conceptualization of knowledge about John will be one of the following choices:

1. A completely vague concept YOUNG whose denotation is the field of indefiniteness, characterized like family $\mathcal{F}(Y_{25}, \text{John})$, and consisting of many sets, not necessarily of the same limits (as, e.g. $\mathcal{Y}_1$, $\mathcal{Y}_3$, $\mathcal{Y}_4$);

2. A vague concept $YOUNG^l_k$ whose denotation is the field of indefiniteness consisting of vague sets with the same lower limit with index k and the same upper limit with index l (it is therefore characterized like family $\mathcal{F}'(Y_{25}, \text{John}) = \{\mathcal{Y}_1, \mathcal{Y}_2\}$ consisting of vague sets with the same lower limits $Y_{<19}$ and same upper limits $Y_{\leq 30}$; superscript l at $YOUNG$ is here '≤ 30' and subscript $k =$ '< 19');

3. A relatively vague concept YOUNG^l_k whose denotation is the concrete vague set of the type Y^l_k belonging to the denotation of the concept $YOUNG^l_k$, e.g. family $\mathcal{Y}_2$, which is an element of the field $\mathcal{F}'(Y_{25}, \text{John})$[1]. ♦

It is as if in this way we presented three different ways or levels of grasping the vagueness of concepts. It can be said that to each of them correspond different *meaning families* which are their denotation (Wittgenstein, 1953; Pawłowski, 1978; 1988; Koj, 1969). It is obvious that the essential differences in understanding vagueness are revealed in the essential meaning differences of vague terms. These meanings do not seem to be made sufficiently clear by the users of a given system of signs, especially by the users of a language, who use vague terms. As a result, answers to the questions asked at the beginning are for them ambiguous or undecidable.

There is no correct answer, given the character of a logical sentence, to the question whether the expression

$$\text{John is YOUNG} \tag{5}$$

is used correctly when John is 25, because the answer would have to be either positive or negative, and sentence (5) a logical sentence. The expression (5) is not, however, such a sentence, being a sentence of an indefinite or an undetermined meaning. The vague termen 'YOUNG' occurring in it is not really a name in a logical sense. It can be treated as a *quasi-name*, and an ambigious one, unless we determine the level of its vagueness. It can be completely vague, vague or relatively vague depending on whether a completely vague, vague or relatively vague denotation corresponds to it.

Example 6. Consider the reality $\Im$ whose universe contains John. *Completely vague knowledge* with respect to AGE about 25-year-old John can be represented by the

[1] The font style used in Cases 1, 2 and 3 corresponds to the particular concepts (terms): completely vague (PLAIN), vague (*ITALIC*) and relatively vague (BOLD), respectively. The superscript/subscript notation l/k in the words $YOUNG^l_k$ and YOUNG^l_k is connected with indices l and k determining the upper/lower limit of vague sets belonging to the denotations of vague concepts. We also use a similar notation for vague sets.

sentence

$$\text{John is YOUNG,} \tag{6}$$

vague knowledge about John, with respect to AGE by, for example, the sentence

$$\text{John is } YOUNG^{\leq 30}_{<19}, \tag{7}$$

and *relatively vague knowledge* about him, with regard to the same aspect by, for example, the sentence

$$\text{John is } \mathsf{YOUNG}^{\leq 30}_{<19}. \tag{8}$$

The denotation of the relatively vague term (concept corresponding to it) in sentence (8) is, for example, the vague set $\mathcal{Y}_2$, i.e. the family of all the sets $Y_{<l}$ of the persons of an age not greater than l, where $l = 19, 20, \ldots, 31$. The denotation of the vague term (concept corresponding to it) in sentence (7) is the family of the vague sets with the same limits: $Y_{<19}$ (the lower limit) and $Y_{\leq 30}$ (the upper limit), e.g. the family consisting of denotation $\mathcal{Y}_2$ of the relatively vague term present in sentence (7) and of the denotations of some other relatively vague terms, e.g. family $\mathcal{Y}_1 = \{Y_{<19}, Y_{<25}, Y_{\leq 30}\}$. The denotation of the *completely vague term* (concept corresponding to it) in sentence (6) is the family of the vague sets of not necessarily the same limits, among which there are vague sets of the same limits, determining the denotation of the vague term appearing in sentence (7).

Expression (8) is then a scheme (a sentential function) whose relatively vague term '$\mathsf{YOUNG}^{\leq 30}_{<19}$' is a variable representing sharp names with the extensions being elements of denotation $\mathcal{Y}_2$ of this term. Expressions (7) and (6) are then schemes of sentential forms, including the sentential form (8). The vague terms '$YOUNG^{\leq 30}_{<19}$' and 'YOUNG' in (7) and (6), respectively, are *variables representing relatively vague terms*, especially the term '$\mathsf{YOUNG}^{\leq 30}_{<19}$'. ♦

A sentence, i.e. a sentential function of the type (8), containing a relatively vague term, i.e. a quasi-name, represents one of the three classes of sentences:

1. the class of sentences which are exclusively true (if John were in the lower age limit, i.e. if he were no more than 18),
2. the class of sentences which are exclusively false (if John exceeded the upper age limit, i.e. if he were over 30), and
3. the class of true or false sentences (if John were at the boundary age, i.e. at the age from 19 up to and including 30).

Therefore John is undoubtedly a *designatum* of the relatively vague term '$YOUNG^{l}_{k}$', where k and l indicate respectively the lower and upper limits of the vague set which is the denotation of this term, when John is k years old. He is certainly *not a designatum* of this quasi-name when he has exceeded the limit of l years, and he is a *quasi-designatum* of this quasi-name when he has exceeded the lower age limit k but has not exceeded the upper age limit l.

If John is a designatum of the relatively vague term 'YOUNG_k^l', the *multiplicity of his membership to* its denotation, i.e. the vague set of the type $\mathcal{Y}_k^l$, is the highest and equal to the cardinality of this set. Then also the *degree of his membership to* set $\mathcal{Y}_k^l$ is the highest and equal to 1. When $\operatorname{card}(\mathcal{Y}_k^l) = n$ $(n > 1)$ and John is a quasi-designatum of this term he belongs only to t exact sets $(1 \le t < n)$ determining the vague set of the type $\mathcal{Y}_k^l$. We can assume that the degree of John's membership to this set equals $1/(n+1)$ if he is an element with multiplicity $t = 1$, and when he is a multiple element (with multiplicity $t > 1$), the degree of his membership to $\mathcal{Y}_k^l$ can be defined by the number $1 - 1/t$. When John is not a designatum of this term, the degree of his membership to the vague set of the type $\mathcal{Y}_k^l$ is zero.

Making our consideration general, we can assert that

P1: A vague set is also a fuzzy set in Zadeh's sense.

Moreover, as possessing multiple elements,

P2: A vague set is also a multiset or a bag

(see Cerf *et al.*, 1971; Peterson, 1976; 1981; Blizard, 1989).

Justification of these statements requires a consistent formalism, beginning with axiomatic assumptions of the theory of approximating sets (vague sets). The multiplicity of the element of such a vague set $\mathcal{V}$ is defined as the number of exact sets determining the set $\mathcal{V}$ to which this element belongs. Consequently, the membership relation of a multiset assumes non-negative integer values. Zadeh's membership functions μ for any $u \in U$ can be defined as the degree of membership of u to the vague set $\mathcal{V}$ in the following way:

$$\mu(u) = \begin{cases} 0 & \text{if multiplicity of } u \text{ is } 0, \\[2ex] \dfrac{1}{n+1} & \text{if multiplicity of } u \text{ is } 1, \\[2ex] 1 - \dfrac{1}{t} & \text{if multiplicity of } u \text{ is } t > 1 \text{ and } t < n, \\[2ex] 1 & \text{if multiplicity of } u \text{ is } n, \end{cases}$$

where $n = \operatorname{card}(\mathcal{V}) > 1$.

Example 7. In connection with Example 6, 25-year-old John was the element of multiplicity 2 of the sets $\mathcal{Y}_3$ (a quasi-designatum of the quasi-name '$\mathsf{YOUNG}_{20}^{16-30}$') and $\mathcal{Y}_4$ (a quasi-designatum of the quasi-name '$\mathsf{YOUNG}_{20}^{18-25}$'), the element of multiplicity 5 of the vague set $\mathcal{Y}_2$ (a quasi-designatum of the quasi-name '$\mathsf{YOUNG}_{<19}^{\le 30}$'), and merely the element with multiplicity 1 of the vague set $\mathcal{Y}_1$ (a quasi-designatum of the quasi-name '$\mathsf{YOUNG}_{<19}^{\le 30}$'), but this set also possesses multiple elements. The degree of John's membership to $\mathcal{Y}_2$, $\mathcal{Y}_3$, $\mathcal{Y}_4$ and $\mathcal{Y}_1$ equals 4/5, 1/2, 1/2 and 1/13, respectively. ⧫

Moreover, if the limits which characterize a vague set are the so-called *lower* and *upper approximations* of each exact set determining this set (see Pawlak, 1982; 1991), then

P3: A vague set is a subset of a rough set with the same limits.

Rough sets are families of all sets with the same lower approximations and with the same upper approximations. The lower approximations of a set are defined here as a union of unions of the equivalence classes (elementary classes) of a given equivalence relation in U, which are included in this set, and the upper approximations of this set are defined as an intersection of all these unions of equivalence classes in which this set is included.

It can also be observed that

P4: A rough set is the greatest vague set, in the sense of inclusion, of the field of indefiniteness made up of the vague sets with the same limits, being at the same time an appropriate approximation of each element of these sets.

5. Conceptualization of Complex Knowledge

What concepts and thus also what denotations correspond to the *union*, *intersection* and *negation of the knowledge* about an object u of reality $\Re$? The *union* and *intersection of the knowledge* about object u of reality $\Re$ are determined by the relations (properties) being respectively the union and intersection of two relations (properties) of this reality; *negation of the knowledge* about object u with respect to its relation R_i is the knowledge about this object with respect to complement $R_i' = U - R_i$.

If a concept with denotation $\mathcal{F}\{R_i, u\} = \{\mathcal{V}_s\}_{s<\xi}$ corresponds to completely vague or vague knowledge about object u with respect to relation (property) R_i, and a concept with denotation $\mathcal{F}(R_j, u\} = \{\mathcal{W}_t\}_{t<\varphi}$ corresponds to such knowledge about object u with respect to relation R_j, then the *union* (resp. *intersection*) of the knowledge about the object is the knowledge to which corresponds the concept with denotation being the family of unions (resp. intersections) of all possible pairs of sets chosen from both $\mathcal{F}(R_i, u)$ and $\mathcal{F}(R_j, u)$. We define the above-mentioned operations on completely vague knowledge and vague knowledge by means of operations $\Box$ and $\circ$ on denotations of the concepts corresponding to knowledge of these kinds. Definitions of the operations $\Box$ and $\circ$ are the folowing:

$$\{\mathcal{V}_s\}_{s<\xi} \Box \{\mathcal{W}_t\}_{t<\varphi} = \{\mathcal{V}_s \cup \mathcal{W}_t\}_{\substack{s<\xi \\ t<\varphi}},$$

$$\{\mathcal{V}_s\}_{s<\xi} \circ \{\mathcal{W}_t\}_{t<\varphi} = \{\mathcal{V}_s \cap \mathcal{W}_t\}_{\substack{s<\xi \\ t<\varphi}},$$

and negation of the completely vague knowledge about object u with respect to relation R_i determining a concept with denotation $\{\mathcal{V}_s\}_{s<\xi}$ to which corresponds knowledge, designating the concept with denotation

$$\neg\{\mathcal{V}_s\}_{s<\xi} = \{\mathcal{P}(U) - \mathcal{V}_s\}_{s<\xi}.$$

It can easily be observed that

P5: The class of all the fields of the knowledge vagueness, together with the families $\{\{\emptyset\}\}$ (*zero of this class*) and $\{\{\mathcal{P}(U)\}\}$ (*unit of this class*), with respect to the operations $\square$, $\circ$, $\neg$ on these fields, constitutes a Boolean algebra.

By analogy, we could describe the *union*, *intersection* and *negation of relatively vague knowledge*. When we determine that the lower limits of the approximating sets $\mathcal{V} = \{R_i\}_{i \in I}$ and $\mathcal{W} = \{R_j\}_{j \in J}$ consisting of relations (properties) of reality $\Re$ are respectively sets $\underline{\mathcal{V}}$ and $\underline{\mathcal{W}}$, and their upper limits are respectively sets $\overline{\mathcal{V}}$ and $\overline{\mathcal{W}}$, then the *union* $\oplus$, *intersection* $\otimes$ and *negation* $\sim$ of the approximating sets $\mathcal{V}$ and $\mathcal{W}$ are defined as follows:

$$\{R_i\}_{i \in I} \oplus \{R_j\}_{j \in J} = \{R_i \cup R_j\}_{\substack{i \in I \\ j \in J}},$$

$$\{R_i\}_{i \in I} \otimes \{R_j\}_{j \in J} = \{R_i \cap R_j\}_{\substack{i \in I \\ j \in J}},$$

$$\sim \{R_i\}_{i \in I} = \{U - R_i\}_{i \in I}.$$

The limits of the union (resp. intersection) of approximating sets $\mathcal{V}$ and $\mathcal{W}$ do not have to be the union (resp. intersection) of their limits $\underline{\mathcal{V}}$ and $\underline{\mathcal{W}}$ or $\overline{\mathcal{V}}$ and $\overline{\mathcal{W}}$. The lower limit of such a union is the intersection $\bigcap\{R_i \cup R_j\}_{\substack{i \in I \\ j \in J}}$ containing the union $\underline{\mathcal{V}} \cup \underline{\mathcal{W}}$ of the lower bounds of the sets $\mathcal{V}$ and $\mathcal{W}$. The upper limit of these sets is the union of the upper bounds of these sets, i.e. the set $\overline{\mathcal{V}} \cup \overline{\mathcal{W}}$, because for approximating sets we assumed that they are coverings of the upper bound (the union of approximating sets is also the covering of its own upper bound). The lower limit of the intersection of the approximating sets $\mathcal{V}$ and $\mathcal{W}$ is the intersection of their lower bounds, i.e. the set $\underline{\mathcal{V}} \cap \underline{\mathcal{W}}$, and the upper limit is the set $\bigcup\{R_i \cap R_j\}_{\substack{i \in I \\ j \in J}}$, including itself in the intersection of the limits of these sets, i.e. in the set $\overline{\mathcal{V}} \cap \overline{\mathcal{W}}$. Thus, the intersection of approximating sets is the covering of its upper bound.

The lower limit of the negation of the approximating set $\mathcal{V}$ is $U - \overline{\mathcal{V}} = \overline{\mathcal{V}}'$, and the upper limit of this set is $U - \underline{\mathcal{V}} = \underline{\mathcal{V}}'$. The negation of a vague set is at the same time the covering of its upper limit $\underline{\mathcal{V}}'$.

The functions on approximating sets are well defined in the class of all such sets. It can easily be observed that

P6: The class of all approximating sets together with families $\{\emptyset\}$ (*zero of this class*) and $\{U\}$ (*unit of this class*) with respect to functions $\oplus$, $\otimes$ and $\sim$ constitute a Boolean algebra.

To the union (resp. intersection) of doubly relatively vague knowledge about object u of universe U, with respect to two definite relations (properties), corresponds a concept with the denotation being the union (resp. intersection) of the denotations of both the concepts determined by this double knowledge:

P7: The union of doubly relatively vague knowledge can be complete knowledge when the concept corresponding to it has the denotation $\{U\}$, identified with the entire universe U; this takes place when we deal with knowledge determined by contrary properties.

Similarly,

P8: The knowledge defined by the intersection of relatively vague knowledge with respect to two contrary properties is empty knowledge, and it occurs when the concept corresponding to it has a denotation being the set $\{\emptyset\}$, identified with an empty set.

Also note that

P9: The union of relatively vague knowledge with respect to two opposite properties R_i and R_j ($R_i \cap R_j = \emptyset$ and $R_i \cup R_j \neq U$) is not complete knowledge.

Similarly,

P10: The intersection of relatively vague knowledge with respect to opposite properties is not empty knowledge.

Example 8. Let us consider reality $\Re$ with respect to AGE and determine that relatively vague knowledge about John with respect to property Y_{25} of being 25 years old assigns John a vague set $\mathcal{Y}_4$, whereas relatively vague knowledge with respect to the property Y_{30} of being 30 years old assigns him a vague set $\mathcal{Y}_3$. The union $\mathcal{Y}_4 \oplus \mathcal{Y}_3$ is not then the set $\{U\}$, and the intersection $\mathcal{Y}_4 \otimes \mathcal{Y}_3$ is not the set $\{\emptyset\}$. ♦

This is connected with the following open question: Is the use of sentence (1) correct when the upper limit of being young is determined once every 30 years, and in another case, once every 25 years? When John is 30, he can then be YOUNG and non-YOUNG, when once 'YOUNG' designates the quasi-name 'YOUNG^{30}', and in another case the quasi-name 'YOUNG^{25}'. Similarly, a 25-year-old John can be neither YOUNG, nor non-YOUNG. It is shown here that the term 'YOUNG' functions then as a variable representing different quasi-names which, in turn, represent sharp names.

As can be seen, the use of a vague name in its different representations (i.e. also meanings) may be a source of essential problems and lead to misunderstandings.

6. Conclusions

We defined two elementary types of vagueness: (i) complete vagueness or vagueness, and (ii) relative vagueness. The concept of vagueness was defined here as a certain property corresponding to the knowledge of the agent discovering the objective reality, i.e. on an epistemic level.

Vague knowledge is represented in a language, or in other systems of knowledge representation, in the form of expressions containing vague terms and expressing the vagueness of type (i) or (ii). Vague terms are of two kinds: (i) completely vague or vague, or (ii) relatively vague. Completely vague or vague terms denote whole fields of indefiniteness, being families consisting of families of subsets of the universe, called the sets approximating this field; these are sets approximated by their bounds (with respect to inclusion), called limits, being at the same time the coverings of their upper limits. When approximating sets are at least two-element ones, they are vague sets with limits defined by their bounds.

The term 'relatively vague' denotes a vague set with determined limits and is treated as a *quasi-name* or a *vague variable* representing sharp names, whose extensions form this vague set. A vague or complety vague term can be treated as a *quasi-quasi-name* or a vague variable representing quasi-names, i.e. relatively vague terms. Relatively vague expressions, i.e. expressions containing relatively vague names, can be treated as sentential functions containing these vague variables. Such expressions represent true or false sentences, not being so themselves, e.g.

$$\text{John is } \mathsf{YOUNG}^{\leq 30}_{<19}.$$

Completely vague or vague expressions, i.e. expressions containing completely vague or vague terms (quasi-quasi-names), can be treated as schemes of relatively vague expressions, i.e. schemes of sentential forms containing relatively vague terms (quasi-names, vague variables).

The above statements allow us to formulate the following conclusions:

1. Relatively vague expressions satisfy all the laws of classical logic. It is so because
   (a) in laws of classical sentential calculus, replacing sentence variables with these sentential expression forms, we obtain sentential expression forms which represent exclusively true sentences, and
   (b) in laws of classical quantifiers calculus, replacing predicate variables with predicate variables corresponding to relatively vague terms, i.e. vague variables, we obtain exclusively 'true' relatively vague expressions, i.e. sentential forms which contain vague terms and represent true sentences exclusively.
2. Completely or almost completely vague expressions, as schemes of relatively vague expressions which satisfy all the laws of classical logic, satisfy them themselves.
3. 'Mixed' expressions, containing at least one relatively vague term and at least one completely vague or vague term, and possibly, ordinary logical sentences, satisfy all the laws of classical logic.

The above conclusions justify the opinion that

> *To grasp vagueness, analyzed in a language aspect, one should extend classical two-valued logic, building into it, in a natural way, 'classical logic of relatively vague expressions', and into that 'classical logic of completely vague expressions'.*

Obviously, these classical logics of vague expressions are not two-valued logics because the counterparts of logical connectives are not extensional: two sentential forms belonging to the class of sentential forms representing sometimes true sentences and sometimes false ones, connected, e.g. by an alternative, can give this class of sentential forms or the class of sentential forms representing solely true sentences (Słupecki *et al.*, 1976).

> *Such an extension of classical logic, i.e. sentential logic containing vague sentences, has an interpretation in extended algebra of sets, the algebra of sets containing an ordinary set calculus together with the Boolean algebra of approximating sets and the Boolean algebra of indefinite fields.*

The formal approach to the problem of vagueness suggested here pertains to the classical trend in research, represented e.g. by Fine (1975) (see also Cresswell, 1973). The approach presented here is considered a conservative one in the reasearch on vagueness, but it seems that it reflects well the essence of how our brain works using vague information:

> *Logic which is basic in the research on vagueness is classical logic.*

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