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ON DELAY-DEPENDENT STABILITY FOR NEUTRAL DELAY-DIFFERENTIAL SYSTEMS

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This paper deals with the stability problem for a class of linear neutral delay-differential systems. The time delay is assumed constant and known. Delay-dependent criteria are derived. The criteria are given in the form of linear matrix inequalities which are easy to use when checking the stability of the systems considered. Numerical examples indicate significant improvements over some existing results.

Keywords: stability, time delay, neutral system, linear matrix inequality (LMI)

1. Introduction

The problems of stability and stabilization of time-delay systems of neutral type have received considerable attention in the last two decades, see, e.g. (Byrnes *et al.*, 1984; Chukwu and Simpson, 1989; Hale and Verduyn Lunel, 1993; Logemann and Pandolfi, 1994; Logemann and Townley, 1996; Spong, 1985). Practical examples of such systems include distributed networks containing lossless transmission lines (Brayton, 1966), and population ecology (Kuang, 1993). Current efforts regarding this topic can be divided into two categories (Mori, 1985), namely, *delay-independent* stability criteria and *delay-dependent* stability criteria. For linear time-delay systems of neutral type, some *delay-independent* stability conditions were obtained. They were formulated in terms of a matrix measure and a matrix norm (Hu and Hu, 1996; Park and Won, 1999), or the existence of a positive definite solution to an auxiliary algebraic Riccati matrix equation (Slemrod and Infante, 1972; Verriest and Niculescu, 1997). Although these conditions are easy to check, they require the matrix measure to be negative or the parameters to be tuned. Moreover, the abandonment of information on the delay necessarily causes the conservativeness of the criteria, especially when the delay is small. Delay-dependent stability results, which take the delay into account, are usually less conservative than the delay-independent stability ones. Park and Won (2000) proposed a delay-dependent stability criterion. A numerical example illustrated that the result in (Khusainov and Yun'kova, 1988) was improved. Recently, an LMI approach has been widely used to study the stability of time-delay systems, see, e.g.

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(Bliman, 2001; Han and Gu, 2001a; 2001b; Li and de Souza, 1997), because it has the advantage that it can be implemented numerically very efficiently using standard LMI algorithms (Boyd *et al.*, 1994).

In this paper, based on some model transformation techniques and Lyapunov-Krasovskii's functional approach, the delay-dependent stability problem of the considered system is transformed into that of the *existence* of some symmetric positive-definite matrices. The stability criteria are formulated in the form of linear matrix inequalities (LMIs). Numerical examples show that the results obtained in this paper are less conservative than those in (Khusainov and Yun'kova, 1988; Park and Won, 2000).

Notation. For a symmetric matrix W, $W > 0$ means that W is a positive definite matrix. I is the identity matrix of appropriate dimensions. $C([-h,0],\mathbb{R}^n)$ stands for the set of continuous $\mathbb{R}^n$ valued functions on $[-h,0]$, $x_t \in C([-h,0],\mathbb{R}^n)$ is a segment of the system trajectory defined by $x_t(\theta) = x(t+\theta)$, $-h \le \theta \le 0$, and $\|\varphi\|_c = \sup_{-h\le\theta\le 0}\|\varphi(\theta)\|$ denotes the norm of $\varphi \in C([-h,0],\mathbb{R}^n)$. Let $\bar{\mathbb{C}}_+$ be the closed right-half plane. For a matrix $A = (a_{ij})_{n\times n} \in \mathbb{R}^{n\times n}$, we use $|A|$ to denote $|A| = (|a_{ij}|)_{n\times n} \in \mathbb{R}^{n\times n}$. The symbol $\|\cdot\|$ stands for the Euclidean vector norm and $\lambda_{\max}(W)$ $(\lambda_{\min}(W))$ denotes the maximum (minimum) eigenvalue of a symmetric matrix W. Moreover, $\rho(W)$ denotes the spectral radius of a matrix W.

2. Problem Statement

Consider the following linear neutral delay-differential system:

$$\dot{x}(t) - C\dot{x}(t-h) = Ax(t) + Bx(t-h), \tag{1}$$

$$x(t_0+\theta) = \varphi(\theta), \quad \forall\, \theta \in [-h,0], \tag{2}$$

where $x(t) \in \mathbb{R}^n$ is the state, $h > 0$ is a constant time-delay, $\varphi(\cdot)$ is a continuous vector-valued initial function, $A, B \in \mathbb{R}^{n\times n}$ and $C \in \mathbb{R}^{n\times n}$ are known real constant matrices. For given initial conditions of the form (2), system (1) admits a unique solution $x(t,t_0,\varphi(\cdot))$ which is defined on $[t_0-h,\infty)$.

Definition 1.

(i) The solution $x = 0$ of eqn. (1) is said to be *stable* if for any $\varepsilon > 0$ there exists a $\delta = \delta(t_0,\varepsilon) > 0$ such that if $\|\varphi(\cdot)\| < \delta$, then $\|x(t,t_0,\varphi(\cdot))\| < \varepsilon$ for all $t > t_0$.

(ii) The solution $x = 0$ of eqn. (1) is said to be *asymptotically stable* if it is stable and there exists a $\Delta = \Delta(t_0) > 0$ such that $x(t,t_0,\varphi(\cdot)) \to 0$ as $t \to \infty$.

The stability property of system (1), (2) can be described by its characteristic equation. The system (1), (2) is asymptotically stable if and only if

$$\det(sI - sCe^{-hs} - A - Be^{-hs}) \neq 0, \quad \forall\, s \in \bar{\mathbb{C}}_+.$$

It is difficult to directly solve the above equation. One of the most general approaches to the stability analysis of (1) is the Lyapunov-Krasovskii functional approach. To derive delay-dependent stability conditions, which include the information of the time-delay h, one usually uses the dependence (Hale and Verduyn Lunel, 1993)

$$x(t-h) = x(t) - \int_{-h}^{0} \dot{x}(t+\theta)\,\mathrm{d}\theta$$

to transform the original system (1) to a system of neutral type or a system with a distributed delay.

In this paper, we shall attempt to formulate two practically computable criteria to check the stability of system (1), (2).

3. Main Results

We now use the following neutral type representation of system (1) that leads to our first result:

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big[x(t) - Cx(t-h) + B\int_{t-h}^{t} x(\xi)\,\mathrm{d}\xi\Big] = (A+B)x(t). \tag{3}$$

Remark 1. Using an argument similar to Niculescu *et al.* (1994), it is easy to prove that the stability of system (3) (or (6) after Remark 2) implies that of system (1).

Theorem 1. *System (1), (2) is asymptotically stable if the difference-integral system* $x(t)-Cx(t-h)+B\int_{t-h}^{t} x(\xi)\,\mathrm{d}\xi = 0$ *is asymptotically stable and there exist symmetric positive definite matrices* P, R *and* W *satisfying the following LMI:*

$$\Xi = \begin{bmatrix} -(A+B)^T P - P(A+B) - hR - W & (A+B)^T PC & -h(A+B)^T PB \\ C^T P(A+B) & W & 0 \\ -hB^T P(A+B) & 0 & hR \end{bmatrix} > 0. \tag{4}$$

In order to prove Theorem 1, we need the following integral inequality:

Lemma 1. (Gu, 2000) *For any constant symmetric matrix* $M \in \mathbb{R}^{n\times n}$, $M = M^T > 0$, *scalar* $\gamma > 0$, *vector function* $\omega:\ [0,\gamma] \to \mathbb{R}^m$ *such that the integration in the following is well-defined, we have*

$$\gamma \int_0^{\gamma} \omega^T(\beta) M \omega(\beta)\,\mathrm{d}\beta \ge \Big(\int_0^{\gamma} \omega(\beta)\,\mathrm{d}\beta\Big)^T M \Big(\int_0^{\gamma} \omega(\beta)\,\mathrm{d}\beta\Big).$$

Proof of Theorem 1. Consider the Lyapunov-Krasovskii functional candidate $V = V_1 + V_2 + V_3$, where

$$V_1 = \Big[x(t) - Cx(t-h) + B\int_{t-h}^{t} x(\xi)\,\mathrm{d}\xi\Big]^T P \Big[x(t) - Cx(t-h) + B\int_{t-h}^{t} x(\xi)\,\mathrm{d}\xi\Big],$$

$$V_2 = \int_{t-h}^{t} (h - t + \xi) x^T(\xi) R x(\xi)\, \mathrm{d}\xi,$$

$$V_3 = \int_{t-h}^{t} x^T(\xi) W x(\xi)\, \mathrm{d}\xi,$$

P, W and R being symmetric positive-definite solutions of (4).

It is easy to see that the functional V satisfies the condition

$$\alpha_1 \Big\| \Big[x(t) - Cx(t-h) + B \int_{t-h}^{t} x(s)\, \mathrm{d}s \Big] \Big\|^2 \leq V \leq \alpha_2 \|x_t\|_c^2,$$

where $\alpha_1 = \lambda_{\min}(P)$ and $\alpha_2 = \lambda_{\max}(P)(1 + \|C\| + h\|B\|) + \frac{1}{2} h^2 \lambda_{\max}(R) + h\lambda_{\max}(W)$.

The derivative of V along the trajectory of system (3) is given by $\dot{V} = \dot{V}_1 + \dot{V}_2 + \dot{V}_3$, where

$$\begin{aligned}
\dot{V}_1 &= 2x^T(t)(A+B)^T P \Big[x(t) - Cx(t-h) + B \int_{t-h}^{t} x(\xi)\, \mathrm{d}\xi \Big] \\
&= x^T(t) \left[(A+B)^T P + P(A+B) \right] x(t) - 2x^T(t)(A+B)^T P C x(t-h) \\
&\quad + 2x^T(t)(A+B)^T P B \int_{t-h}^{t} x(\xi)\, \mathrm{d}\xi, \\
\dot{V}_2 &= h x^T(t) R x(t) - \int_{t-h}^{t} x^T(\xi) R x(\xi)\, \mathrm{d}\xi, \\
\dot{V}_3 &= x^T(t) W x(t) - x^T(t-h) W x(t-h).
\end{aligned}$$

Then we have

$$\begin{aligned}
\dot{V} &= x^T(t) \left[(A+B)^T P + P(A+B) + hR + W \right] x(t) \\
&\quad - 2x^T(t)(A+B)^T P C x(t-h) - x^T(t-h) W x(t-h) \\
&\quad + 2x^T(t)(A+B)^T P B \int_{t-h}^{t} x(\xi)\, \mathrm{d}\xi - \int_{t-h}^{t} x^T(\xi) R x(\xi)\, \mathrm{d}\xi.
\end{aligned}$$

Using Lemma 1, obtain

$$\int_{t-h}^{t} x^T(\xi) R x(\xi)\, \mathrm{d}\xi \geq \Big(\frac{1}{h} \int_{t-h}^{t} x(\xi)\, \mathrm{d}\xi \Big)^T (hR) \Big(\frac{1}{h} \int_{t-h}^{t} x(\xi)\, \mathrm{d}\xi \Big).$$

From the above inequality it follows that

$$\begin{aligned}\dot V &\le -x^T(t)\left[-(A+B)^TP-P(A+B)-hR-W\right]x(t)\\&\quad-2x^T(t)(A+B)^TPCx(t-h)-x^T(t-h)Wx(t-h)\\&\quad-2x^T(t)\left[-h(A+B)^TPB\right]\Big(\frac{1}{h}\int_{t-h}^{t}x(\xi)\,\mathrm{d}\xi\Big)\\&\quad-\Big(\frac{1}{h}\int_{t-h}^{t}x(\xi)\,\mathrm{d}\xi\Big)^T(hR)\Big(\frac{1}{h}\int_{t-h}^{t}x(\xi)\,\mathrm{d}\xi\Big)\\&=-\left(x^T(t)\;x^T(t-h)\;\Big(\frac{1}{h}\int_{t-h}^{t}x(\xi)\,\mathrm{d}\xi\Big)^T\right)\Xi\begin{pmatrix}x(t)\\x(t-h)\\(\frac{1}{h}\int_{t-h}^{t}x(\xi)d\xi)\end{pmatrix}\end{aligned}$$

In light of (4), $\dot V$ is negative definite. Since the difference-integral system $x(t)-Cx(t-h)+B\int_{t-h}^{t}x(\xi)\,\mathrm{d}\xi=0$ is asymptotically stable, so is system (1), (2), according to Theorem 8.1 of (Hale and Verduyn Lunel, 1993, pp.292–293). ■

Remark 2. The difference-integral system $x(t)-Cx(t-h)+B\int_{t-h}^{t}x(\xi)\,\mathrm{d}\xi=0$ is asymptotically stable if there exists a $\delta>0$ such that all the solutions λ of the characteristic equation

$$\det\left[I-Ce^{-h\lambda}+B\int_{-h}^{0}e^{\lambda\theta}\,\mathrm{d}\theta\right]=0$$

satisfy $\mathrm{Re}(\lambda)\le-\delta<0$. Through simple computation, the above equation can be written as

$$\det\left[I-Ce^{-h\lambda}+B\frac{1-e^{-h\lambda}}{\lambda}\right]=0.$$

It is easy to see that a sufficient condition for the considered difference-integral system to be asymptotically stable is that $\rho(|C|+h|B|)<1$, which is equivalent to the existence of a symmetric positive-definite matrix Q satisfying the matrix inequality

$$\big(|C|+h|B|\big)^TQ\big(|C|+h|B|\big)-Q<0. \tag{5}$$

This inequality is not an LMI concerning the variable h. However, for a fixed h, (5) is an LMI.

Let us rewrite (1) as

$$\dot x(t)-C\dot x(t-h)=(A+B)x(t)-B\int_{t-h}^{t}\dot x(\xi)\,\mathrm{d}\xi \tag{6}$$

Theorem 2. *System (1), (2) is asymptotically stable if the difference system $x(t) - Cx(t-h) = 0$ is asymptotically stable and there exist symmetric positive-definite matrices P, R, S and W satisfying the following LMI:*

$$\Sigma = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} & \Sigma_{13} & \Sigma_{14} \\ \Sigma_{12}^T & \Sigma_{22} & \Sigma_{23} & \Sigma_{24} \\ \Sigma_{13}^T & \Sigma_{23}^T & \Sigma_{33} & 0 \\ \Sigma_{14}^T & \Sigma_{24}^T & 0 & \Sigma_{44} \end{bmatrix} > 0 \tag{7}$$

where

$$\Sigma_{11} = -(A+B)^T P - P(A+B) - W - A^T(hR+S)A,$$

$$\Sigma_{12} = (A+B)^T PC - A^T(hR+S)B, \quad \Sigma_{13} = -A^T(hR+S)C,$$

$$\Sigma_{14} = hPB, \quad \Sigma_{22} = W - B^T(hR+S)B, \quad \Sigma_{23} = -B^T(hR+S)C,$$

$$\Sigma_{24} = -hC^T PB, \quad \Sigma_{33} = S - C^T(hR+S)C, \quad \Sigma_{44} = hR.$$

Proof. Consider the Lyapunov-Krasovskii functional candidate $V = V_1 + V_2 + V_3 + V_4$, where

$$V_1 = \big[x(t) - Cx(t-h)\big]^T P \big[x(t) - Cx(t-h)\big],$$

$$V_2 = \int_{t-h}^{t} (h - t + \xi)\dot{x}^T(\xi) R \dot{x}(\xi)\, \mathrm{d}\xi, \quad V_3 = \int_{t-h}^{t} \dot{x}^T(\xi) S \dot{x}(\xi)\, \mathrm{d}\xi,$$

$$V_4 = \int_{t-h}^{t} x^T(\xi) W x(\xi)\, \mathrm{d}\xi,$$

P, R, S and W being symmetric positive-definite solutions of (7).

The functional V satisfies the condition

$$\alpha_3 \big\|[x(t) - Cx(t-h)]\big\|^2 \le V \le \alpha_4 \|x_t\|_{c1}^2,$$

where $\|x_t\|_{c1} = \sup_{-h \le \theta \le 0}\{\|x(t+\theta)\|, \|\dot{x}(t+\theta)\|\}$ and $\alpha_3 = \lambda_{\min}(P)$, $\alpha_4 = \lambda_{\max}(P)(1 + \|C\|) + \frac{1}{2}h^2\lambda_{\max}(R) + h\lambda_{\max}(S) + h\lambda_{\max}(W)$. Some connections between the stability results obtained using the norms $\|\cdot\|_c$ and $\|\cdot\|_{c1}$ can be found in (Els'golts' and Norkin, 1973).

The derivative of V along the trajectory of system (6) is given by $\dot{V} = \dot{V}_1 + \dot{V}_2 + \dot{V}_3 + \dot{V}_4$, where

$$\begin{aligned}
\dot{V}_1 &= 2\big[x(t) - Cx(t-h)\big]^T P \left[(A+B)x(t) - B\int_{t-h}^{t} \dot{x}(\xi)\,\mathrm{d}\xi\right] \\
&= x^T(t)\left[(A+B)^T P + P(A+B)\right]x(t) - 2x^T(t)PB\int_{t-h}^{t} \dot{x}(\xi)\,\mathrm{d}\xi \\
&\quad - 2x^T(t)(A+B)^T PCx(t-h) + 2x^T(t-h)C^T PB\int_{t-h}^{t} \dot{x}(\xi)\,\mathrm{d}\xi, \\
\dot{V}_2 &= h\dot{x}^T(t)R\dot{x}(t) - \int_{t-h}^{t} \dot{x}^T(\xi)R\dot{x}(\xi)\,\mathrm{d}\xi, \\
\dot{V}_3 &= \dot{x}^T(t)S\dot{x}(t) - \dot{x}^T(t-h)S\dot{x}(t-h), \\
\dot{V}_4 &= x^T(t)Wx(t) - x^T(t-h)Wx(t-h).
\end{aligned}$$

Then

$$\begin{aligned}
\dot{V} &= x^T(t)\left[(A+B)^T P + P(A+B) + W\right]x(t) - 2x^T(t)(A+B)^T \\
&\quad \times PCx(t-h) - x^T(t-h)Wx(t-h) + \dot{x}^T(t)(hR+S)\dot{x}(t) \\
&\quad - \dot{x}^T(t-h)S\dot{x}(t-h) - 2x^T(t)PB\int_{t-h}^{t} \dot{x}(\xi)\,\mathrm{d}\xi \\
&\quad + 2x^T(t-h)C^T PB\int_{t-h}^{t} \dot{x}(\xi)\,\mathrm{d}\xi - \int_{t-h}^{t} \dot{x}^T(\xi)R\dot{x}(\xi)\,\mathrm{d}\xi.
\end{aligned}$$

Since $\dot{x}(t) = Ax(t) + Bx(t-h) + C\dot{x}(t-h)$, we deduce that

$$\begin{aligned}
&\dot{x}^T(t)(hR+S)\dot{x}(t) \\
&\qquad = x^T(t)A^T(hR+S)Ax(t) + 2x^T(t)A^T(hR+S)Bx(t-h) \\
&\qquad\quad + 2x^T(t)A^T(hR+S)C\dot{x}(t-h) + x^T(t-h)B^T(hR+S)Bx(t-h) \\
&\qquad\quad + 2x^T(t-h)B^T(hR+S)C\dot{x}(t-h) \\
&\qquad\quad + \dot{x}^T(t-h)C^T(hR+S)C\dot{x}(t-h)
\end{aligned}$$

and, using Lemma 1, we obtain

$$\int_{t-h}^{t} \dot{x}^T(\xi)R\dot{x}(\xi)\,\mathrm{d}\xi \geq \left(\frac{1}{h}\int_{t-h}^{t} \dot{x}(\xi)\,\mathrm{d}\xi\right)^T (hR)\left(\frac{1}{h}\int_{t-h}^{t} \dot{x}(\xi)\,\mathrm{d}\xi\right).$$

We thus get

$$
\begin{aligned}
\dot{V} \le & -x^T(t)[-(A+B)^T P - P(A+B) - W - A^T(hR+S)A]x(t) \\
& - 2x^T(t)\left[(A+B)^T PC - A^T(hR+S)B\right]x(t-h) \\
& - 2x^T(t)\left[-A^T(hR+S)C\right]\dot{x}(t-h) \\
& - 2x^T(t)hPB\left(\frac{1}{h}\int_{t-h}^{t}\dot{x}(\xi)\,\mathrm{d}\xi\right) \\
& - x^T(t-h)\left[W - B^T(hR+S)B\right]x(t-h) \\
& - 2x^T(t-h)\left[-B^T(hR+S)C\right]\dot{x}(t-h) \\
& - 2x^T(t-h)\left[-hC^T PB\right]\left(\frac{1}{h}\int_{t-h}^{t}\dot{x}(\xi)\,\mathrm{d}\xi\right) \\
& - \dot{x}^T(t-h)\left[S - C^T(hR+S)C\right]\dot{x}(t-h) \\
& - \left(\frac{1}{h}\int_{t-h}^{t}\dot{x}(\xi)\,\mathrm{d}\xi\right)^T (hR)\left(\frac{1}{h}\int_{t-h}^{t}\dot{x}(\xi)\,\mathrm{d}\xi\right) \\
= & -\begin{pmatrix} x^T(t) & x^T(t-h) & \dot{x}^T(t-h) & \left(\frac{1}{h}\int_{t-h}^{t}\dot{x}(\xi)\,\mathrm{d}\xi\right)^T \end{pmatrix} \\
& \times \Sigma \begin{pmatrix} x(t) \\ x(t-h) \\ \dot{x}(t-h) \\ \left(\frac{1}{h}\int_{t-h}^{t}\dot{x}(\xi)\,\mathrm{d}\xi\right) \end{pmatrix}.
\end{aligned}
$$

This derivative is negative definite in light of (7). Just as in Theorem 1, we can conclude that if the assumptions of Theorem 2 are satisfied, system (1), (2) is asymptotically stable. ■

Remark 3. A sufficient condition for the difference system $x(t) - Cx(t-h) = 0$ to be asymptotically stable is $\rho(C) < 1$, or there exists a symmetric positive-definite matrix Q satisfying the following LMI:

$$C^T QC - Q < 0. \tag{8}$$

Let us further consider a more general neutral-type system

$$\dot{x}(t) - C\dot{x}(t-\tau) = Ax(t) + Bx(t-h), \tag{9}$$

$$x(t_0+\theta) = \varphi(\theta), \quad \forall\, \theta \in [-\max(h,\tau), 0], \tag{10}$$

where $h > 0$ is a discrete-delay and $\tau > 0$ is a neutral delay.

Rewrite (9) as

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[x(t) - Cx(t-\tau) + B\int_{t-h}^{t} x(\xi)\,\mathrm{d}\xi\right] = (A+B)x(t). \tag{11}$$

Choose the Lyapunov-Krasovskii functional candidate $V = V_1 + V_2 + V_3$, where

$$V_1 = \left[x(t) - Cx(t-\tau) + B\int_{t-h}^{t} x(\xi)\,\mathrm{d}\xi\right]^T P\left[x(t) - Cx(t-\tau) + B\int_{t-h}^{t} x(\xi)\,\mathrm{d}\xi\right],$$

$$V_2 = \int_{t-h}^{t} (h-t+\xi)x^T(\xi)Rx(\xi)\,\mathrm{d}\xi, \quad V_3 = \int_{t-\tau}^{t} x^T(\xi)Wx(\xi)\,\mathrm{d}\xi.$$

In much the same way as in the proof of Theorem 1, one can easily obtain the following result:

Theorem 3. *System (9), (10) is asymptotically stable if the difference-integral system* $x(t)-Cx(t-\tau)+B\int_{t-h}^{t} x(\xi)\,\mathrm{d}\xi = 0$ *is asymptotically stable and there exist symmetric positive-definite matrices* P, R *and* W *satisfying the following LMI:*

$$\begin{bmatrix} -(A+B)^TP-P(A+B)-hR-W & (A+B)^TPC & -h(A+B)^TPB \\ C^TP(A+B) & W & 0 \\ -hB^TP(A+B) & 0 & hR \end{bmatrix} > 0. \tag{12}$$

Remark 4. From Theorem 3, it is easy to see that (12) does not include any information on the neutral delay $\tau > 0$. This means that Theorem 3 gives *neutral-delay-independent* stability conditions. Finding both *neutral-delay-dependent* and *discrete-delay-dependent* criteria constitutes the subject of further work.

Remark 5. The system under consideration here is a nominal system. If the system is subject to a norm-bounded uncertainty, based on the stability criteria in this paper, one can easily reformulate the results to appropriate LMIs following the idea given in (Han and Gu, 2001a).

4. Examples

In order to use Theorems 1 and 2 to test the stability of system (1), (2), there have been written two MATLAB m-functions which automatically generate LMIs (4), (5) and (7), (8), respectively, and then solve this set of LMIs using the LMI Solver FEASP in the MATLAB LMI toolbox (Gahinet *et al.*, 1995). The inputs to the functions are the system matrices and the time delay. The functions verify whether the LMIs are feasible. If so, they also give matrices P, Q, R and W (for Theorem 1) or P, Q, R, S and W (for Theorem 2) as the outputs. The following examples are generated using these MATLAB m-functions to illustrate the effectiveness of the approaches.

Example 1. Consider the time-delay system

$$\dot{x}(t) = [A + \Delta A(t)]x(t) + [B + \Delta B(t)]x(t - r),$$

where

$$A = \begin{bmatrix} -3 & -2.5 \\ 1 & 0.5 \end{bmatrix}, \quad B = \begin{bmatrix} 1.5 & 2.5 \\ -0.5 & -1.5 \end{bmatrix}, \quad C = \begin{bmatrix} c & 0 \\ 0 & c \end{bmatrix}, \quad |c| < 1.$$

Using the methods from (Khusainov and Yun'kova, 1988; Park and Won, 2000), no conclusion can be made since the corresponding conditions are not satisfied. The maximum time delay for asymptotic stability $h_{\max}$ as estimated by Theorem 1 is listed in Table 1. It is clear that if $|c|$ increases, then $h_{\max}$ decreases. Hence, for this example, the criterion in this paper gives a less conservative result that the ones set forth in (Khusainov and Yun'kova, 1988; Park and Won, 2000).

Table 1. Bound $h_{\max}$ for various c.

c	-0.50	-0.30	-0.10	0.00	0.10	0.30	0.50
$h_{\max}$	0.500	0.700	0.900	1.000	0.900	0.700	0.500

Example 2. Consider the system

$$\dot{x}(t) - \begin{bmatrix} c & 0 \\ 0 & c \end{bmatrix} \dot{x}(t - h) = \begin{bmatrix} -0.8 & 0.2 \\ -0.2 & -0.8 \end{bmatrix} x(t - h),$$

where $-1 < c < 1$.

The maximum time delay for asymptotic stability $h_{\max}$ is illustrated in Table 2. It is seen that for $c = 0$, Theorems 1 and 2 lead to the same bound 1.1764. For $0 < |c| < 1$, the results got from Theorem 1 are less conservative than those obtained from Theorem 2. As $|c|$ increases from 0 to 1, $h_{\max}$ decreases from 1.1764 to 0. This example also shows that different LMIs in Theorems 1 and 2 are not equivalent.

Table 2. Bound $h_{\max}$ for various $-1 < c < 1$.

$\lvert c \rvert$	0.00	0.10	0.20	0.30	0.40
Theorem 2	1.176	0.863	0.622	0.439	0.302
Theorem 1	1.176	1.055	0.933	0.812	0.691
$\lvert c \rvert$	0.50	0.60	0.70	0.80	0.90
Theorem 2	0.201	0.128	0.076	0.039	0.013
Theorem 1	0.570	0.448	0.327	0.206	0.085

5. Conclusion

The stability problem of linear neutral delay-differential systems has been addressed. Delay-dependent stability criteria have been obtained. Numerical examples have shown significant improvements over some existing results.

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