COMPUTING GENERALIZED INVERSE SYSTEMS USING MATRIX PENCIL METHODS

Andras VARGA*

We address the numerically reliable computation of generalized inverses of rational matrices in descriptor state-space representation. We put particular emphasis on two classes of inverses: the weak generalized inverse and the Moore-Penrose pseudoinverse. By combining the underlying computational techniques, other types of inverses of rational matrices can be computed as well. The main computational ingredient to determine generalized inverses is the orthogonal reduction of the system matrix pencil to appropriate Kronecker-like forms.

Keywords: system inversion, rational matrices, descriptor systems, numerical methods

1. Introduction

Inverse systems have many important applications in areas such as control theory, filtering and coding theory. The computation of the so-called *zero initial state* system inverses for linear time-invariant state-space systems is essentially equivalent to determing generalized inverses of the associated transfer-function matrices. For square and invertible systems, the computation of inverses can be performed by explicit formulas either in the standard state-space or in a descriptor system formulation. For non-square systems, explicit formulas can be employed only in the full-rank case to determine left or right inverses, provided that the system feedthrough matrix has full rank too. However, these direct formulas do not allow to arbitrarily choose the spurious poles which appear in the computed left or right inverses. In the more general case of systems with transfer-function matrices of arbitrary ranks, no explicit formulas can be used.

In this paper we address the numerically reliable computation of generalized inverses of rational matrices by using orthogonal matrix pencil reduction techniques. We put particular emphasis on the computation of two classes of inverses: the weak generalized inverse, also known as the (1,2)-inverse, and the Moore-Penrose pseudoinverse. The (1,2)-inverses can be computed using a numerically reliable approach based on the reduction of the system matrix pencil to a particular Kronecker-like form. The computation of the Moore-Penrose pseudoinverse is performed by employing full-rank

* German Aerospace Center, DLR – Oberpfaffenhofen, Institute of Robotics and Mechatronics, D-82234 Wessling, Germany, e-mail: `Andras.Varga@dlr.de`

factorizations resulting from appropriate column/row compressions with all-pass factors. By combining the underlying computational techniques, other types of inverses of rational matrices can be computed as well. We present some numerical examples to illustrate our methods, and report on recently developed MATLAB software devised by the author to compute some generalized inverses.

2. Generalized Inverses

Consider a $p \times m$ rational matrix $G(\lambda)$ with real coefficients. Throughout the paper we assume that $\operatorname{rank} G(\lambda) = r$ over rationals (i.e., $G(\lambda)$ has rank r for almost all values of λ). The *zeros* of $G(\lambda)$ are those values of λ (finite or infinite) where $G(\lambda)$ loses its maximal rank r. The *poles* of $G(\lambda)$ are those values of λ (finite or infinite) where the elements of $G(\lambda)$ become infinite. We call $G(\lambda)$ *proper* if it has only finite zeros (i.e., $G(\infty)$ is finite). In a system theoretical setting, $G(\lambda)$ can be interpreted as the transfer-function matrix either of a continuous-time system, if λ is the complex variable s appearing in the Laplace transform, or of a discrete-time system, if λ is the complex variable z appearing in the Z-transform. Accordingly, we call $G(\lambda)$ *stable* if all its poles lie in the appropriate stability domain (i.e., the open left half complex plane for a continuous-time system or the interior of the unit circle for a discrete-time system). Depending on the type of the system, the *conjugate* of $G(\lambda)$ is the matrix $G^\sim(\lambda)$ defined as $G^\sim(s) = G^T(-s)$ for a continuous-time system or $G^\sim(z) = G^T(1/z)$ for a discrete-time system. We say that $G(\lambda)$ is *all-pass* if $G^\sim(\lambda)G(\lambda) = I_m$. A stable all-pass matrix is called an *inner* matrix.

Let $X(\lambda)$ be an $m \times p$ real rational matrix. Consider the following Moore-Penrose relations (see, e.g., (Ben Israel and Greville, 1976)):

1) $G(\lambda)X(\lambda)G(\lambda) = G(\lambda)$,

2) $X(\lambda)G(\lambda)X(\lambda) = X(\lambda)$,

3) $G(\lambda)X(\lambda) = (G(\lambda)X(\lambda))^\sim$,

4) $X(\lambda)G(\lambda) = (X(\lambda)G(\lambda))^\sim$.

The well-known *Moore-Penrose pseudoinverse* $X(\lambda) = G^\#(\lambda)$ is unique and satisfies all the four Moore-Penrose relations. Depending on the interpretation of λ, for the same rational matrix $G(\lambda)$ we have two different pseudoinverses $G^\#(s)$ and $G^\#(z)$ for a continuous-time and a discrete-time system, respectively.

In solving practical problems, the uniqueness of the Moore-Penrose pseudoinverse is rather a disadvantage, since no flexibility is provided, for example, in assigning the poles of the corresponding inverse system to desired locations. Therefore, often other types of generalized inverses are preferred. The *weak generalized inverse* $X(\lambda) = G^+(\lambda)$ satisfies only the first two Moore-Penrose conditions and is therefore called a (1,2)-inverse. Weak generalized inverses are useful in solving rational matrix equations or in computing inner-outer factorizations (Varga, 1998).

In what follows we will use the notation $G^+(\lambda)$ for all types of inverses satisfying one or more Moore-Penrose conditions. Of particular importance for applications are

(Ben Israel and Greville, 1976; Campbell and Meyer, 1991): the (1)-inverse to solve systems of equations with rational matrices, the (1,3)-inverse to compute least-squares solutions of rational matrix equations, the (1,4)-inverse, also known as the *minimum norm inverse*, to compute minimum norm solutions of matrix equations. With this nomenclature, the Moore-Penrose pseudoinverse is an (1,2,3,4)-inverse. The *left* and *right inverses*, frequently used in the control literature, are particular (1,2)-inverses of full column rank or full row rank matrices, respectively.

Note. Our approach to determine generalized inverses relies on the standard system inversion concepts widely used in the control literature. Therefore, to define the generalized inverses using the Moore-Penrose conditions, we will not assume that $G(\lambda)$ has constant rank for all λ (as is done elsewhere (Sontag, 1980)). It follows that the generalized inverses computed by our methods satisfy pointwise conditions 1–4 (or part of them) for almost all values of λ, except a finite set of points, which includes certainly the poles and zeros of $G(\lambda)$.

3. Computation of Weak Inverses

Weak or (1,2)-inverses of proper rational matrices appear frequently in control, filtering and coding applications. An important class of (1,2)-inverses consists of the left and right inverses. Their computations and properties drew considerable attention in the control literature (see, e.g., (Wang and Davison, 1973) and the references cited therein). The properties of more general weak inverses were studied in (Rakowski, 1991) (for an application of these concepts to solving differential-algebraic equations see also (Campbell and Rakowski, 1994)).

In this section we consider the most general case of computing the (1,2)-inverse of an arbitrary rank rational matrix $G(\lambda)$, without the restriction that $G(\lambda)$ is proper. Thus, the proposed approach is applicable to a general improper $G(\lambda)$ and, in particular, to polynomial matrices. Important associated problems are the computation of stable (1,2)-inverses (whenever they exist) or of least McMillan degree (1,2)-inverses. A problem yet to be solved is the computation of stable (1,2)-inverses with the least McMillan degree.

Let $G(\lambda)$ be a real rational matrix of rank r, having an n-th order regular descriptor representation $\mathbf{G}$, denoted by $\mathbf{G} := (A - \lambda E, B, C, D)$, which satisfies

$$G(\lambda) = C(\lambda E - A)^{-1}B + D.$$

We assume that the descriptor representation of $G(\lambda)$ is minimal, so that n is the least integer for which the above relation holds. Let $S(\lambda)$ be the system pencil matrix associated with the descriptor system $\mathbf{G} = (A - \lambda E, B, C, D)$,

$$S(\lambda) = \left[\begin{array}{c|c} A - \lambda E & B \\ \hline C & D \end{array} \right].$$

From the straightforward formula

$$\left[\begin{array}{cc} A - \lambda E & 0 \\ 0 & G(\lambda) \end{array} \right] = \left[\begin{array}{cc} I_n & 0 \\ -C(A - \lambda E)^{-1} & I_p \end{array} \right] S(\lambda) \left[\begin{array}{cc} I_n & -(A - \lambda E)^{-1}B \\ 0 & I_m \end{array} \right]$$

it follows that $S(\lambda)$ has rank $n + r$ over rationals. Further, the computation of (1,2)-inverses of $G(\lambda)$ can be performed as

$$G^+(\lambda) = \left[\begin{array}{cc} 0 & I_m \end{array} \right] S^+(\lambda) \left[\begin{array}{c} 0 \\ I_p \end{array} \right], \tag{1}$$

where $S^+(\lambda)$ is an (1,2)-inverse of $S(\lambda)$. By using this formula, the computation of the (1,2)-inverse of any rational matrix can be accomplished by determining the (1,2)-inverse of the associated system pencil. Note that if $m = p = r$, then the (1,2)-inverse of $S(\lambda)$ is the ordinary inverse of this matrix, and thus (1) represents the descriptor inverse of an invertible rational matrix. Further, if D is invertible, then the inverse system can be alternatively expressed as

$$\mathbf{G}^{-1} := (A - BD^{-1}C - \lambda E, \ -BD^{-1}, \ D^{-1}C, \ D^{-1}).$$

In the general case, we will use the following result to compute (1,2)-inverses (Ben Israel and Greville, 1976):

Lemma 1. *Let $S(\lambda)$ be a rational matrix of rank k and let P_l and P_r, permutation matrices such that*

$$P_l S(\lambda) P_r = \left[\begin{array}{cc} S_{11}(\lambda) & S_{21}(\lambda) \\ S_{21}(\lambda) & S_{22}(\lambda) \end{array} \right],$$

where $\operatorname{rank} S_{11}(\lambda) = k$. *Then an (1,2)-inverse of $S(\lambda)$ is*

$$S^+(\lambda) = P_r \left[\begin{array}{cc} S_{11}^{-1}(\lambda) & 0 \\ 0 & 0 \end{array} \right] P_l.$$

The computation of $S^+(\lambda)$ can be performed by reducing $S(\lambda)$ to an appropriate Kronecker-like form from which a regular sub-pencil of maximum rank $k = n + r$ can easily be separated. Let Q and Z be orthogonal matrices to reduce $S(\lambda)$ to the Kronecker-like form

$$\overline{S}(\lambda) := QS(\lambda)Z = \left[\begin{array}{c|c} \overline{S}_{11} & \overline{S}_{12}(\lambda) \\ \hline 0 & \overline{S}_{22} \end{array} \right]$$

$$= \left[\begin{array}{c|cccc} B_r & A_r - \lambda E_r & A_{r,\mathrm{reg}} - \lambda E_{r,\mathrm{reg}} & A_{r,l} - \lambda E_{r,l} \\ 0 & 0 & A_{\mathrm{reg}} - \lambda E_{\mathrm{reg}} & A_{\mathrm{reg},l} - \lambda E_{\mathrm{reg},l} \\ 0 & 0 & 0 & A_l - \lambda E_l \\ \hline 0 & 0 & 0 & C_l \end{array} \right],$$

where the regular part $A_{\mathrm{reg}} - \lambda E_{\mathrm{reg}}$ contains the finite and infinite system zeros, the pair $(A_r - \lambda E_r, B_r)$ is controllable with E_r nonsingular, and the pair $(C_l, A_l - \lambda E_l)$ is observable with E_l nonsingular (see (Varga, 1996) for how to obtain such a Kronecker-like form). The controllability of the pair $(A_r - \lambda E_r, B_r)$ and the observability of

the pair $(C_l, A_l - \lambda E_l)$ are the consequences of full row rank and full column rank conditions involving these pairs.

Since

$$\operatorname{rank} \overline{S}(\lambda) = \operatorname{rank} \overline{S}_{12}(\lambda),$$

by applying Lemma 1 with an obvious choice of the permutation matrices, we obtain an (1,2)-inverse of $G(\lambda)$ defined by the descriptor system

$$\mathbf{G}^+ := (\overline{A}_{12} - \lambda \overline{E}_{12}, \overline{B}_1, \overline{C}_2, 0),$$

where

$$\overline{A}_{12} - \lambda \overline{E}_{12} := \overline{S}_{12}(\lambda), \quad \left[\begin{array}{c} \overline{B}_1 \\ \overline{B}_2 \end{array} \right] := Q \left[\begin{array}{c} 0 \\ I_p \end{array} \right], \quad [\overline{C}_1\ \overline{C}_2] := [0\ -I_m]Z.$$

The eigenvalues of the inverse are

$$\Lambda(\overline{A}_{12}, \overline{E}_{12}) = \Lambda_{\text{fixed}} \cup \Lambda_{\text{spurious}},$$

where $\Lambda_{\text{fixed}} = \Lambda(A_{\text{reg}}, E_{\text{reg}})$ are the eigenvalues of the regular part thus containing the system zeros, and $\Lambda_{\text{spurious}} = \Lambda(A_r, E_r) \cup \Lambda(A_l, E_l)$ are the finite "spurious" poles originating from the column/row singularities of $G(\lambda)$.

While the system zeros are always among the poles of the inverse, the spurious poles can be chosen arbitrarily. To show this, consider two transformation matrices U and V of the form

$$U = \left[\begin{array}{ccc|c} I & 0 & 0 & K_1 \\ 0 & I & 0 & K_2 \\ 0 & 0 & I & K_3 \\ \hline 0 & 0 & 0 & I \end{array} \right] := \left[\begin{array}{c|c} I & K \\ \hline 0 & I \end{array} \right],$$

$$V = \left[\begin{array}{c|ccc} I & F_1 & F_2 & F_3 \\ \hline 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{array} \right] := \left[\begin{array}{c|c} I & F \\ \hline 0 & I \end{array} \right].$$

Then the transformed system pencil is given by

$$\widehat{S}(\lambda) := U \overline{S}(\lambda) V = \left[\begin{array}{c|c} \overline{S}_{11} & \widehat{S}_{12}(\lambda) \\ 0 & \overline{S}_{22} \end{array} \right]$$

$$= \left[\begin{array}{c|ccc} B_r & A_r + B_r F_1 - \lambda E_r & A_{r,\text{reg}} + B_r F_2 - \lambda E_{r,\text{reg}} & A_{r,l} + B_r F_3 + K_1 C_l - \lambda E_{r,l} \\ 0 & 0 & A_{\text{reg}} - \lambda E_{\text{reg}} & A_{\text{reg},l} + K_2 C_l - \lambda E_{\text{reg},l} \\ 0 & 0 & 0 & A_l + K_3 C_l - \lambda E_l \\ \hline 0 & 0 & 0 & C_l \end{array} \right], \quad (2)$$

where $\widehat{S}_{12}(\lambda) = \overline{S}_{12}(\lambda) + \overline{S}_{11}F + K\overline{S}_{22}$. The inverse is defined by

$$\mathbf{G}^+ := (\widehat{A}_{12} - \lambda\widehat{E}_{12}, \widehat{B}_1, \widehat{C}_2, 0),$$

where

$$\widehat{A}_{12} - \lambda\widehat{E}_{12} := \overline{A}_{12} + \overline{S}_{11}F + K\overline{S}_{22} - \lambda\overline{E}_{12},$$

$$\begin{bmatrix} \widehat{B}_1 \\ \widehat{B}_2 \end{bmatrix} := UQ \begin{bmatrix} 0 \\ I_p \end{bmatrix} = \begin{bmatrix} \overline{B}_1 + K\overline{B}_2 \\ \overline{B}_2 \end{bmatrix}, \tag{3}$$

$$[\, \widehat{C}_1 \ \widehat{C}_2 \,] := [\, 0 \ -I_m \,]ZV = [\, \overline{C}_1 \ \overline{C}_2 + \overline{C}_1 F \,].$$

An important aspect in some applications is to obtain inverses with the spurious poles lying in a "good" domain $\mathbb{C}_g$ of the complex plane (e.g., stable or antistable domain). For the transformed pencil $\widehat{S}(\lambda)$, the spurious poles of the inverse are

$$\Lambda_{\text{spurious}} = \Lambda(A_r + B_r F_1, E_r) \cup \Lambda(A_l + K_3 C_l, E_l).$$

Thus, with $F_2 = 0$, $F_3 = 0$, $K_1 = 0$, $K_2 = 0$ and by choosing F_1 such that $\Lambda(A_r + B_r F_1, E_r) \subset \mathbb{C}_g$ and K_3 such that $\Lambda(A_l + K_3 C_l, E_l) \subset \mathbb{C}_g$, all spurious poles can be moved to $\mathbb{C}_g$. This is always possible because the pair $(A_r - \lambda E_r, B_r)$ is controllable and the pair $(C_l, A_l - \lambda E_l)$ is observable. Thus our approach provides, in the most general setting, a complete solution to the stable left/right inverse problems solved in (Antsaklis, 1978).

Another interesting aspect is the computation of inverses with the least McMillan degree. This aspect can be addressed by our approach in the following way. We can choose F and/or K to make some of spurious poles of $G^+(\lambda)$ unobservable or uncontrollable. For example, for a fixed K, choosing F such that all spurious poles corresponding to the pair $(A_r + B_r F_1, E_r)$ become unobservable is the disturbance rejection (or output-nulling) problem (DRP) (Wonham, 1979). Similarly, cancelling the spurious poles of the pair $(A_l + K_3 C_l, E_l)$ is the dual of a DRP (unknown-input problem). The cancellation of all spurious poles is usually not possible, but to find the (1,2)-inverse with the least McMillan degree, we can always cancel a maximum number of poles by an appropriate choice of F and K. The computational problem is essentially that of finding an (A,B)-invariant subspace of least dimension which contains a given subspace (Morse, 1976) and for whose solution reliable algorithms can be devised. A much more difficult problem is to compute the least McMillan degree inverses with the additional constraint that the spurious poles lie in a given domain $\mathbb{C}_g$. As far as we know, there is no complete solution to this problem, this being equivalent to the difficult question of stabilizing a linear system via constant gain output feedback (see also (Wolovich et al., 1977)).

An important advantage of our computational approach to determine the (1,2)-inverse is that a descriptor representation of the inverse is determined without explicitly inverting the system matrix $S(\lambda)$. The reduction to the Kronecker-like form can be performed by using a structure preserving numerically stable reduction algorithm

similar to that proposed in (Misra *et al.*, 1994) (see also (Varga, 1996)). The eigen-value placement problems can be solved by using stabilization or pole assignment techniques for descriptor systems as those proposed in (Varga, 1995). Note that this computation must be performed only if the spurious eigenvalues must lie in a specific region $\mathbb{C}_g$ of the complex plane, as, for instance, in the stability domain as necessary in computing certain inner-outer factorizations (Varga, 1998; Varga and Katayama, 1998).

4. Computation of Pseudoinverses

The numerical computation of Moore-Penrose pseudoinverses was only recently addressed in the control literature (Oară and Varga, 2000). A numerically reliable approach to compute pseudoinverses of rational matrices can be devised along the lines of the recently developed general methods to compress rational matrices to full row/column-rank matrices by using left/right multiplication with inner factors. This computation is the main part of the recently developed algorithms to compute inner-outer factorizations of rational matrices (Oară and Varga, 1999; 2000). By using "orthogonal" compression techniques with inner factors, the computation of the Moore-Penrose pseudoinverse of a given $G(\lambda)$ can be performed as follows:

1. Compute the full row-rank factorization

$$G(\lambda) = U(\lambda) \left[\begin{array}{c} G_1(\lambda) \\ O \end{array} \right],$$

where $G_1(\lambda)$ has full row-rank and $U(\lambda)$ is square inner $(U^\sim(\lambda)U(\lambda) = I_p)$.

2. Compute the full row-rank factorization

$$G_1^T(\lambda) = V^T(\lambda) \left[\begin{array}{c} G_2^T(\lambda) \\ O \end{array} \right],$$

where $G_2(\lambda)$ is invertible and $V(\lambda)$ is square inner $(V^\sim(\lambda)V(\lambda) = I_m)$.
Note. We have now the overall "orthogonal" decomposition of $G(\lambda)$ as

$$G(\lambda) = U(\lambda) \left[\begin{array}{cc} G_2(\lambda) & O \\ O & O \end{array} \right] V(\lambda). \tag{4}$$

3. Compute

$$G^\#(\lambda) := V^\sim(\lambda) \left[\begin{array}{cc} G_2^{-1}(\lambda) & O \\ O & O \end{array} \right] U^\sim(\lambda). \tag{5}$$

This computational approach employing the recent compression techniques developed in (Oară and Varga, 1999; 2000) is applicable to an arbitrary rank rational matrix $G(\lambda)$ regardless of whether it is polynomial, proper or improper. Furthermore, $G(\lambda)$ can have stable or unstable poles and zeros, and even poles and zeros on the imaginary axis or on the unit circle.

The main computations in this approach are the row compressions performed in Steps 1 and 2. The methods proposed in (Oară and Varga, 1999; 2000) to perform this compression are based on reducing the system pencil $S(\lambda)$ to a particular Kronecker-like form which isolates the left singular structure of $G(\lambda)$. Then the compression is achieved by solving for the stabilizing solution a standard algebraic Riccati equation of order n_ℓ, where n_ℓ is the sum of the left minimal indices of $G(\lambda)$. Note that n_ℓ is usually much smaller than the order n of a minimal descriptor realization of $G(\lambda)$.

5. Computation of Other Type of Inverses

In this section we show how we can compute (1,2,3)- and (1,2,4)-inverses by combining the underlying techniques to determine (1,2)- and (1,2,3,4)-inverses. We also discuss shortly a general approach to determining inverses using row/column compression techniques. For notational convenience, in this section we will denote by G simply a rational matrix $G(\lambda)$.

In the previous section we discussed compression techniques of rational matrices by premultiplying them with all-pass factors. The following procedure can be used to compute a (1,2,3)-inverse of a given G by combining row compression with inner factors and (1,2)-inverse computation:

1. Compute the full row-rank factorization

$$G = U \begin{bmatrix} R \\ O \end{bmatrix},$$

   where R has full row-rank and U is square inner.

2. Compute R^+, a (1,2)-inverse (right inverse) of R, using the approach of Section 3.

3. Compute

$$G^+ := \begin{bmatrix} R^+ & O \end{bmatrix} U^\sim.$$

It is easy to check that G^+ is a (1,2,3)-inverse. Indeed,

$$GG^+ = U \begin{bmatrix} R \\ O \end{bmatrix} \begin{bmatrix} R^+ & O \end{bmatrix} U^\sim = U \begin{bmatrix} I & O \\ O & O \end{bmatrix} U^\sim$$

and the first three Moore-Penrose relations are satisfied. To compute a (1,2,4)-inverse, the above procedure can be applied to the transposed matrix G^T.

An alternative approach to calculation of generalized inverses can be devised using exclusively row/column compression techniques. With a row compression followed by a column compression, we can determine a full rank factorization of G in the form

$$G = L \begin{bmatrix} R & 0 \\ 0 & 0 \end{bmatrix} T \tag{6}$$

with L, T and R being invertible rational matrices. This decomposition of G allows us to determine several inverses with a specific choice of the transformation matrices L and T. In general, a (1,2)-inverse of G can simply be computed as

$$G^+ = T^{-1} \begin{bmatrix} R^{-1} & 0 \\ 0 & 0 \end{bmatrix} L^{-1}.$$

If we use an inner matrix L for the row compression and a general matrix T for column compression, then the corresponding inverse G^+ is a (1,2,3)-inverse, while for general L and inner T the corresponding inverse is a (1,2,4)-inverse. If L and T are inner, then the inverse G^+ is the Moore-Penrose pseudoinverse.

Compression techniques with inner matrices were recently developed in (Oară and Varga, 1999; 2000). For compression with general matrices, methods proposed in (Oară, 2000) can be employed. These methods rely also on pencil algorithms using the equivalent descriptor representation of the rational matrix G. A particular advantage of this approach is the flexibility in choosing the poles and zeros of the compressing factor.

6. Numerical Issues

All presented computational methods to determine a generalized inverse of a rational matrix $G(\lambda)$ have overall computational complexity $O(n^3)$, where n is the order of the underlying descriptor representation $\mathbf{G}$. The main advantage of using pencil techniques to solve computational problems involving rational matrices is that the whole arsenal of well-developed linear algebra techniques to manipulate matrix pencils can be employed to devise numerically reliable algorithms for these rather complex problems. In contrast, rational matrix manipulations using polynomial techniques are generally considered to be numerically less robust than methods based on state-space representations. It is to be expected that all presented techniques can be extended to more general systems, as, for instance, periodic time-varying and even general time-varying systems.

For the computation of weak generalized inverses with arbitrarily assigned spurious poles and for the computation of Moore-Penrose pseudoinverses, MATLAB functions are available in a recently developed DESCRIPTOR SYSTEMS toolbox (Varga, 2000). The main functions rely on a collection of `mex`-functions providing easy-to-use gateways to highly optimized Fortran codes from the SLICOT library (Benner *et al.*, 1999) developed within the NICONET project.[1] The computational layer of basic `mex`-functions provides efficient and numerically robust computational tools to perform reductions to Kronecker-like forms, reordering of generalized real Schur forms, balancing of descriptor systems, minimal realization of rational matrices, and other computations.

7. Example

Consider the rational matrix from (Wang and Davison, 1973),

$$G(s) = \begin{bmatrix} \dfrac{1}{s+2} & \dfrac{1}{s+1} \\[2mm] \dfrac{s+3}{s^2+3s+2} & \dfrac{s}{s+1} \\[2mm] \dfrac{s^2+3s}{s^2+3s+2} & 0 \end{bmatrix}.$$

This matrix has full column rank and no zeros. A state space realization for $G(s)$ is given by

$$A = \begin{bmatrix} -3 & -2 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 3 & -1 \\ 0 & -2 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

For $G(s)$ above, we compute three left-inverses with particular properties and the unique Moore-Penrose pseudoinverse.

We determine the left-inverses using the approach to compute (1,2)-inverses of Section 3. For notation, we also refer to this section. To allow for an easy reproducibility of the results, we illustrate our method by using non-orthogonal transformation

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[1] see `http://www.win.tue.nl/niconet/niconet.html`

matrices Q and Z to obtain the Kronecker-like form of the system pencil $S(\lambda)$ with "nice" numbers. For the transformation matrices

$$U = \begin{bmatrix} I_5 & K \\ O & 1 \end{bmatrix}, \quad Q = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & -1 \\ 1 & 1 & 0 & 0 & 0 & -1 \\ 1 & 1 & 1 & 0 & -1 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix},$$

$$Z = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}, \quad V = I,$$

where

$$K = \begin{bmatrix} K_2 \\ \hline K_3 \end{bmatrix} := \begin{bmatrix} k_1 \\ k_2 \\ \hline k_3 \\ k_4 \\ k_5 \end{bmatrix},$$

we obtain the reduced pencil $\widehat{S}(\lambda) = UQS(\lambda)ZV$ in (2) such that

$$\widehat{S}(\lambda) = \begin{bmatrix} A_{\mathrm{reg}} & A_{\mathrm{reg},l} + K_2 C_l \\ \hline O & A_l + K_3 C_l - \lambda I \\ \hline O & C_l \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 & -2 & k_1 \\ 0 & 1 & -2 & 4 & k_2 - 1 \\ \hline 0 & 0 & -3-\lambda & 0 & k_3 \\ 0 & 0 & -2 & -\lambda & k_4 \\ 0 & 0 & 0 & -3 & -\lambda + k_5 \\ \hline 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

From (3) we obtain

$$
\left[\begin{array}{c} \widehat{B}_1 \\ \hline \widehat{B}_2 \end{array}\right] = \left[\begin{array}{ccc} k_1 & 0 & 1 \\ k_2 & 1 & 0 \\ k_3 & 0 & -1 \\ k_4 & 0 & -1 \\ k_5 & -1 & -1 \\ \hline 1 & 0 & 0 \end{array}\right], \quad \widehat{C}_2 = \left[\begin{array}{ccccc} -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \end{array}\right].
$$

The (1,2)-inverse corresponding to $\widehat{S}(\lambda)$ has a standard state-space realization $(A_{\mathrm{inv}}, B_{\mathrm{inv}}, C_{\mathrm{inv}}, D_{\mathrm{inv}})$ of order 3, with

$$
A_{\mathrm{inv}} = \left[\begin{array}{ccc} -3 & 0 & k_3 \\ -2 & 0 & k_4 \\ 0 & -3 & k_5 \end{array}\right], \quad B_{\mathrm{inv}} = \left[\begin{array}{ccc} k_3 & 0 & -1 \\ k_4 & 0 & -1 \\ k_5 & -1 & -1 \end{array}\right],
$$

$$
C_{\mathrm{inv}} = \left[\begin{array}{ccc} 2 & -2 & k_1 \\ -2 & 4 & k_2 - 1 \end{array}\right], \quad D_{\mathrm{inv}} = \left[\begin{array}{ccc} k_1 & 0 & 1 \\ k_2 & 1 & 0 \end{array}\right].
$$

This realization can be easily retrieved from the particular forms of $\widehat{S}(\lambda)$, $\widehat{B}_1$ and $\widehat{C}_2$.

With $K = 0$, the corresponding (1,2)-inverse is

$$
G^+(s) = \left[\begin{array}{ccc} 0 & 0 & \dfrac{s^2 + 3s + 2}{s(s+3)} \\[3mm] 0 & \dfrac{s+1}{s} & -\dfrac{s+1}{s^2} \end{array}\right]
$$

and has McMillan degree 3. This inverse is not stable, having poles in the origin. With

$$
K = \left[\begin{array}{ccccc} 0 & 0 & 0 & 2 & -5 \end{array}\right]^T
$$

we can assign the spurious poles of the inverse to $\{-2, -3, -3\}$. The corresponding stable (1,2)-inverse

$$
G^+(s) = \left[\begin{array}{ccc} \dfrac{-4s}{(s+2)(s+3)} & \dfrac{4}{(s+2)(s+3)} & \dfrac{s^3 + 8s^2 + 27s + 28}{(s+2)(s+3)^2} \\[4mm] \dfrac{13s + 6}{(s+2)(s+3)} & \dfrac{s^2 + 6s - 2}{(s+2)(s+3)} & -\dfrac{s^2 + 22s + 35}{(s+2)(s+3)^2} \end{array}\right]
$$

has McMillan degree 3.

A stable left inverse of the least McMillan order can be obtained by choosing

$$K = \begin{bmatrix} -\dfrac{2}{3} & 1 & \dfrac{4}{3} & 1 & 0 \end{bmatrix}^T.$$

The corresponding inverse is

$$G^+(s) = \begin{bmatrix} -\dfrac{2}{3}\dfrac{s}{s+1} & \dfrac{2}{3}\dfrac{1}{s+1} & \dfrac{1}{3}\dfrac{3\,s+5}{s+1} \\[2ex] \dfrac{1}{3}\dfrac{10\,s+3\,s^2+3}{s^2+2\,s+1} & \dfrac{1}{3}\dfrac{3\,s^2+6\,s-1}{s^2+2\,s+1} & -\dfrac{2}{3}\dfrac{3\,s+5}{s^2+2\,s+1} \end{bmatrix}$$

and has McMillan degree 2. This is the same inverse as that of (Wang and Davison, 1973, p.224), corresponding to the free parameters set to $b_{01} = 0$ and $b_{02} = 1$. We also computed the unique Moore-Penrose pseudoinverse of $G(s)$

$$G^\#(s) = \begin{bmatrix} \dfrac{-s^5-3s^4+s^3+9s^2+6s}{p(s)} & \dfrac{s^4+3s^3-s^2-9s-6}{p(s)} & \dfrac{s^3+8s^2+27s+28}{p(s)} \\[2ex] \dfrac{s^6+s^5-10s^4-11s^3+2s^2+6s+3}{p(s)} & \dfrac{-s^5+13s^3+9s^2-12s-9}{p(s)} & \dfrac{-s^5+8s^3+4s^2-3s}{p(s)} \end{bmatrix},$$

where $p(s) = s^6 - 11s^4 + 15s^2 - 9$. $G^\#(s)$ has McMillan degree 6 and is unstable.

8. Conclusions

We have proposed numerically reliable methods to compute two classes of generalized inverses of rational matrices: the (1,2)- or weak inverses and the Moore-Penrose pseudoinverses. The proposed methods are completely general, being applicable to rational matrices of arbitrary ranks regardless of whether they are polynomial, proper or improper. Particular emphasis has been put on reliably computing (1,2)-inverses, because of their relevance to many practical applications. The proposed approach provides flexibility to cope with various conditions on the spurious poles of the computed (1,2)-inverses. For instance, a stable (1,2)-inverse can easily be computed whenever exists. The computation of other types of inverses has been also addressed, by combining "orthogonal" compression techniques using inner factors with left/right inverse computation. For the proposed methods, robust numerical software has been implemented.

Interesting open computational problems in the context of determining various types of inverses are posed by the exploitation of intrinsic parametric freedom. Two aspects could be relevant for control applications: (1) determining particular inverses with special properties (e.g., a minimal Mc-Millan degree or stable with minimal Mc-Millan degree), and (2) generating the whole class of inverses by exploiting the parametrizations of inverses (see, e.g., (Campbell and Meyer, 1991)). In this paper we partially addressed only the first aspect. However, a detailed elaboration of algorithms to compute least order inverses is still necessary.

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