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EXACT CONTROLLABILITY OF AN ELASTIC MEMBRANE COUPLED WITH A POTENTIAL FLUID

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We consider the problem of boundary control of an elastic system with coupling to a potential equation. The potential equation represents the linearized motions of an incompressible inviscid fluid in a cavity bounded in part by an elastic membrane. Sufficient control is placed on a portion of the elastic membrane to insure that the uncoupled membrane is exactly controllable. The main result is that if the density of the fluid is sufficiently small, then the coupled system is exactly controllable.

Keywords: exact controllability, fluid-elastic interaction, fluid-structure interaction, potential fluid

1. Introduction

In this article we consider the problem of controlling an elastic membrane that is adjacent to a linear potential fluid. There has recently been much research in this direction, concerning the case of an elastic system with acoustic coupling, see (Avalos, 1996; Banks *et al.*, 1993; Lions and Zuazua, 1995; Micu and Zuazua, 1997) and references therein. Other papers consider the case where the adjacent fluid is a Stokes fluid (Osses and Puel, 1998; 1999). In this paper the fluid is modeled as a linearized potential fluid (with a harmonic velocity potential). This model has been used, for example, to analyze the dynamics of the cochlea in the inner ear (Lighthill, 1981). A comparison of these various models can be found in (Conca *et al.*, 1998).

The system we consider involves a body of fluid bounded at least partly by a flexible membrane. A potential equation is used to model the fluid while a wave equation is used to model the membrane. The two equations are coupled by matching velocities of the fluid with that of the membrane, and using the fluid pressure as a forcing term for the membrane. Furthermore, the incompressibility of the fluid introduces constraints upon the possible motions of the membrane.

It is well-known for the wave equation that the Dirichlet boundary control on a "sufficiently large" portion of the boundary is sufficient for exact controlliblity. The main result of this article is that if the fluid density is sufficiently small, the

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same holds true for the coupled fluid-membrane system. A similar result, although regarding some special cases involving a two-dimensional potential fluid *surrounding* a one-dimensional elastic system, appeared in (Hansen and Lyashenko, 1997). There, the moment method was applied and hence that approach does not work for the case of a three-dimensional fluid considered here. Here, a modification of the classical "multiplier method" is applied. The modifications involve handling the additional terms from the fluid coupling, and finding suitable multipliers that are valid in consideration of the incompressibility constraint.

1.1. Problem Formulation

Consider the situation of a fluid in a cavity in which a portion (at least) of the boundary is flexible. Then the domain of the fluid has a boundary consisting of a rigid part and a flexible part such that the fluid is on one side of the flexible boundary. (This requirement is for simplicity only.) The fluid in the cavity is assumed to be incompressible and irrotational (inviscid), and velocities are small enough so that linearization about the motionless state is valid. The membrane is forced by the pressure of the fluid and the velocity of the fluid is matched with the velocity of the boundary. Control is exercised on a portion of the boundary of the membrane.

To describe the situation mathematically, we let Ω denote a bounded domain in $\mathbb{R}^3$ ($\mathbb{R}^2$ is OK, with obvious adjustments) with Lipschitz boundary Γ. It is assumed that Γ consists of an *inflexible* part Γ_0 and a *flexible* part ω. For simplicity, it is assumed that at equilibrium ω is a subdomain of the plane $x_3 = 0$ that has a sufficiently smooth (C^2 is fine) boundary $\gamma \neq \emptyset$, which consists of a controlled part γ_1 and an uncontrolled part γ_0. To avoid a discussion of singularities, we assume that $\gamma = \gamma_0 \cup \gamma_1$ with $\bar{\gamma}_0 \cap \bar{\gamma}_1 = \emptyset$. We also need a condition that guarantees that control is active on a large enough portion of γ. One can assume γ_0 to be empty, otherwise it is assumed that the uncontrolled region satisfies the standard geometric condition:

$$(x_1, x_2) \cdot \mathbf{n} \leq 0 \quad \text{on} \quad \gamma_0, \tag{1}$$

where $\mathbf{n}$ denotes the unit outward normal vector to γ (in the plane $x_3 = 0$). Due to the incompressibility, the fluid velocity $\mathbf{q}$ satisfies

$$\operatorname{div} \mathbf{q} = 0 \quad \text{on} \quad \Omega \times \mathbb{R}^+$$

and the assumption that the fluid is irrotational (inviscid) implies that

$$\operatorname{curl} \mathbf{q} = 0 \quad \text{on} \quad \Omega \times \mathbb{R}^+.$$

Consequently ,

$$\mathbf{q} = \nabla \Phi, \qquad \text{where} \ \ \Delta \Phi = 0 \ \ \text{on} \ \ \Omega \times \mathbb{R}^+. \tag{2}$$

Matching velocities of the fluid and membrane at $x_3 = 0$ leads to

$$\frac{\partial \Phi}{\partial \mathbf{n}} = \begin{cases} 0 & \text{on} \ \ \Gamma_0 \times \mathbb{R}^+, \\ w_t & \text{on} \ \ \omega \times \mathbb{R}^+. \end{cases} \tag{3}$$

Actually, if velocities are matched on the deformed boundary ω, we obtain a free boundary problem. However, since small vibrations are under consideration, we linearize by matching velocities on the boundary of the (fixed) equilibrium domain Ω. Henceforth, Ω and its boundary will be assumed to be fixed, as is the case in all the papers mentioned earlier.

The energy $\mathcal{E}(t)$ is the sum of the kinetic $\mathcal{K}(t)$ and potential $\mathcal{P}(t)$ energies where

$$\mathcal{K} = \frac{1}{2}\int_\Omega \rho|\nabla_3\Phi|^2 \,\mathrm{d}\Omega + \frac{1}{2}\int_\omega |w_t|^2 \,\mathrm{d}\omega, \quad \mathcal{P} = \frac{1}{2}\int_\omega |\nabla_2 w|^2 \,\mathrm{d}\omega,$$

∇_k being the gradient in the first k coordinate directions.

The equations of motion can be obtained from Hamilton's principle. That is, the first variation, with respect to a class of admissible variations, of the Lagrangian $\mathcal{L} = \int_0^T (\mathcal{K} - \mathcal{P})\,\mathrm{d}t$ is set to zero. The class of variation functions $\{\hat{w}, \hat{\Phi}\}$ we consider includes those that satisfy the constraints (2), (3), $\hat{w}\big|_\gamma = 0$, and vanish near $t = 0$ and $t = T$. We obtain

$$\begin{aligned} 0 \;&=\; \int_0^T \left\{\int_\omega w_t\hat{w}_t - \nabla w\nabla\hat{w}\,\mathrm{d}\omega + \rho\int_\Omega \nabla\Phi\nabla\hat{\Phi}\,\mathrm{d}\Omega\right\}\mathrm{d}t \\ &=\; \int_0^T\!\!\int_\omega \left\{(-w_{tt} + \Delta w)\hat{w} + \Phi\hat{w}_t\right\}\mathrm{d}\omega\,\mathrm{d}t \\ &=\; \int_0^T\!\!\int_\omega \left\{(-w_{tt} + \Delta w - \rho\Phi_t)\hat{w}\,\mathrm{d}\omega\right\}\mathrm{d}t. \end{aligned} \tag{4}$$

Note that since $\hat{\Phi}$ is determined by a Neumann problem, $\int_\omega \hat{w}_t\,\mathrm{d}\omega = \int_\omega \partial\hat{\Phi}/\partial x_3\,\mathrm{d}\omega = 0$. Therefore $\int_\omega \hat{w}\,\mathrm{d}\omega$ is constant and equal to its initial value. Thus

$$\int_\omega \hat{w}\,\mathrm{d}\omega = 0, \quad \forall t \geq 0.$$

Consequently, the equations (in the strong form) are only determined up to an additive constant (denoted by C).

In the strong form the equations of motion become

$$w_{tt} + \rho\Phi_t - \Delta w \;=\; C \qquad \text{in } \omega\times\mathbb{R}^+, \tag{5}$$

$$\Delta\Phi \;=\; 0 \qquad \text{in } \Omega\times\mathbb{R}^+, \tag{6}$$

$$\frac{\partial\Phi}{\partial\mathbf{n}} \;=\; \begin{cases} 0 & \text{on } \Gamma_0\times\mathbb{R}^+, \\ w_t & \text{on } \omega\times\mathbb{R}^+, \end{cases} \tag{7}$$

$$w \;=\; \begin{cases} 0 & \text{on } \gamma_0\times\mathbb{R}^+, \\ f & \text{on } \gamma_1\times\mathbb{R}^+. \end{cases} \tag{8}$$

Initial conditions are of the form

$$(w, w_t)\big|_{t=0} = (w_0, w_1), \quad \text{where} \int_\omega w_1 \, \mathrm{d}\omega = 0. \tag{9}$$

Due to (7) we have $\int_\omega w \, \mathrm{d}\omega = \int_\omega w_0 \, \mathrm{d}\omega$. One can rewrite the equations in terms of $\tilde{w} = w - w^*$, where w^* is the steady state solution determined by (5) and (8) (with zero boundary data) and the condition $\int_\omega w^* \, \mathrm{d}\omega = \int_\omega w_0 \, \mathrm{d}\omega$. Thus, without loss of generality,

$$\int_\omega w_0 \, \mathrm{d}\omega = 0. \tag{10}$$

The natural energy space E for the system is

$$E = \big\{(w, w_t, \Phi) \in \tilde{H}_0^1(\omega) \times \tilde{L}^2(\omega) \times (H^1(\Omega)/C) : \Phi \text{ satisfies (6), (7)}\big\}, \tag{11}$$

where $\tilde{H}_0^1(\omega)$ and $\tilde{L}^2(\omega)$ denote the subspaces of functions in $H_0^1(\omega)$ and $L^2(\omega)$, respectively, which are orthogonal to constants. The space $H^1(\Omega)/C$ denotes the equivalence classes of functions in $H^1(\Omega)$ that are identified up to an additive constant.

As regards the uncontrolled system, we will prove the following:

Theorem 1. *Assume that $f = 0$ and $(w_0, w_1) \in \tilde{H}_0^1(\omega) \times \tilde{L}^2(\omega)$. Then (5)–(8), (9) have a unique solution with*

$$(w, w_t, \Phi) \in C\big([0, \infty); E\big). \tag{12}$$

Moreover, the energy $\mathcal{E}(t) = \mathcal{K}(t) + \mathcal{P}(t)$ is conserved along solution trajectories. If, in addition, $(w_0, w_1) \in \mathcal{V} := H^2(\omega) \cap \tilde{H}^1(\omega) \times \tilde{H}_0^1(\omega)$, then

$$(w, w_t) \in C\big([0, \infty); \mathcal{V}\big). \tag{13}$$

We will, however, wish to utilize a control $f \in L^2((0,T) \times \gamma_1)$ and hence it is necessary to work with weak solutions defined on $\mathcal{V}'$ (the dual of $\mathcal{V}$ relative to an inner product defined on $\mathcal{H}$; see Section 2). Later we will show that for $f \in L^2((0,T) \times \gamma_1)$ and any initial condition $(w_0, w_1) \in \mathcal{V}'$ there is a unique weak solution that satisfies

$$(w, w_t) \in C\big([0, T); \mathcal{V}'\big).$$

Regarding our control problem, under all the geometric conditions described earlier, we have the following main result.

Theorem 2. *There exists $\rho_0 > 0$ such that if $0 \le \rho < \rho_0$, then the system (5)–(8) is exactly controllable on the space $\tilde{L}^2(\omega) \times \tilde{H}^{-1}(\omega)$. That is, for T large enough, if $0 \le \rho < \rho_0$, given any initial data $\{w_0, w_1\} \in \mathcal{V}'$, there exists an $f \in L^2((0,T) \times \omega)$ such that $\{w, w_t\}\big|_{t=T} = \{0, 0\}$ and $\Phi\big|_{t=T}$ is constant.*

Remark 1. Clearly, due to the time-reversibility of the system, one can equivalently find an L^2 control which drives the given initial state to any desired terminal state in $\mathcal{V}'$ in the same time.

Remark 2. Actually, we will prove the following "observability inequality", which by duality (i.e., "Hilbert's Uniqueness Method", cf. (Lions, 1988)) is equivalent to the controllability in Theorem 2: *Let $\{w^0, w^0\} \in \tilde{H}_0^1(\omega) \times \tilde{L}^2(\omega)$ and suppose that T and ρ are as in Theorem 2. Let w denote a solution to (5)–(10) with $f = 0$. Then there exists $c > 0$ such that*

$$\int_0^T \int_{\gamma_1} \left| \frac{\partial w}{\partial \mathbf{n}} \right|^2 \mathrm{d}\gamma \, \mathrm{d}t \geq c\mathcal{E}(0). \tag{14}$$

Remark 3. It unknown whether ρ_0 in Theorem 2 can be taken arbitrarily large, or whether the control time T is as small as that of the uncoupled wave equation on ω. On the other hand, the proof provides an explicit lower bound for ρ_0 in terms of the geometry of Ω and an explicit estimate for T in terms of ρ and the geometry of Ω.

2. Existence, Uniqueness and Regularity

In this section we prove the well-posedness of the coupled elastic-fluid system.

2.1. Regularity of Fluid Pressure on the Beam

We first discuss some properties of the Neumann problem for the Laplacian. We assume that Ω is a bounded domain in $\mathbb{R}^3$ with boundary Γ.

Consider the following Neumann problem:

$$\begin{cases} \Delta\Phi = 0 \quad \text{in } \Omega, \\ \dfrac{\partial \Phi}{\partial \mathbf{n}} = f \quad \text{on } \Gamma. \end{cases} \tag{15}$$

The solvability condition for (15) is

$$\int_\Gamma f \, \mathrm{d}\Gamma = 0. \tag{16}$$

It is well-known that when Γ and f are regular and f satisfies (16), there exist classical solutions to (15) which are unique up to an arbitrary additive constant. If f or Γ is less regular, then variational solutions may be defined in some cases. For our purposes the following regularity results (Necas, 1967) will be sufficient.

Proposition 1. *For (15), (16) suppose that Γ is Lipschitz and $-1 \leq s \leq 0$. If $f \in H^s(\Gamma)$, (with zero average), then $\Phi \in H^{s+3/2}(\Omega)/C$ and $\Phi\big|_\Gamma \in H^{s+1}(\Gamma)/C$, where X/C denotes the quotient space of functions in X identified up to an additive constant.*

For $\phi \in \tilde{L}^2(\omega)$ define the *Neumann to Dirichlet map* Λ by

$$\Lambda\phi = \Phi\big|_\omega,$$

where

$$\begin{aligned} \Delta\Phi &= 0 \quad \text{in } \Omega, \\ \Phi &= 0 \quad \text{on } \Gamma_0, \\ \frac{\partial\Phi}{\partial\mathbf{n}} &= \phi \quad \text{on } \omega. \end{aligned}$$

Proposition 2. *Assume that* Γ *is Lipschitz. Then* $\Lambda : \tilde{L}^2(\omega) \to H^1(\omega)/C$ *continuously and*

$$\|\Lambda\phi\|_{\tilde{H}^1(\omega)} \leq C_\Omega \|\phi\|_{L^2(\omega)}, \quad \forall\, \phi \in \tilde{L}^2(\omega). \tag{17}$$

Furthermore, Λ *is positive and self-adjoint on* $\tilde{L}^2(\omega)$ *in the sense that*

$$\int_\omega (\Lambda\phi)\psi \,\mathrm{d}\omega = \int_\omega (\Lambda\psi)\phi \,\mathrm{d}\omega, \quad \forall\, \phi, \psi \in \tilde{L}^2(\omega), \tag{18}$$

$$\int_\omega (\Lambda\phi)\phi \,\mathrm{d}\omega \geq 0, \quad \forall\, \phi \in \tilde{L}^2(\omega). \tag{19}$$

Proof. The estimate (17) follows immediately from Proposition 1. To prove the second part, let F and G be harmonic functions on Ω with

$$\frac{\partial F}{\partial\mathbf{n}} = \begin{cases} 0 & \text{on } \Gamma_0, \\ \phi & \text{on } \omega, \end{cases} \quad \text{and} \quad \frac{\partial G}{\partial\mathbf{n}} = \begin{cases} 0 & \text{on } \Gamma_0, \\ \psi & \text{on } \omega. \end{cases} \tag{20}$$

Integration by parts, using the definition of Λ yields

$$\int_\omega (\Lambda\phi)\psi \,\mathrm{d}\omega = \int_\omega F\frac{\partial G}{\partial\mathbf{n}} \,\mathrm{d}\omega = \int_\Omega \nabla F \nabla G \,\mathrm{d}\omega = \int_\omega \phi\Lambda\psi \,\mathrm{d}\omega. \tag{21}$$

The same equations with $\psi = \phi$ establish the positivity in (19). This completes the proof. ■

We now can, at least formally, rewrite the system (5)–(8) as

$$w_{tt} + \rho(\Lambda w_t)_t - \Delta w = C \qquad \text{in } \omega \times \mathbb{R}^+, \tag{22}$$

$$w = \begin{cases} 0 & \text{on } \gamma_0 \times \mathbb{R}^+, \\ f & \text{on } \gamma_1 \times \mathbb{R}^+. \end{cases} \tag{23}$$

2.2. Finite Energy Solutions

In this section we use the semigroup theory to prove the existence and uniqueness of finite energy solutions to (22)–(23).

Let us define $\mathcal{C}_\rho : \tilde{L}^2(\omega) \to L^2(\omega)/C$ by

$$\mathcal{C}_\rho \phi = \phi + \rho \Lambda \phi.$$

From Proposition 2 it is clear that $\mathcal{C}_\rho$ is positive and self-adjoint for $\rho \geq 0$ in the same sense Λ is in (18), (19). For this reason it is convenient to identify the space $\tilde{L}^2(\omega)$ with its dual $L^2(\omega)/C$. Under this identification each element $\phi \in L^2(\omega)/C$ is identified with an element $\phi_0 \in \tilde{L}^2(\omega)$ by $\phi_0 = \phi + C_0$, where C_0 is picked so that $\int_\omega (\phi + C_0)\,\mathrm{d}\omega = 0$. Thus $\mathcal{C}_\rho$ is positive and self-adjoint on $\tilde{L}^2(\omega)$ in the usual sense.

Define the following forms:

$$\begin{aligned}
(\phi, \psi) &= \int_\omega \phi \cdot \bar{\psi}\,\mathrm{d}\omega, \quad \forall \phi, \psi \in \left(L^2(\omega)\right)^3, \\
\langle \phi, \psi \rangle_\rho &= \int_\omega (\mathcal{C}_\rho \phi) \bar{\psi}\,\mathrm{d}\omega, \quad \forall \phi, \psi \in \tilde{L}^2(\omega), \\
\big\langle \{\phi_1, \phi_2\}, \{\psi_1, \psi_2\} \big\rangle_{\mathcal{E}} &= (\nabla \phi_1, \nabla \psi_1) + \langle \phi_2, \psi_2 \rangle_\rho, \quad \forall \{\phi_1, \phi_2\}, \{\psi_1, \psi_2\} \in \mathcal{H}.
\end{aligned}$$

Due to Poincaré's inequality and the above mentioned properties of the operator $\mathcal{C}_\rho$, $(\nabla \cdot\,, \nabla \cdot)$ is easily seen to be an inner product on $\tilde{H}_0^1(\omega)$. Likewise, $\langle \cdot\,, \cdot \rangle_\rho$ is an inner product on $\tilde{L}^2(\omega)$. Consequently, $\langle \cdot\,, \cdot \rangle_{\mathcal{E}}$ is an inner product on the finite energy space $\mathcal{H}$.

Define $y = w$ and $v = w_t$. The first-order form of (22) is

$$\frac{\mathrm{d}}{\mathrm{d}t} \begin{pmatrix} y \\ v \end{pmatrix} = \begin{pmatrix} 0 & I \\ \mathcal{C}_\rho^{-1} \Delta_C & 0 \end{pmatrix} \begin{pmatrix} y \\ v \end{pmatrix} =: \mathcal{A}\{y, v\}, \tag{24}$$

where we have written Δ_C to emphasize that the range must be considered as a quotient space, or equivalently, we set $\Delta_C \phi = \Delta \phi + C$ such that $\Delta \phi + C$ has zero average (as is done when $\tilde{L}^2(\omega)$ is identified with its dual $L^2(\omega)/C$).

Also, define the spaces

$$\mathcal{H} = \tilde{H}_0^1(\omega) \times \tilde{L}^2(\omega), \quad \mathcal{V} = \Big(H^2(\omega) \cap \tilde{H}_0^1(\omega) \Big) \times \tilde{L}^2(\omega).$$

Proposition 3. *Let $\rho > 0$. The operator $\mathcal{A}$ is the generator of a strongly continuous group of isometries with respect to the energy inner product $\langle \cdot, \cdot \rangle_{\mathcal{E}}$ on the finite energy space $\mathcal{H}$. Consequently, given the initial conditions*

$$\big\{ y(0), v(0) \big\} = \{ y^0, v^0 \} \in \mathcal{H}, \tag{25}$$

there exists a unique solution $\{y, v\}$ to (24) that satisfies

$$\{y, v\} \in C(\mathbb{R}; \mathcal{H}). \tag{26}$$

Proof. With $\mathcal{D}(\mathcal{A}) = \mathcal{V}$ it is clear that $\mathcal{A}$ is densely defined. Let us verify that $\mathcal{A}$ is closed. We let $\{\phi_n, \psi_n\} \to \{\phi, \psi\}$ in $\mathcal{H}$, with $\{\phi_n, \psi_n\} \in \mathcal{V}$. Assume that $\{y_n, v_n\} := \mathcal{A}\{\phi_n, \psi_n\} \to \{y, v\}$ in $\mathcal{H}$. We need to see that $\{\phi, \psi\} \in \mathcal{V}$ and $\mathcal{A}\{\phi, \psi\} = \{y, v\}$. Since $y_n = \psi_n \to y$ in $\tilde{H}_0^1(\omega)$ and $\psi_n \to \psi$ in $\tilde{L}^2(\omega)$, it follows that $\psi = y \in \tilde{H}_0^1(\omega)$. For the other variable we have $v_n = \mathcal{C}_\rho^{-1}\Delta_C \phi_n \to v$ in $\tilde{L}^2(\omega)$ and $\phi_n \to \phi$ in $\tilde{H}_0^1(\omega)$. First note that $-\Delta_C$ is associated with a coercive symmetric quadratic form on $\tilde{H}_0^1(\omega)$ since for all f, g in $H^2(\omega) \cap \tilde{H}_0^1(\omega)$ one has

$$\int_\omega -\Delta_C f g \,\mathrm{d}\omega = \int_\omega \nabla f \cdot \nabla g \,\mathrm{d}\omega = -\int_\omega f \Delta_C g \,\mathrm{d}\omega.$$

The form is strictly positive since

$$\min_{u \in \tilde{H}_0^1(\omega)} \frac{(\nabla u, \nabla u)}{\|u\|_{L^2}^2} \geq \min_{u \in H_0^1(\omega)} \frac{(\nabla u, \nabla u)}{\|u\|_{L^2}^2} = \lambda_1 > 0,$$

where λ_1 is the first eigenvalue of $-\Delta$ (the usual Laplacian operator with homogeneous Dirichlet boundary conditions). From the Lax-Milgram theorem it follows that Δ_C is an isomorphism from $H^2(\omega) \cap \tilde{H}_0^1(\omega)$ to $\tilde{L}^2(\omega)$. Due to the positivity of Λ, $\mathcal{C}_\rho$ is also an isomorphism on $\tilde{L}^2(\omega)$. From this we conclude that $\{\phi_n\}$ is convergent in $H^2(\omega)$. Since $\phi_n \to \phi$ in $\tilde{H}_0^1(\omega)$, we see that $\phi = \lim_{n\to\infty} \Delta_C^{-1}\mathcal{C}_\rho v_n = \Delta_C^{-1}\mathcal{C}_\rho v$. Thus $\mathcal{A}$ is closed.

To show that $\mathcal{A}$ generates a group, we apply the Lumer-Phillips theorem (see e.g., Pazy, 1983) to $\mathcal{A}$ and $-\mathcal{A}$, i.e., it is enough to show that $\mathcal{A}$, $\mathcal{A}^*$, $-\mathcal{A}$, $-\mathcal{A}^*$ are all dissipative. However, this follows if we show that $\mathcal{A}$ is anti-Hermitian with respect to the energy inner product. For all $\{y_i, v_i\} \in \mathcal{V}$, $i = 1, 2$ we calculate

$$\begin{aligned}
&\big\langle \mathcal{A}\{y_1, v_1\}, \{y_2, v_2\}\big\rangle_{\mathcal{E}} \\
&\quad = \big\langle \{v_1, \mathcal{C}_\rho^{-1}\Delta_C y_1\}, \{y_2, v_2\}\big\rangle_{\mathcal{E}} = (\nabla v_1, \nabla y_2) + \langle \mathcal{C}_\rho^{-1}\Delta_C y_1, v_2\rangle_\rho \\
&\quad = -(v_1, \Delta y_2) + (\Delta_C y_1, v_2) = -(v_1, \Delta_C y_2) + (\Delta y_1, v_2) \\
&\quad = -(\mathcal{C}_\rho v_1, \mathcal{C}_\rho^{-1} y_2) - (\nabla y_1, \nabla v_2) = -\langle v_1, \mathcal{C}_\rho^{-1} y_2\rangle_\rho - (\nabla y_1, \nabla v_2) \\
&\quad = -\big\langle \{y_1, v_1\}, \mathcal{A}\{y_2, v_2\}\big\rangle_{\mathcal{E}}.
\end{aligned} \tag{27}$$

Thus $\mathcal{A}$ is antisymmetric and $\mathcal{D}(\mathcal{A}) \subset \mathcal{D}(\mathcal{A}^*)$. However, as we have previously shown, $\mathcal{C}_\rho\Delta_C$ is an isomorphism from $H^2(\omega) \cap \tilde{H}_0^1(\omega)$ to $\tilde{L}^2(\omega)$. It follows that $\mathcal{A}$ is surjective from $\mathcal{V}$ to $\mathcal{H}$ and hence $\mathcal{D}(\mathcal{A}) = \mathcal{D}(\mathcal{A}^*)$. Thus the semigroup generated by $\mathcal{A}$ is actually a group and is easily shown to be unitary. The other statements in the theorem are immediate consequences. ■

2.3. Weak Solutions

Let us denote by $\tilde{H}^{-1}(\omega)$ the dual space to $\tilde{H}_0^1(\omega)$ relative to $\langle \cdot, \cdot \rangle_\rho$. Since we have proven that $\mathcal{A} : \mathcal{V} \to \mathcal{H}$ is one to one and onto, when $\mathcal{V}$ is endowed with the graph norm $\|\{y, v\}\|_{\mathcal{V}} = \|\mathcal{A}\{y, v\}\|_{\mathcal{E}}$, $\mathcal{A}$ becomes a topological isomorphism as well. We can define by duality an extension of $\mathcal{A}$, temporarily denoted by $\hat{\mathcal{A}}$, from $\mathcal{H}$ to $(\mathcal{D}(\mathcal{A}^*))' = (\mathcal{D}(\mathcal{A}))' =: \mathcal{V}'$ as follows:

$$\big\langle \hat{\mathcal{A}}\{y,v\},\{Y,V\}\big\rangle_{\mathcal{E}} = \big\langle \{h,v\},-\mathcal{A}\{Y,V\}\big\rangle_{\mathcal{E}}, \quad \forall\,\{Y,V\}\in\mathcal{V}.$$

Since $(\nabla y,\nabla Y) = (y,\Delta_C Y)$ for all $y\in\tilde{H}^1_0(\omega)$ and all $Y\in H^2(\omega)\cap H^1_0(\omega)$, we see that the first component of $(\mathcal{D}(\mathcal{A}))'$ is $\tilde{L}^2(\omega)$. The second component is the dual space to $\tilde{H}^1_0(\omega)$ relative to $\langle\cdot,\cdot\rangle_\rho$. Denote this by $\tilde{H}^{-1}(\omega)$.

The extended operator $\hat{\mathcal{A}}$ can be shown to be the generator of a strongly continuous semigroup of unitary operators isomorphic to the original one. Henceforth we make no distinction between $\mathcal{A}$ and its possible extensions. As regards the system (24) or, equivalently, (22)–(23) with $f=0$, we have the following:

Corollary 1. *The semigroup defined in Proposition 3 extends continuously to a strongly continuous, unitary group on the space* $\mathcal{V}' := \tilde{L}^2(\omega)\times\tilde{H}^{-1}(\omega)$. *Consequently, given the initial data* $\{y^0,v^0\}\in\mathcal{V}'$, *there is a uniquely defined solution to (22), (23) with* $f=0$ *which satisfies*

$$y\in C\big((-\infty,\infty),\tilde{L}^2(\omega)\big)\cap C^1\big((-\infty,\infty),\tilde{H}^{-1}(\omega)\big).$$

The nonhomogeneous system we are interested in is

$$\mathcal{C}_\rho w_{tt}-\Delta w \;=\; C \quad \text{in } \omega\times(0,\infty), \tag{28}$$

$$w \;=\; f \quad \text{on } \gamma_0\times(0,\infty), \tag{29}$$

$$w \;=\; 0 \quad \text{on } \gamma_1\times(0,\infty), \tag{30}$$

$$\{w,w_t\}\big|_{t=0} \;=\; \{w^0,w^1\}\in\tilde{L}^2(\omega)\times\tilde{H}^{-1}(\omega). \tag{31}$$

To define a weak solution, we write $w=w_0+z$, where w_0 satisfies (28)–(31) with $f=0$ in (29) and z satisfies (28)–(31) with, however, $\{w^0,w^1\}=\{0,0\}$. Formally multiplying (28) (with z in place of w) by a smooth function ϕ with

$$\phi\in C\big([0,T],H^2(\omega)\cap\tilde{H}^1_0(\omega)\big)\cap C^1\big([0,T],\tilde{H}^1_0(\omega)\big)$$

results in

$$\begin{aligned}
0 \;&=\; \int_0^T\!\!\int_\omega C\phi\,\mathrm{d}x\,\mathrm{d}t = \int_0^T\!\!\int_\omega(\mathcal{C}_\rho z_{tt}\phi-\Delta z\phi)\,\mathrm{d}x\,\mathrm{d}t\\
&=\; \int_\omega\big(\mathcal{C}_\rho z_t(T)\phi(T)-\mathcal{C}_\rho z(T)\phi_t(T)\big)\,\mathrm{d}x\,\mathrm{d}t + \int_0^T\!\!\int_{\gamma_0} f\frac{\partial\phi}{\partial\mathbf{n}}\,\mathrm{d}\gamma\,\mathrm{d}t\\
&\quad+\int_{Q_T} z(\mathcal{C}_\rho\phi_{tt}-z\Delta\phi)\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}$$

Hence, if ϕ is a solution to the *dual system*

$$\mathcal{C}_\rho \phi_{tt} - \Delta\phi = C \quad \text{in } \omega \times (-\infty, \infty), \tag{32}$$

$$\phi = 0 \quad \text{on } \gamma_0 \times (-\infty, \infty), \tag{33}$$

$$\phi = 0 \quad \text{on } \gamma_1 \times (-\infty, \infty), \tag{34}$$

$$\{\phi, \phi_t\}\big|_{t=T} = \{\phi^0, \phi^1\} \tag{35}$$

with $\{\phi^0, \phi^1\} \in H^2(\omega) \cap \tilde{H}_0^1(\omega) \times \tilde{H}_0^1(\omega)$, we obtain the identity

$$\left\langle z_t(T), \phi^0 \right\rangle_\rho - \left\langle z(T), \phi^1 \right\rangle_\rho = -\int_0^T\!\!\int_{\gamma_0} f \frac{\partial\phi}{\partial\mathbf{n}} \, \mathrm{d}\gamma \, \mathrm{d}t. \tag{36}$$

We see that for $f \in L^2((0,T) \times (\gamma_0))$ we have

$$\begin{aligned} \Big| \int_0^T\!\!\int_{\gamma_0} f \frac{\partial\phi}{\partial\mathbf{n}} \, \mathrm{d}\gamma \, \mathrm{d}t \Big| &\leq K \|f\|_{L^2\left(\gamma_0 \times (0,T)\right)} \Big\| \frac{\partial\phi}{\partial\mathbf{n}} \Big\|_{L^2(\gamma_0 \times (0,T))} \\ &\leq K \|f\|_{L^2\left(\gamma_0 \times (0,T)\right)} \left\| \{\phi^0, \phi^1\} \right\|_{\mathcal{V}}, \end{aligned}$$

where K is a constant that may change from line to line, and where we have used Proposition 3. Consequently, the right-hand side of (36) can be viewed as a continuous linear functional acting on the space $(H^2(\omega) \cap \tilde{H}_0^1(\omega)) \times \tilde{H}_0^1(\omega)$. Therefore, by considering all possible $\{\phi^0, \phi^1\} \in (H^2(\omega) \cap \tilde{H}_0^1(\omega)) \times \tilde{H}_0^1(\omega)$, the identity (36) defines a unique element $\{z(T), z_t(T)\}$ as an element of the dual space. If we define $\tilde{H}^{-2}(\omega) = (H^2(\omega) \cap \tilde{H}_0^1(\omega))'$, where the duality is with respect to $\langle \cdot, \cdot \rangle_\rho$, we see that for $0 \leq t \leq T$ (36) defines a unique solution z with $z \in C([0,T], \tilde{H}^{-1}(\omega)$ and $z_t \in L^\infty(\tilde{H}^{-2}(\omega))$. Results on "regularity lifting" actually provide that $z_t \in C([0,T], \tilde{H}^{-2}(\omega))$. We therefore have the following (suboptimal) result, which will be improved shortly.

Proposition 4. *Let* $\{w^0, w^1\} \in \mathcal{V}'$ *and* $f \in L^2((0,T) \times \gamma_0)$. *Then there is a unique weak solution* w *to (28)–(31) for which*

$$w \in C\big([0,T], \tilde{H}^{-1}(\omega)\big) \cap C^1\big([0,T], \tilde{H}^{-2}(\omega)\big). \tag{37}$$

3. Optimal Regularity

We will see that the regularity in (37) of Proposition 4 can actually be improved by one degree. This occurs in the (uncoupled) wave equation with Dirichlet control and the usual argument involves multiplying the wave equation $w_{tt} - \Delta w = 0$ by the multiplier $h \cdot \nabla w$, where h is a $C^2(\bar{\omega})$ vector field which is equal to the normal to γ on γ. However, in the present situation the equation of motion occurs in a quotient space and hence, in addition to the previous conditions on the multiplier $h \cdot \nabla w$, we need $\int_\omega h \cdot \nabla w \, \mathrm{d}x = 0$. For smooth solutions $w = 0$ on γ, and hence we need equivalently $\int_\omega w \nabla h \, \mathrm{d}x = 0$ for each solution w. A sufficient condition, since solutions w have zero average value, is that $\int_\omega \nabla\cdot h\,\mathrm{d}x = C$, where C is any constant. Fortunately, the problem of obtaining such a multiplier h is solved in (Galdi, 1994). We thus have the following:

Lemma 1. *Under our assumptions on ω (ω is a bounded domain in $\mathbb{R}^2$ with C^2 boundary) there exists a vector field $h:\bar{\omega}\to\mathbb{R}^2$ such that*

(i) h is C^2 on $\bar{\omega}$,

(ii) $h=\mathbf{n}$ on γ, where $\mathbf{n}$ is the outward unit normal vector to γ,

(iii) $\nabla\cdot h = C$ on ω, where C is a constant.

Using the multiplier from Lemma 1 applied to the homogeneous system (32)–(35), we obtain the following result:

Proposition 5. *Let ϕ be a solution to the homogeneous backwards problem (32)–(35) with the data $\{\phi^0,\phi^1\}$ given in $\mathcal{V}$. Then there exist $K_1>0$, $K_2>0$ independent of T and the data such that*

$$\int_0^T\int_\gamma \left|\frac{\partial\phi}{\partial\mathbf{n}}\right|^2 \le (K_1+K_2T)\,\left\|\{\phi^0,\phi^1\}\right\|^2_{\mathcal{E}}. \tag{38}$$

Proof. Let us introduce the notation

$$\iint\psi = \int_{\omega\times(0,T)}\psi\,\mathrm{d}\omega\,\mathrm{d}t,\qquad \int\psi = \int_{\gamma\times(0,T)}\psi\,\mathrm{d}\gamma\,\mathrm{d}t. \tag{39}$$

Also, write

$$X = (\phi_t+\rho\Lambda\phi_t, h\cdot\nabla\phi)\Big|_0^T,\quad Y=(\phi_t+\Lambda\rho\phi_t,\phi)\Big|_0^T,$$

where $(\psi,\phi)=\int_\omega\psi\phi\,\mathrm{d}\omega$.

Let ϕ be as in the hypothesis and h as in Lemma 1. Using Lemma 1 and integration by parts, we calculate the following:

$$\begin{aligned} 0 &= \iint\left\{(\mathcal{C}_\rho\phi_{tt}-\Delta\phi+C)(h\cdot\nabla\phi)\right\}\\ &= X-\int\left\{\frac{\partial\phi}{\partial\mathbf{n}}h\cdot\nabla\phi\right\}+\iint\left\{\nabla\phi\nabla(h\cdot\nabla\phi)-\mathcal{C}_\rho\phi_t(h\cdot\nabla\phi_t)\right\}\\ &= X-\int\left\{\left|\frac{\partial\phi}{\partial\mathbf{n}}\right|^2\right\}+\iint\left\{\nabla\phi\nabla(h\cdot\nabla\phi)-\mathcal{C}_\rho\phi_t(h\cdot\nabla\phi_t)\right\}. \end{aligned} \tag{40}$$

Let us define the gradient of a vector $h = (h_i)$ to be the matrix $\nabla h = ((\nabla h)_{ij})$, where $(\nabla h)_{ij} = \partial h_i/\partial x_j$. Then we have

$$\begin{aligned}\iint \Big\{\nabla\phi\cdot\nabla(h\cdot\nabla\phi)\} &= \iint\{\nabla\phi\cdot(\nabla h\nabla\phi)+\frac{1}{2}\nabla|\nabla\phi|^2\cdot h\Big\}\\ &= \iint\Big\{\nabla\phi\cdot(\nabla h\nabla\phi)-\frac{1}{2}(\nabla\cdot h)|\nabla\phi|^2\Big\}+\frac{1}{2}\int|\nabla\phi|^2\\ &= \iint\Big\{\nabla\phi\cdot(\nabla h\nabla\phi)-\frac{1}{2}(\nabla\cdot h)|\nabla\phi|^2\Big\}+\frac{1}{2}\int\Big|\frac{\partial\phi}{\partial\mathbf{n}}\Big|^2, \qquad (41)\end{aligned}$$

where we have used the fact that ϕ vanishes on γ in the last line.

Define the operator $\mathcal{M}$ initially on $\tilde{H}_0^1(\omega)$ by

$$\mathcal{M}\psi = \Lambda(h\cdot\nabla\psi) - h\cdot\nabla\Lambda\psi. \qquad (42)$$

We also have

$$\begin{aligned}\iint(\mathcal{C}_\rho\phi_t)h\cdot\nabla\phi_t &= \iint(\phi_t+\rho\Lambda\phi_t)h\cdot\nabla\phi_t\\ &= \iint\Big\{\frac{1}{2}h\cdot\nabla\phi_t^2+\frac{\rho}{2}h\cdot\nabla(\phi_t\Lambda\phi_t)\\ &\quad+\frac{\rho}{2}(h\cdot\nabla\phi_t)\Lambda\phi_t-\phi_t h\cdot\nabla(\Lambda\phi_t)\Big\}\\ &= \frac{-C_1}{2}\Big[\iint\{\phi_t^2+\rho(\Lambda\phi_t)\phi_t\}\Big]+\frac{\rho}{2}\iint\phi_t\mathcal{M}\phi_t\\ &= \int_0^T\Big\{\frac{-C_1}{2}\langle\phi_t,\phi_t\rangle_\rho+\frac{\rho}{2}(\phi_t,\mathcal{M}\phi_t)\Big\}\,\mathrm{d}t. \qquad (43)\end{aligned}$$

In the previous lines, C_1 is the constant of Lemma 1. Putting (41) and (43) together, we obtain

$$\begin{aligned}\int\Big|\frac{\partial\phi}{\partial\mathbf{n}}\Big|^2 &= 2X+2\iint\{\nabla\phi\cdot(\nabla h)\nabla\phi\}\\ &\quad+C_1\int_0^T\big\{\langle\phi_t,\phi_t\rangle_\rho-(\nabla\phi,\nabla\phi)\big\}\,\mathrm{d}t-\rho\int_0^T(\phi_t,\mathcal{M}\phi_t)\,\mathrm{d}t. \qquad (44)\end{aligned}$$

Multiplication of the dual system (32)–(35) by ϕ followed by integration by parts gives

$$\begin{aligned}\iint(\mathcal{C}_\rho\phi_{tt}-\Delta\phi-C)\phi &= C\iint\phi=0\\ &= (\mathcal{C}_\rho\phi_t,\phi)\big|_0^T-\iint\phi_t\mathcal{C}_\rho\phi_t+\iint|\nabla\phi|^2.\end{aligned}$$

Therefore

$$\int_0^T \big\{\langle\phi_t,\phi_t\rangle_\rho - (\nabla\phi,\nabla\phi)\big\}\,\mathrm{d}t = (\mathcal{C}_\rho\phi_t,\phi)\Big|_0^T = Y. \tag{45}$$

Combining (44) and (45), we obtain

$$\int \Big|\frac{\partial\phi}{\partial\mathbf{n}}\Big|^2 = 2X + C_1 Y + \iint \{2\nabla\phi\cdot\nabla h\nabla\phi - \rho\phi_t\mathcal{M}\phi_t\}. \tag{46}$$

Each of the terms on the right-hand side of (46) can be bounded by a multiple of the energy $\mathcal{E}(T) := \|\{\phi^0,\phi^1\}\|_{\mathcal{E}}$. For example, with Y we use conservation of energy and Poincaré's inequality. Let K_1 be such that

$$\int_\omega |\psi|^2\,\mathrm{d}\omega \le K_1 \int_\omega |\nabla\psi|^2\,\mathrm{d}\omega.$$

We obtain

$$\begin{aligned}
|Y| &= \big|\langle\phi_t(T),\phi(T)\rangle_\rho - \langle\phi_t(0),\phi(0)\rangle_\rho\big| \\
&\le \langle\phi_t,\phi_t\rangle_\rho^{1/2}\langle\phi,\phi\rangle_\rho^{1/2}\{|_{t=T} + |_{t=0}\} \\
&\le \langle\phi_t,\phi_t\rangle_\rho^{1/2}\big(1+\rho\|\Lambda\|\big)^{1/2}(\phi,\phi)^{1/2}\{|_{t=T} + |_{t=0}\} \\
&\le \mathcal{E}^{1/2}\big(K_1(1+\rho\|\Lambda\|)\big)^{1/2}\Big(\int_\omega |\nabla\phi|^2\,\mathrm{d}\omega\Big)^{1/2}\{|_{t=T} + |_{t=0}\} \\
&\le 2\big(K_1(1+\rho\|\Lambda\|)\big)^{1/2}\mathcal{E}(T) =: C_2\mathcal{E}(T).
\end{aligned} \tag{47}$$

For $|X|$ we use the fact that h is bounded on $\bar\omega$ and similar estimates to obtain

$$|X| \le C_3\mathcal{E}(T). \tag{48}$$

Since h is C^2 on $\bar\omega$, the matrix norm of ∇h is bounded. Consequently, there exists C_4 such that

$$\iint \nabla\phi\cdot\nabla h\nabla\phi \le TC_4\mathcal{E}(T). \tag{49}$$

To estimate the term involving $\mathcal{M}$, we first note that for $\psi\in\tilde H_0^1$ we have

$$\begin{aligned}
(\psi,\mathcal{M}\psi) &= (\psi,\Lambda(h\cdot\nabla\psi) - h\cdot\nabla\Lambda\psi) \\
&= (h\Lambda\psi,\nabla\psi) - (\psi,h\cdot\nabla\Lambda\psi) \\
&= -\big(\nabla\cdot(h\Lambda\psi),\psi\big) - (h\cdot\nabla\Lambda\psi,\psi) \\
&= -\big(h\cdot\nabla\Lambda\psi + (\nabla\cdot h)\Lambda\psi,\psi\big) - (h\cdot\nabla\Lambda\psi,\psi) \\
&= -2(h\cdot\nabla\Lambda\psi,\psi) - C_1(\Lambda\psi,\psi),
\end{aligned} \tag{50}$$

where C_1 is again the constant in Lemma 1. By Proposition 1 the operator Λ is continuous from $\tilde L^2(\omega)$ to $\tilde H^1(\omega)/C$ and hence $h\cdot\nabla\Lambda$ is continuous from $\tilde L^2(\omega)$ to

$L^2(\omega)$. It follows that $\mathcal{M}$ extends to a continuous operator on $\tilde{L}^2(\omega)$ (when $\tilde{L}^2(\omega)$ is identified with its dual). We then have that there exists a C such that

$$\iint \phi_t \mathcal{M} \phi_t \leq CT\mathcal{E}(T). \tag{51}$$

Combining (47)–(51) with (46), we obtain

$$\int \Big|\frac{\partial \phi}{\partial \mathbf{n}}\Big|^2 \leq K_1\mathcal{E}(T) + TK_2\mathcal{E}(T) = (K_1 + TK_2)\mathcal{E}(0), \tag{52}$$

where the constants K_1 and K_2 are independent of T and the initial data. This completes the proof. ■

From this estimate one can easily prove the following:

Corollary 2. *If $\{\phi^0, \phi^1\} \in \mathcal{V}$, then $\partial\phi/\partial\mathbf{n}$ satisfies (52). Equivalently, the observation operator $\Psi : \mathcal{V} \to L^2((0,t) \times \gamma)$ defined by $\Psi\{\phi^0, \phi^1\} = \partial\phi/\partial\mathbf{n}\big|_{\gamma\times(0,t)}$ extends continuously to $\mathcal{E}$.*

4. Exact Controllability

Consider the homogeneous problem

$$\begin{aligned} y_{tt} + \rho(\Lambda y_t)_t - \Delta y &= C && \text{in } \omega \times \mathbb{R}^+, && (53)\\ y &= 0 && \text{on } \gamma \times (0,T), && (54)\\ \{y, y_t\}\big|_{t=0} &= \{y^0, y^1\} && \text{given in } \tilde{H}_0^1(\omega) \times \tilde{L}^2(\omega). && (55) \end{aligned}$$

The dual problem to the control problem (5)–(8) (equivalently, (28)–(31)) consists of (53), (54) with terminal data specified in $\tilde{H}_0^1(\omega) \times \tilde{L}^2(\omega)$ together with the observation

$$z = \frac{\partial y}{\partial \mathbf{n}} \quad \text{on } \gamma_1. \tag{56}$$

Due to the time-reversibility of the problem, we may equivalently consider (53), (54) with the initial data given at time 0 as in (55).

The goal is to prove the exact observability of (53)–(56), i.e., that for some $c > 0$ one has

$$\int_0^T\int_{\gamma_1} |z|^2 \,\mathrm{d}\gamma\,\mathrm{d}t \geq c\mathcal{E}(0). \tag{57}$$

We apply the standard multiplier $(\mathbf{x}) \cdot \nabla y$ (Lions, 1988) to solutions of (53)–(55).

Let $(\psi, \phi) = \int_\omega \psi\phi \,\mathrm{d}\omega$ and set

$$\begin{aligned} X &= (y_t + \rho\Lambda y_t, \mathbf{x}\cdot\nabla y)\big|_0^T, \quad Y = (y_t + \rho\Lambda y_t, y)\big|_0^T\\ \mathcal{M}\psi &= \Lambda(\mathbf{x}\cdot\nabla\psi) - \mathbf{x}\cdot\nabla(\Lambda\psi). \end{aligned}$$

Using the notation in (39), we multiply (53) by the standard multiplier $\mathbf{x}\cdot\nabla y$ to obtain

$$0 = X - \iint \big\{(y_t+\rho\Lambda y_t)\mathbf{x}\cdot\nabla y_t + (\Delta y)\mathbf{x}\cdot\nabla y\big\} = X - T_1 - T_2, \qquad (58)$$

where

$$\begin{aligned}
T_1 &= \frac{1}{2}\iint \{\mathbf{x}\cdot\nabla(y_t)^2 + \rho\mathbf{x}\cdot\nabla(y_t\Lambda y_t) + \rho(\mathbf{x}\cdot\nabla y_t)\Lambda y_t - \rho\mathbf{x}\cdot\nabla(\Lambda y_t)y_t\} \\
&= -\frac{n}{2}\iint \{y_t + \rho(\Lambda y_t)y_t\} + \frac{\rho}{2}\iint\{y_t\mathcal{M}y_t\}, \\
T_2 &= \int \Big\{\frac{\partial y}{\partial \mathbf{n}}\mathbf{x}\cdot\nabla y\Big\} - \iint \{\nabla y\cdot(\mathbf{x}\cdot\nabla y)\} \\
&= \int \frac{\partial y}{\partial \mathbf{n}}\mathbf{x}\cdot\nabla y - \frac{1}{2}\int \mathbf{x}\cdot\mathbf{n}|\nabla y|^2 + \frac{n-2}{2}\iint |\nabla y|^2 \\
&= \frac{1}{2}\int\Big\{\frac{\partial y}{\partial \mathbf{n}}^2\mathbf{x}\cdot\mathbf{n}\Big\} + \frac{n-2}{2}\iint|\nabla y|^2,
\end{aligned}$$

n being the dimension of ω, which is 2 for the present situation. In the calculation of T_2, we have used the fact that $y=0$ on γ to combine the boundary terms.

Recall the energies

$$\begin{aligned}
&\mathcal{P}(t) = \frac{1}{2}\int_\omega |\nabla w|^2\,\mathrm{d}\omega, \quad \mathcal{K}(t) = \frac{1}{2}\int_\omega (w_t+\rho\Lambda w_t)w_t\,\mathrm{d}\omega, \\
&\mathcal{E}(t) = \|\{w,w_t\}\|_{\mathcal{E}}^2 = 2\big(\mathcal{P}(t)+\mathcal{K}(t)\big).
\end{aligned}$$

Combining them with (58), we obtain

$$\begin{aligned}
0 &= X + n\int_0^T \mathcal{K}(t)\,\mathrm{d}t - (n-2)\int_0^T \mathcal{P}(t)\,\mathrm{d}t \\
&= X + (n-1)\int_0^T \mathcal{K}(t)-\mathcal{P}(t)\,\mathrm{d}t + \frac{1}{2}\int_0^T \mathcal{E}(t)\,\mathrm{d}t \\
&\quad - \frac{\rho}{2}\iint y_t\mathcal{M}y_t - \frac{1}{2}\int \mathbf{x}\cdot\mathbf{n}\Big|\frac{\partial y}{\partial \mathbf{n}}\Big|^2 \\
&= X + \frac{n-1}{2}Y + \frac{1}{2}T\mathcal{E}(0) - \frac{\rho}{2}\iint\{y_t\mathcal{M}y_t\} - \frac{1}{2}\int \mathbf{x}\cdot\mathbf{n}\Big|\frac{\partial y}{\partial \mathbf{n}}\Big|^2,
\end{aligned}$$

where the last line results from the identity

$$\begin{aligned}
0 &= \iint \big(y_{tt}+\rho(\Lambda y)_{tt} - \Delta y + C\big)y = Y + \iint \big(|\nabla y|^2 - y_t(y_t+\rho\Lambda y_t)\big) \\
&= Y + 2\int_0^T (\mathcal{P}-\mathcal{K})\,\mathrm{d}t.
\end{aligned}$$

Next we use the geometrical condition on the control region. On γ_0 we have $\mathbf{x}\cdot\mathbf{n} \le 0$, and on γ_1 we have $\mathbf{x}\cdot\mathbf{n} \le R := \max_{\mathbf{x}\in\gamma_1} |\mathbf{x}|$. Therefore we obtain

$$\frac{T}{2}\mathcal{E}(0) \le \frac{n-1}{2}|Y| + |X| + \frac{\rho}{2}\Big|\iint y_t \mathcal{M} y_t\Big| + \frac{R}{2}\int_{\gamma_1\times(0,T)} \Big|\frac{\partial y}{\partial \mathbf{n}}\Big|^2 \mathrm{d}\gamma\,\mathrm{d}t. \tag{59}$$

Let s_1 be the first eigenvalue of $-\mathcal{C}_\rho\Delta_C$ and let λ_1 be the first eigenvalue of the Dirichlet Laplacian on ω. We get

$$\begin{aligned} s_1 &= \min_{u\in H^2(\omega)\cap\tilde{H}_0^1(\Omega)} \frac{(\nabla u,\nabla u)}{\langle u,u\rangle_\rho} \ge \min_{u\in H^2(\omega)\cap\tilde{H}_0^1(\Omega)} \frac{(\nabla u,\nabla u)}{\|\mathcal{C}_\rho\|\langle u,u\rangle_{L^2}} \\ &\ge \min_{u\in H^2(\omega)\cap H_0^1(\omega)} \frac{(\nabla u,\nabla u)}{\|\mathcal{C}_\rho\|\langle u,u\rangle_{L^2}} = \frac{\lambda_1}{\|C_\rho\|}. \end{aligned}$$

Hence one has the following estimate:

$$\langle u,u\rangle_\rho \le \frac{\|\mathcal{C}_\rho\|}{\lambda_1}(\nabla u,\nabla u). \tag{60}$$

Using (60) together with fact that $\langle\cdot\,,\cdot\rangle_\rho$ is an inner product gives the estimate

$$\begin{aligned} \big|(\mathcal{C}_\rho y_t, y)\big| &\le \frac{1}{2}(\mathcal{C}_\rho y_t, y_t) + \frac{1}{2}(\mathcal{C}_\rho y, y) \\ &\le \frac{1}{2}(\mathcal{C}_\rho y_t, y_t) + \frac{\|\mathcal{C}_\rho\|}{2\lambda_1}(\nabla y,\nabla y). \end{aligned} \tag{61}$$

From (61) one easily obtains the following bound on Y:

$$|Y| \le \frac{\|\mathcal{C}_\rho\|}{\lambda_1}\mathcal{E}(0). \tag{62}$$

Similarly, $|X|$ can be bounded in terms of the energy:

$$|X| \le R\|\mathcal{C}_\rho\|\mathcal{E}(0). \tag{63}$$

Finally, the term involving $\mathcal{M}$ can be handled in the same way that the corresponding term was handled in (50) in the proof of Proposition 5. One obtains the estimate

$$\Big|\iint y_t\mathcal{M}y_t| \le T\big(n\|\Lambda\| + RC_\Omega\big)\mathcal{E}(0), \tag{64}$$

where (recall $n=2$ here) C_Ω is the constant appearing in Proposition 2.

Combining (59)–(64), we see that

$$\begin{aligned} R\int_0^T\!\!\int_{\gamma_1}\Big|\frac{\partial y}{\partial\mathbf{n}}\Big|^2 \mathrm{d}\gamma\,\mathrm{d}t &\ge \Big(T - \rho T\big(RC_\Omega + n\|\Lambda\|\big) - (n-1)\frac{\|\mathcal{C}_\rho\|}{\lambda_1} - 2R\|\mathcal{C}_\rho\|\Big)\mathcal{E}(0) \\ &= \big(T(1-\rho/\rho_0) - K\big)\mathcal{E}(0), \end{aligned} \tag{65}$$

where

$$\rho_0 = \big(RC_\Omega + n\|\Lambda\|\big)^{-1}, \quad K = (n-1)\frac{\|\mathcal{C}_\rho\|}{\lambda_1} + 2R\|\mathcal{C}_\rho\|. \tag{66}$$

Thus, for $\rho < \rho_0$, the observability estimate (57) for (53)–(56) holds provided that

$$T > \frac{K\rho_0}{\rho_0 - \rho}. \tag{67}$$

Finally, the exact controllability of Theorem 2 follows by duality. This completes the proof of Theorem 2. ■

5. Concluding Remarks

We have shown that the classical multiplier argument used to establish the exact controllability of the wave equation with Dirichlet boundary control can also be applied to the coupled fluid-elastic system (5)–(8) to obtain the necessary observability estimate mentioned in Remark 2, provided at least that the fluid density ρ is sufficiently small.

As has been mentioned in Remark 3, it is unknown whether exact controllability holds for all ρ. However, in the sufficient condition that $\rho < \rho_0$, ρ_0 exhibits the same inverse dependence upon the geometric constant C_Ω that was obtained in (Hansen and Lyashenko, 1997), where the moment method was applied (for the case where ω is one-dimensional).

It is almost certainly true that the control time T can be improved. In the limit as $\rho \to 0$, the control time obtained here tends to $2R + (n-1)/\lambda_1$, which is the value obtained in (Lions, 1988) for the wave equation. (This value can in turn be improved to the optimal value of $2R$ in various ways that do not apply here, cf. see (Lions, 1988).)

As for the geometry of Ω, we have taken Ω to be simply connected to obtain the existence of a velocity potential Φ. However, this constraint can be eliminated as in (Hansen and Lyashenko, 1997) by working directly with the pressure instead. Of course, the simple connectivity of Ω does not imply that ω need be simply connected. If control is active only on a portion of the boundary γ_1, we have taken $\bar\gamma_1 \cap \bar\gamma_0 = \emptyset$ to avoid a discussion of singularities. In the case of boundary control of the wave equation, various methods have been developed to eliminate the necessity of the geometric condition (1), e.g., results on "propagation of singularities" (Bardos *et al.*, 1992).

In the present problem, many properties of hyperbolic partial differential equations, e.g., unique continuation properties, finite propagation speed, etc., do not apply due to the nonlocal nature of the incompressibility constraint. Thus the extent to which the results available for the wave equation apply to the system of this paper is unclear.

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