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AN APPLICATION OF THE FOURIER TRANSFORM TO OPTIMIZATION OF CONTINUOUS 2-D SYSTEMS

VITALI DYMKOU*, MICHAEL DYMKOV**

* Graduate College, University of Erlangen-Nürnberg
Cauerstreet 7, D–91058, Erlangen, Germany
e-mail: dymkou@lnt.de

** Institute of Mathematics, National Academy of Sciences,
Surganov, 11, 220072, Minsk, Belarus
e-mail: dymkov@im.bas-net.by

This paper uses the theory of entire functions to study the linear quadratic optimization problem for a class of continuous $2D$ systems. We show that in some cases optimal control can be given by an analytical formula. A simple method is also proposed to find an approximate solution with preassigned accuracy. Some application to the $1D$ optimization problem is presented, too. The obtained results form a theoretical background for the design problem of optimal controllers for relevant processes.

Keywords: 2D systems, optimization, entire functions, Fourier transform, approximation

1. Introduction

The research termed 'multidimensional systems' was initially motivated by the need for a mathematical description of some problems that had arisen in the area of circuits and multidimensional signal, image and video processing (Bose, 1982; Fornasini and Marchesini, 1978). The next studies showed that also many information processes in various fields posses such a unique mathematical nature and they can be fully described in the form of multidimensional dynamical systems (Kaczorek, 1985; Gałkowski and Wood, 2001). The unique key feature of an mD system is that the process dynamics depend on m indeterminates and hence information is propagated in many independent directions. A natural way is the representation of mD systems by a polynomial-based description of the process dynamics. Although very promising, it is related to serious numerical problems. One of the principal advantages of a dynamic system formulation is that it provides a framework in which it is possible to examine traditional optimal control concepts. In the case of mD systems the propagation of dynamics in the independent directions can be realized by either (i) functions of discrete variables, (ii) continuous variables, or (iii) continuous variables in one direction and discrete variables in the other. Recently, close attention has been paid to discrete-continuous mD processes (Kaczorek, 1995; Dymkov, 2001) where at least along one direction system dynamics are defined in terms of continuous variables. On the other hand, a few scientific works (Shankar and Willems, 2000; Idczak and Walczak, 2000) are devoted to continuous mD systems.

This paper reports an application of the theory of entire functions to control problems. This approach has been used, in particular, in optimization problems of some classes of continuous-discrete 2D models (Dymkov, 1999). It is shown that in some cases the optimization problem can be reduced to a linear programming problem in the appropriate Hilbert space of entire functions. This paper uses entire function theory to study the linear quadratic optimization problem for continuous 2D systems. It is shown that in the scalar case the optimal control can be given by an analytical formula. We discuss a method of finding an approximate solution with preassigned accuracy and also indicate some applications of entire functions to the 1D optimization problem. The obtained results provide a theoretical background for the design problem of optimal controllers for relevant processes.

1.1. Preliminaries and Motivation

The simplest classes of linear 2D discrete systems used in applied problems and mathematical theory can be written as follows:

$$x(t+1,s) = Ax(t,s) + Dx(t,s+1) + g(t,s), \qquad (1)$$

or as a couple of equations

$$\begin{cases} x(t+1,s) = A_{11}x(t,s) + A_{12}y(t,s) \\ \qquad\qquad +D_{12}x(t,s+1) + g_1(t,s), \\ y(t,s+1) = A_{21}x(t,s) + A_{22}y(t,s) \\ \qquad\qquad +D_{21}y(t+1,s) + g_2(t,s), \end{cases} \tag{2}$$

given on the space of the functions defined on the integer-valued lattice $\mathbb{Z}_+$. Another state-space 2D objects were investigated by (Gaishun, 1983). In the simplest case they can be given in the form

$$\begin{cases} x(t+1,s) = A_1x(t,s) + g_1(t,s), \\ x(t,s+1) = A_2x(t,s) + g_2(t,s). \end{cases} \tag{3}$$

The main characteristic feature of such models is their overdetermination (in the sense that the number of equations for this case is greater than that of the unknown functions) and, as a consequence, it is a problem to correctly define the notion of the solution. In this sense, such a system is similar to a one-dimensional discrete-time system with parametric uncertainty. For this reason the classes of completely integrable systems for which the boundary Cauchy problem has a unique solution are of the strongest interest. These models can be also treated as discrete versions of Pfaff partial differential equations that have been used in elasticity theory, magnetohydrodynamics and other engineering problems (see, e.g., Perov, 1975).

Recently, in modern m-D theory, continuous and continuous-discrete versions of discrete multidimensional systems were actively investigated. Some of these, e.g.,

$$x(t+1,s) = \sum_{j\in Z_+} A_j \frac{\mathrm{d}^{(j)}x(t,s)}{\mathrm{d}s^j} + Bu(t,s), \tag{4}$$

$$\frac{\mathrm{d}x(t,s)}{\mathrm{d}s} = Ax(t,s) + Dx(t-1,s) + Bu(t,s), \tag{5}$$

were considered in (Kaczorek, 1995; Dymkov, 1999). The continuous version of Roesser's systems of the form

$$\begin{cases} \partial x(t,s)/\partial t = A_{11}x(t,s) + A_{12}y(t,s) + B_1u(t,s), \\ \partial y(t,s)/\partial s = A_{21}x(t,s) + A_{22}y(t,s) + B_2u(t,s) \end{cases} \tag{6}$$

was investigated by Idczak and Walczak (2000), and others.

In this paper we consider a continuous version of the system (3). First applications of such equations were connected with differential geometry to find manifolds with a given tangential subspace (Rashevski, 1947). In electrodynamics, for example, this model describes the electric potentials for the given electric field (Armand, 1977; Perov, 1975). Some details concerning stability theory and related topics can be found in (Gaishun, 1983).

This paper reports an application of a subclass of entire functions, i.e. functions regular in the complex plane $\mathbb{C}$ except the point $z = \infty$ (Ibragimov, 1984), to control systems. This class has a complex topological structure but we only employ a simpler subclass of entire functions, i.e. the space of entire functions of exponential type and finite degree.

We say that a complex function $f\colon \mathbb{C} \to \mathbb{C}$ is an entire function of the exponential type and a finite degree σ if f is regular on $\mathbb{C}$ and for any $\varepsilon > 0$ there is a constant $M = M(\varepsilon)$ such that the inequality $M\exp\{(\sigma - \varepsilon)|z_s|\} < |f(z)| < M\exp\{(\sigma+\varepsilon)|z|\}$ holds for all $z \in \mathbb{C}$ and some $z_s \in \mathbb{C}, z_s \to \infty, s \to \infty$. Let W_σ denote the set of entire functions of exponential type and a finite degree σ non-exceeding π such that its restriction to $\mathbb{R}$ consists of some functions from the space $L_2(\mathbb{R})$. Then it is known that W_σ is a Hilbert space (also termed the Wiener-Paley space (Ibragimov, 1984)) where the inner product is defined by $(f,g)_W = \int_{\mathbb{R}} f(x)\overline{g(x)}\,\mathrm{d}x$ and the over-bar means the complex conjugate.

Some properties of the Wiener-Paley space are relevant to optimization theory. In particular, according to Wiener's theorem, functions from this space admit the following description. The set W_σ coincides with the set of the analytical (regular) extension $F(z)$ for the Fourier transformation of the functions $f(t)$ from $L_2([-\sigma,\sigma],\mathbb{R})$: $F(z) = (1/\sqrt{2\pi})\int_{-\sigma}^{\sigma} f(t)e^{-izt}\,\mathrm{d}t$. Moreover, the space W_σ is compact in the sense that for any sequence $\{f_n(z)\}$ of functions from W_σ there exists a subsequence $\{f_{n_k}(z)\}$ that is uniformly convergent on every compact set K from $\mathbb{C}$ (with respect to the L_2-norm) to some function from the space W_σ. Note that there is also another property of the Wiener-Paley space which can be used for solving optimization problems. In particular, according to the Kotelnikov theorem there is an isomorphism between W_σ and the space of square summable sequences of complex numbers l_2:

$$f \in W_\sigma \leftrightarrow \{c_k\} \in l_2, \quad f(z) = \sum_{k=-\infty}^{\infty} (-1)^k c_k \frac{\sin\pi(z-k)}{\pi(z-k)}.$$

Otherwise, the function f is determined by numbers c_k. This fact is used to give the complete solution to the 1D optimal control problem.

2. Linear Quadratic Optimization for Continuous 2D Systems

We consider the linear time-invariant continuous $2D$ system described by the equations

$$\begin{cases} \dfrac{\partial x(t_1,t_2)}{\partial t_1} = A_1x(t_1,t_2) + B_1u(t_1,t_2), \\ \dfrac{\partial x(t_1,t_2)}{\partial t_2} = A_2x(t_1,t_2) + B_2u(t_1,t_2), \end{cases} \tag{7}$$

where $(t_1,t_2) \in S = [-\pi,\pi]\times[-\pi,\pi]$, $x \in \mathbb{R}^n$ is the state vector depending on parameters t_1 and t_2, $u \in \mathbb{R}^m$ is the input control vector of the same parameters t_1, t_2; A_i and B_i, $i = 1,2$ are constant matrices of dimensions $(n \times n)$ and $(n \times m)$, respectively. Also assume that $u(t_1,t_2)$ is a function from the space $C^1(\Omega,\mathbb{R}^m)$ of continuously differentiable functions defined on the set Ω, where Ω is some domain in $\mathbb{R}^2$ containing S .

Definition 1. A function x: $S \to \mathbb{R}^n$ is called the solution to (7) for a given function $u(t_1,t_2)$ if $x(\cdot) \in C^1(\Omega,\mathbb{R}^n)$, where Ω is some domain in $\mathbb{R}^2$ including S, and this $x(t_1,t_2)$ satisfies (7) for all $(t_1,t_2) \in S$.

Definition 2. We say that Eqns. (7) are *completly solvable* for a given function $u(t_1,t_2)$ if for each point $x_0 \in \mathbb{R}^n$ there exists a unique solution $x = x(t_1,t_2,x_0)$ of (7) satisfying the initial condition $x(-\pi,-\pi) = x_0$.

It is well known that the following Frobenious commutativity relations (Gaishun, 1983):

$$\begin{aligned} &A_1A_2 = A_2A_1,\\ &A_1B_2u(t_1,t_2)+B_2\frac{\partial u(t_1,t_2)}{\partial t_1}\\ &\quad = A_2B_1u(t_1,t_2)+B_1\frac{\partial u(t_1,t_2)}{\partial t_2}, \quad (t_1,t_2)\in S, \end{aligned} \tag{8}$$

are necessary and sufficient conditions for the complete solvability of (7). For this reason we define the admissible control functions as follows:

Definition 3. A function u: $S \to \mathbb{R}^m$ is called *admissible* if $u(\cdot) \in C^1(S,\mathbb{R}^m)$ and $u(\cdot)$ satisfies (8) for all $(t_1,t_2) \in S$.

The optimization problem is to minimize the cost functional

$$J(u) = \iint_S (|x(t_1,t_2)|^2 + |u(t_1,t_2)|^2)\,\mathrm{d}t_1\mathrm{d}t_2, \tag{9}$$

where $x(t_1,t_2)$ is the solution of (7) corresponding to the given admissible control $u(t_1,t_2)$ and satisfying to following boundary conditions:

$$x(-\pi,-\pi) = x_0, \quad x(\pi,\pi) = x_\pi, \tag{10}$$

where x_π, $x_0 \in \mathbb{R}^n$ are given points. For simplicity, we set $x_\pi = 0$.

Remark 1. To guarantee the existence of admissible controls which solve the controllability problem (10), we have to formulate some additional conditions. The lemma given below presents the conditions which guarantee the existence of the admissible controls defined on some time segment of the form $[-\pi,\pi]\times[t_1^*,t_2^*]$. The proper zero controllability conditions with fixed time segment are not known till now. Nevertheless, we assume that the analysed control system has a nonempty set of admissible controls.

The controllability problem for Pfaff differential equations can be stated in a differ manner. In fact, more than one distinct concepts of controllability can be defined for this case (Chramtzov, 1985). The simplest one is as follows:

Definition 4. The system (7) is called *controllable* if for each $x^0, x^* \in \mathbb{R}^n$ there are a moment $T^* = (t_1^*, t_2^*) \in \mathbb{R}^2$ and an admissible control function $u(t_1,t_2)$, $(0 \le t_1 \le t_1^*,\ 0 \le t_2 \le t_2^*)$ such that the solution $x = x(t_1,t_2,x_0)$ of (7) corresponding to this control satisfies the conditions $x(-\pi,-\pi) = x^0,\ x(T^*) = x(t_1^*,t_2^*) = x^*$.

Denote by Θ the subclass of systems (7) for which the conditions (8) and

$$\operatorname{rank}[B_1,B_2] = \operatorname{rank}[B_1,B_2,P] = m,$$

$$P = A_1B_2 - A_2B_1,$$

$$\exists\,\alpha \in R^1:\ \operatorname{rank}[\alpha B_1 + (1-\alpha)B_2] = m$$

hold. Then the following result gives the required controllability conditions (Chramtzov, 1985):

Lemma 1. *The system (7) of the class Θ is controllable if, and only if, $\operatorname{rank} F(\alpha) = n$ for some $\alpha \in \mathbb{R}^1$, where*

$$\begin{aligned} F(\alpha) &= \{B(\alpha), A(\alpha)B(\alpha), \dots, A^{n-1}(\alpha)B(\alpha)\},\\ B(\alpha) &= \alpha B_1 + (1-\alpha)B_2,\\ A(\alpha) &= \alpha A_1 + (1-\alpha)A_2. \end{aligned}$$

The previous studies of the structural properties of discrete 2D systems were often realized on their representations in the form of 1D dynamical systems (Fornasini and Marchesini, 1978; Dymkov, 1999). Such a kind of representation based on the Fourier transform is applied to the model under consideration. To realize this approach for (7) we use the class of finite functions, whose Fourier transforms belong to the class of entire functions (Ibragimov, 1984). We suppose that the control function in (7) is finite on S in the following sense: for each $t_2 \in [-\pi,\pi]$ the function $u(t_1,t_2) \equiv 0$, $\forall t_1 \notin [-\pi,\pi]$. In accordance with the Wiener-Paley theorem the analytic extension $\tilde{u}(z,t_2)$ to the complex plane $\mathbb{C}$ of the following function (i.e. of the Fourier transform of the function $u(t_1,t_2)$):

$$\tilde{u}(\omega,t_2) = \frac{1}{\sqrt{2\pi}}\int_{-\pi}^{\pi} u(t_1,t_2)e^{-i\omega t_1}\,\mathrm{d}t_1 \tag{11}$$

is an element of the Wiener-Paley space W_π for each $t_2 \in [-\pi,\pi]$. Applying the Fourier transform to (8) for each fixed $t_2 \in [-\pi,\pi]$ yields the following singular differential equation:

$$B_1 \frac{\mathrm{d}\tilde{u}(\omega,t_2)}{\mathrm{d}t_2} + B(\omega)\tilde{u}(\omega,t_2) = 0,$$
$$\omega \in \mathbb{R}, \quad t_2 \in [-\pi,\pi], \tag{12}$$

where $B(\omega) = A_2B_1 - A_1B_2 - i\omega B_2$. It is known that the solvability of singular systems (12) is determined, in general, by the properties of the pencil $L(\lambda,\omega) = \lambda B_1 + B(\omega)$. In this paper we consider the special case of the regular pencil $L(\lambda,\omega)$ when $n = m$ and the matrix B_1 has the inverse B_1^{-1}. In this case the solution of (12) is as follows:

$$\tilde{u}(\omega,t_2) = e^{-\hat{B}(\omega)(t_2+\pi)}v(\omega), \tag{13}$$

where $\hat{B}(\omega) = \hat{A} + i\omega\hat{B} = B_1^{-1}(A_2B_1 - A_1B_2) + i\omega(-B_1^{-1}B_2)$, $v(\omega) = \tilde{u}(\omega,-\pi)$.

Thus the Fourier transforms of the control functions $u(t_1,t_2)$ that are finite on $[-\pi,\pi]$ for a fixed t_2 and satisfy the differential equality of (8) are described by (13), where $v(z)$ is an arbitrary entire function from the Wiener-Paley space W_π.

Remark 2. Note that, in general, the inverse Fourier transformation of the function (13) with $v(z)$ from W_π is not a function from the class $C^1(S,\mathbb{R}^m)$, which is required for the admissible control functions. It is well known that the class $L_2[-\pi,\pi]$ of square integrable functions is invariant under the Fourier transform. In this case we determine first functions $\tilde{u}(t_1,t_2)$, $\tilde{u}(\cdot,t_2) \in L_2[-\pi,\pi]$, $t_2 \in [-\pi,\pi]$, which together with the corresponding solution $\tilde{x}(t_1,t_2)$ of (7), (10) minimize the cost functional (9). Such control functions are called generalized optimal controls for the problem (7), (9)–(10). Then the approximate optimal control $u^{\mathrm{ap}}(t_1,t_2)$ from the required class of admissible functions is determined as a proper approximation of the obtained function $v^0(t_1) = \tilde{u}(t_1,t_2)$, $t_1 \in [-\pi,\pi]$ from $L_2[-\pi,\pi]$ for fixed $t_2 \in [-\pi,\pi]$ by the functions from the space $C^1[-\pi,\pi]$. Hence, the solution $x^{\mathrm{ap}}(t_1,t_2)$ of (7) corresponding to $u^{\mathrm{ap}}(t_1,t_2)$ satisfies approximately the boundary conditions (10) and they provide the approximate optimal cost value. It is shown that the accuracy of this approximation can be easily evaluated.

It is easy to determine the solution of (7) along the two edges of the rectangle S:

$$\begin{aligned}
x(-\pi,t_2) &= e^{A_2(t_2+\pi)}x_0,\\
x(\pi,-\pi) &= e^{2\pi A_1}x_0 + \int_{-\pi}^{\pi} e^{(\pi-\tau)A_1}B_1u(\tau,-\pi)\,\mathrm{d}\tau,\\
x(\pi,t_2) &= e^{A_2(t_2+\pi)}x(\pi,-\pi), \quad t_2 \in [-\pi,\pi].
\end{aligned} \tag{14}$$

Since $x(\pi,\pi) = e^{2\pi A_2}x(\pi,-\pi) = 0$, we have $x(\pi,-\pi) = 0$. We suppose that the matrix A_1 has n single eigenvalues $\{\lambda_1,\dots,\lambda_n\}$. In this case

$$e^{A_1t} = \sum_{i=0}^{n-1}\alpha_i(t)A_1^i,$$

where the $\alpha_i(t)$'s are the coefficients of the Lagrange-Sylvester interpolation polynomial corresponding to A_1. Moreover, for the eigenvalues $\{\lambda_1,\dots,\lambda_n\}$ of the matrix A_1 we have

$$e^{\lambda_k t} = \sum_{i=0}^{n-1}\lambda_k^i\alpha_i(t), \quad k = 1,2,\dots,n.$$

Denote by Λ the Vandermond matrix of the n-th degree defined by the eigenvalues $\{\lambda_1,\dots,\lambda_n\}$ of A_1, and let $D = \operatorname{diag}\{e^{\lambda_1\pi},\dots,e^{\lambda_n\pi}\}$, $G = \{B_1, A_1B_1, \dots, A_1^{n-1}B_1\}$. Then from (14) we have

$$\begin{aligned}
-e^{2\pi A_1}x_0 &= \int_{-\pi}^{\pi} e^{A_1(\pi-\tau)}B_1u(\tau,-\pi)\,\mathrm{d}\tau\\
&= \int_{-\pi}^{\pi}\sum_{i=0}^{n-1}A_1^iB_1\alpha_i(\pi-\tau)u(\tau,-\pi)\,\mathrm{d}\tau\\
&= G\int_{-\pi}^{\pi}\begin{bmatrix}\alpha_0(\pi-\tau)\\ \vdots\\ \alpha_{n-1}(\pi-\tau)\end{bmatrix}u(\tau,-\pi)\,\mathrm{d}\tau\\
&= G\Lambda^{-1}\int_{-\pi}^{\pi}\begin{bmatrix}\sum_{i=0}^{n-1}\lambda_1^i\alpha_i(\pi-\tau)\\ \vdots\\ \sum_{i=0}^{n-1}\lambda_n^i\alpha_i(\pi-\tau)\end{bmatrix}u(\tau,-\pi)\,\mathrm{d}\tau\\
&= G\Lambda^{-1}D\begin{bmatrix}\int_{-\pi}^{\pi}e^{-\lambda_1\tau}u(\tau,-\pi)\,\mathrm{d}\tau\\ \vdots\\ \int_{-\pi}^{\pi}e^{-\lambda_n\tau}u(\tau,-\pi)\,\mathrm{d}\tau\end{bmatrix}\\
&= G\Lambda^{-1}D\begin{bmatrix}\tilde{u}(-i\lambda_1,-\pi)\\ \vdots\\ \tilde{u}(-i\lambda_n,-\pi)\end{bmatrix},
\end{aligned}$$

which yields

$$e^{2\pi A_1}x_0 + G\Lambda^{-1}D\hat{U} = 0,$$

where

$$\hat{U} = \big(\tilde{u}(-i\lambda_1,-\pi),\dots,\tilde{u}(-i\lambda_n,-\pi)\big)$$

and $\tilde{u}(\omega,-\pi)$ is given by (11).

Extend now the function $v(\omega) = \tilde{u}(\omega, -\pi)$ to the complex plane as an entire function of the exponential type from the space W_π. Then the following interpolation problem arises: Find a function $v(z)$ from the space W_π such that the equalities

$$F\hat{v} = f, \tag{15}$$

where $\hat{v} = (v(-i\lambda_1), \dots, v(-i\lambda_n))$, hold at the given points $z_1 = -i\lambda_1, \dots, z_n = -i\lambda_n$ of the complex plane $\mathbb{C}$, and $F = G\Lambda^{-1}D,\ f = -e^{-2\pi A_1}x_0$.

In general, the interpolation problem (15) does not have a unique solution. Let F_H be a nonsingular submatrix defined by the $(i_1, \dots, i_p)$-th rows and $(j_1, \dots, j_p)$-th columns of the matrix F where $p = \operatorname{rank} F$. Then (15) yields

$$\hat{v}_H = F_H^{-1} f - F_H^{-1} F_r \hat{v}_r, \tag{16}$$

where F_r is determined by those rows and columns of the matrix F which are not used in F_H, and the vector v is composed in accordance with this partition as $v = (v_H, v_r)$. The latter can be written in coordinate form

$$v|_{z=z_k} = \beta_k, \quad k = j_1, \dots, j_p. \tag{17}$$

Note that the components of the $(n-p)$-vector $\hat{v}_r$ and, hence, the p-vector β are free variables. The set of all solutions to (15) can be written as

$$u(z) = v_1(z) + Q(z)v(z), \tag{18}$$

where $v_1(z)$ is a particular solution to (15), $Q(z)$ is some polynomial of the p-th degree, whose roots are given numbers $z_{j_k},\ k = 1, 2, \dots, p$, and $v(z)$ is an arbitrary function such that $Q(z)v(z) \in W_\pi$. The set of such functions is denoted by V. By the Lagrange formula, a particular solution to (15) can be chosen as

$$v_1(z) = \sum_{i=1}^{p} \beta_i \frac{\varphi(z)}{\varphi'(z)(z - z_i)},$$
$$\varphi(z) = \prod_{i=1}^{p} \sin \frac{\pi}{p}(z - z_i). \tag{19}$$

Note that $\varphi(z)$ cannot be chosen as the simplest interpolation polynomial of the form $\psi(z) = \prod_{i=1}^{p}(z - z_i)$ since $\psi(z)/(z - z_i) \notin W_\pi$ for every i. Thus the set of all admissible controls (their Fourier transforms) driving the point x_0 to the point x_π is given by the formula

$$\tilde{u}(\omega, t_2) = e^{-(\hat{A} + i\omega\hat{B})(t_2 + \pi)}\big(v_1(\omega) + Q(\omega)v(\omega)\big). \tag{20}$$

The problem is now how to find the function $v(z)$ that minimizes the functional (8). Applying the Fourier transform to the first equation of (7) with respect to the variable t_1 yields

$$e^{-i\omega\pi}(\pi, t_2) + (i\omega I - A_1)\tilde{x}(\omega, t_2) = B_1 \tilde{u}(\omega, t_2).$$

Suppose now that the eigenvalues of the matrix A_1 are located in the unit disc of the complex plane. In this case there exists the inverse $(i\omega I - A_1)^{-1}$ for each $\omega \in \mathbb{R}^1$ which allows writing

$$\begin{aligned}\tilde{x}(\omega, t_2) = (i\omega I - A_1)^{-1}\Big[&B_1 e^{-(\hat{A} + i\omega\hat{B})(t_2 + \pi)} \\ &\times \big(v_1(\omega) + Q(\omega)v(\omega)\big) + e^{i\omega\pi} e^{A_2(t_2+\pi)} \\ &- e^{-i\omega\pi} e^{2\pi A_1 + A_2(t_2+\pi)} x_0 \\ &- e^{-i\omega\pi} e^{(t_2+\pi)A_2} f\Big].\end{aligned} \tag{21}$$

We consider now the particular case when $n = m = 1,\ a_1 b_1 \neq 0$, where we give the complete solution to the problem. In this case, from the formula above we have

$$\begin{aligned}\tilde{u}(\omega, t_2) &= e^{-b(\omega)(t_2+\pi)}\big(v_1(\omega) + Q(\omega)v(\omega)\big), \\ \tilde{x}(\omega, t_2) &= (i\omega - a_1)^{-1} b_1 e^{-b(\omega)(t_2+\pi)} \\ &\quad \times \big(v_1(\omega) + Q(\omega)v(\omega)\big) \\ &\quad + (i\omega - a_1)^{-1} e^{i\omega\pi} e^{a_2(t_2+\pi)} x_0,\end{aligned}$$

where

$$b(\omega) = \frac{a_1 b_1 - a_1 b_2}{b_1} + i\omega \frac{b_2}{b_1} \doteq \hat{a} + i\omega\hat{b}.$$

The isometric property of the Fourier transform in the space $L_2[-\pi, \pi]$ implies that the functional (8) can be rewritten via the function $v(\omega)$ as

$$\begin{aligned}J(u) &= \iint_S \big(|x(t_1, t_2)|^2 + |u(t_1, t_2)|^2\big)\, \mathrm{d}t_1\, \mathrm{d}t_2 \\ &= \int_{-\infty}^{\infty} \mathrm{d}\omega \int_{-\pi}^{\pi} \big(|\tilde{x}(\omega, t_2)|^2 + |\tilde{u}(\omega, t_2)|^2\big)\, \mathrm{d}t_2 \\ &= \int_{-\infty}^{\infty} \varphi^2(\omega) \left| v(\omega) + \frac{\psi(\omega)}{\varphi^2(\omega)} \right|^2 \mathrm{d}\omega \\ &\quad + \int_{-\infty}^{\infty} \left(\nu^2(\omega) - \left| \frac{\psi(\omega)}{\varphi^2(\omega)} \right|^2 \right) \mathrm{d}\omega,\end{aligned}$$

where

$$\varphi^2(\omega) = \left(\frac{b_1^2}{|i\omega - a_1|^2} + 1 \right) \left(\frac{1 - e^{-4\pi\hat{a}}}{2} \right) |Q(\omega)|^2,$$

$$\begin{aligned}\psi(\omega) &= \frac{1 - e^{-4\pi\hat{a}}}{2} \overline{v}_1(\omega) Q(\omega) \\ &\quad + \frac{b_1^2 (e^{-2\pi b(\omega)} - 1)^2}{(i\omega - a_1)^2} v_0(\omega) Q(\omega) \\ &\quad + \frac{b_1 x_0 e^{i\omega\pi} (e^{2\pi a_2} - 1)(e^{-2\pi b(\omega)} - 1)}{(i\omega - a_1)^2} Q(\omega),\end{aligned}$$

$$\nu^2(\omega) = \left(\frac{b_1^2}{|i\omega - a_1|^2} + 1\right)\left(\frac{1-e^{-4\pi\hat{a}}}{2}\right)|v_1(\omega)|^2 + \frac{x_0^2(e^{4\pi a_2}-1)}{2|i\omega - a_1|^2} + 2\mathrm{Re}\frac{b_1(e^{-2\pi b(\omega)}-1)e^{-i\omega\pi}(e^{2\pi a_2}-1)x_0}{(i\omega - a_1)^2}.$$

Since the second integral above is not dependent on $v(\omega)$, the problem is to minimize the functional

$$J(u) = \int_{-\infty}^{\infty} \varphi^2(\omega)|v(\omega) - l(\omega)|^2\,\mathrm{d}\omega \tag{22}$$

in the class V where $l(\omega) = -\psi(\omega)/\varphi^2(\omega)$ is a known function. Now, introduce the Hilbert space of the functions that are square integrable on $\mathbb{R}^1$ with the weight function $\varphi^2(\omega)$ and call it $L_{2,\varphi}$. In this space the inner product is given by $(f,g) = \int_{-\infty}^{\infty}\varphi^2(\omega)f(\omega)\overline{g}(\omega)\,\mathrm{d}(\omega)$. Hence the minimization of (22) is reduced to the following problem: Find a function v from the class V that provides the best approximation to the known function $l(\omega)$ in the space $L_{2,\varphi}$.

Since the set V is a closed subspace from the space $L_{2,\varphi}$, there exists a unique best approximation to $l(\omega)$ and this approximation is the projection of the function $l(\omega)$ onto V. This projection can be written as a linear combination $v = \sum_{k=1}^{\infty} c_k e_k$ of the vectors of some orthonormalized basis $e_1, e_2, \ldots$, chosen in V, where the Fourier coefficients c_k are calculated by the formula $c_k = (l, e_k)$, $k = 1, 2, \ldots$. The basis in V can also be chosen in a different manner. First, use the Kotelnikov theorem to choose the required basis, (Hurgin and Yakovlev, 1971). To highlight this, note that each function $f(z) \in W_\pi$ can be expanded into the following power series:

$$f(z) = \frac{1}{\pi}\sum_{k=-\infty}^{\infty}(-1)^k\frac{\sin\pi z}{z-k}, \qquad \sum_{k=-\infty}^{\infty}|f(k)|^2 < \infty.$$

If $v(z) \in V$ and $Q(z)$ is an arbitrary polynomial of the p-th degree, then $v(z)Q(z) \in W_\pi$ (see the definition of V). Hence

$$v(z) = \frac{1}{\pi}\sum_{k=-\infty}^{\infty} a_k \frac{\sin\pi z}{Q(z)(z-k)}, \tag{23}$$

where $a_k = (-1)^k v(k)Q(k)$. Since the function (23) is the entire function if the numbers $k = 0, 1, \ldots, p-1$ are roots of the polynomial $Q(z)$, we have $Q(z) = z$. Thus the collection of the functions

$$g_k(z) = \frac{\sin\pi z}{Q(z)(z-k)}, \quad k \in P = \mathrm{Z}\backslash\{0, 1, \ldots, p-1\}$$

forms a basis in the space V. Let $f_1, f_2, \ldots, f_n, \ldots$ denote the re-numbered orthonormalized vectors of the basis $\{g_k\}$, $k \in P$. Now the required vectors $\{e_k\}$ can be determined from the formula

$$e_k = y_k(\Gamma_k\Gamma_{k-1})^{-1/2}, \quad k = 1, 2, \ldots, \tag{24}$$

where

$$y_k = \begin{bmatrix} (f_1,f_2)\cdots(f_1,f_{k-1})f_1 \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ (f_k,f_1)\ldots(f_k,f_{k-1})f_k \end{bmatrix},$$

$$\Gamma_k = \begin{bmatrix} (f_1,f_1)\ldots(f_1,f_k) \\ \ldots\ldots\ldots\ldots \\ (f_k,f_1)\ldots(f_k,f_k) \end{bmatrix}, \quad k = 1, 2, \ldots \tag{25}$$

Hence we have proven the following result:

Theorem 1. *Let $n = m = 1$ and $a_1 b_1 \neq 0$. Then the generalized optimal control for the problem (7), (9)–(10) is given as*

$$u^0(t_1,t_2) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\mathrm{Re}\left(e^{s(\omega)}v^0(\omega)\right)\mathrm{d}\omega, \tag{26}$$

where

$$v^0(\omega) = \omega\sum_{k=1}^{\infty}C_k e_k(\omega) + v_1(\omega),$$

$$s(\omega) = \frac{a_1b_2 - a_2b_1}{b_1} + i\omega\Big(t_1 - \frac{b_2}{b_1}(t_2+\pi)\Big), \tag{27}$$

$$C_k = (l, e_k) = (\Gamma_k\Gamma_{k-1})^{-1/2} \times \begin{bmatrix} (f_1,f_1)\ldots(f_1,f_{k-1})(l,l_1) \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ (f_k,f_1)\ldots(f_k,f_{k-1})(l,l_k) \end{bmatrix}, \tag{28}$$

and $v_1(\omega)$ is some particular solution of (23) to the interpolation problem (15).

Note that the inner product (f_i, f_j) can be easily calculated from the residual theory as

$$(f_k,f_l) = \int_{-\infty}^{\infty}\frac{\varphi^2(\omega)\sin^2\pi\omega}{Q^2(\omega)(\omega-k)(\omega-l)}\,\mathrm{d}\omega = \pi^2\sum_{j=0}^{p-1}\frac{\varphi^2(j)}{(j!(p-1-j)!)^2(j-k)(j-l)}, \quad k \neq l,$$

$$(f_k,f_l) = \pi^2\sum_{j=0}^{p-1}\frac{\varphi^2(j)}{(j!(p-1-j)!)^2(j-l)^2} + \pi^2\frac{\varphi^2(k)}{k^2(k-1)^2\cdots(k-p-1)^2}, \quad k = l.$$

Based on the inverse Fourier transform $v^0(t) \in L_2[-\pi,\pi]$ of a given function $v^0(z) \in W_\pi$ we are able to determine the generalized optimal control function for the problem under consideration. The approximate optimal control from the class $C^1 \in [-\pi,\pi]$ can be established as an approximation to the given function $v^0(t)$. In particular, this approximation can be obtained by cutting the power series (27), where we consider the finite sum

$$v^{(s)}(\omega) = v_1(\omega) + Q(\omega)\sum_{k=1}^{s} C_k e_k(\omega).$$

It should be noted that the inverse Fourier transforms for the functions $v_1(\omega)$, $e_k(\omega)$, $k = 1,\dots,s$ are continuously differentiable functions. The accuracy of this approximation can be evaluated from the following inequalities:

$$\begin{aligned}\|v^{(s)}(t) - v^0(t)\|^2_{L_2} &= \Big\|Q(\omega)\sum_{k=s+1}^{\infty} C_k e_k(\omega)\Big\|^2_W \\ &\le \int_{-\infty}^{\infty} |Q(\omega)|^2 \Big|\sum_{k=s+1}^{\infty} C_k e_k(\omega)\Big|^2 \,\mathrm{d}\omega \\ &= \int_{-\infty}^{\infty} \varphi^2(\omega) \Big|\sum_{k=s+1}^{\infty} C_k e_k(\omega)\Big|^2 \,\mathrm{d}\omega \le \sum_{k=s+1}^{\infty} |C_k|^2.\end{aligned}$$

3. Optimal Control of 1D Systems with Energy Performance Criteria

In this section, based on the proposed method, we give a complete solution to the following continuous 1D optimization problem:

$$\begin{aligned}&\int_{-\pi}^{\pi} |u(t)|^2 \,\mathrm{d}t \to \min, \quad \dot{x} = Ax + bu, \\ &t \in [-\pi,\pi], \quad x(-\pi) = x_0, \quad Hx(\pi) = 0.\end{aligned} \tag{29}$$

Here x is an n-phase vector, A is an $(n \times n)$-matrix, b and x_0 are given n-vectors, $u(t)$, $t \in [-\pi,\pi]$ is a control function from the space $L_2[-\pi,\pi]$ of measureable and square summable functions on $[-\pi,\pi]$, H is a given $(m \times n)$-matrix. We suppose that the system is controllable and hence the set of admissible controls is nonempty. The existence of an optimal control for this optimization problem can be stated on the analogy of (Vasiljev, 1981). In addition to that, we suppose that A has single eigenvalues. Write $G = [Hb, HAb, \dots, HA^{n-1}b]$, $R = -[Hx_0, HAx_0, \dots, HA^{n-1}x_0]$. V is the $(n \times n)$ Vandermonde matrix, generated by the eigenvalues $\lambda_1,\dots,\lambda_n$ of A; $F = GV^{-1}\Lambda$, $f = (2\pi)^{-1/2}RV^{-1}g$, $g = (e^{2\lambda_1\pi},\dots,e^{2\lambda_n\pi})'$.

Theorem 2. *The Fourier transform of the optimal control in (29) is given by*

$$u^0(z) = \sum_{s=1}^{m} \beta_s \sum_{j=1}^{n} F_{sj}\overline{D(\pi z_j - \pi z)},$$

where the numbers $\beta_s = \nu_s + i\gamma_s$, *(here* $i^2 = -1$*),* $s = 1,2,\dots,$ *are determined as*

$$\sum_{s=1}^{m} \beta_s \sum_{j=1}^{n}\sum_{i=1}^{n} F_{lj}F_{sj}\overline{D(\pi z_i - \pi z_j)} = f_l,$$

for $l = 1,2,\dots,m$. *Here* $D(z) = \sin z/z$, $D(0) = 1$, F_{lj}, f_l, $l = 1,2,\dots,m$, $j = 1,2,\dots,n$ *are elements of the* $(m\times n)$*-matrix* F *and the* n*-vector* f*, respectively.*

Proof. The solution of (29) for a given control function can be written as follows:

$$x(t) = e^{A(t+\pi)}x_0 + \int_{-\pi}^{t} e^{A(t-\tau)}bu(\tau)\,\mathrm{d}\tau, \quad t \in [-\pi,\pi]. \tag{30}$$

The matrix function e^{At} can be represented in the form

$$e^{At} = \sum_{i=0}^{n-1} \alpha_i(t)A^i,$$

where the $\alpha_i(t)$'s are the coefficients of the Lagrange-Silvester interpolation polynomial $r(A)$ that is determined by the matrix A. Then from (29) we have that the admissible control functions satisfy

$$\begin{aligned}\sum_{i=0}^{n-1} HA^i b \int_{-\pi}^{\pi} \alpha_i(\pi-\tau)u(\tau)\,\mathrm{d}\tau \\ = -\sum_{i=0}^{n-1} \alpha_i(2\pi)HA^i x_0.\end{aligned} \tag{31}$$

Set

$$G = [Hb, HAb, \dots, HA^{n-1}b]$$

and

$$R = -[Hx_0, HAx_0, \dots, HA^{n-1}x_0].$$

Similarly as in the previous section, it can be established that the Fourier transform $\widetilde{u}(w)$ of the admissible control, which solves the controllability problem (29), can be extended to the complex plane as the entire function of the form

$$\widetilde{u}(z) = (2\pi)^{-1/2}\int_{-\pi}^{\pi} u(\tau)e^{-izt}\,\mathrm{d}t. \tag{32}$$

Denote by $\widetilde{w} = (\widetilde{w}_1,\dots,\widetilde{w}_n)$ the n-vector, whose coordinates $\widetilde{w}_k = \widetilde{u}(-i\lambda_k)$, $k = 1,\dots,n$ are the values of the function (32) at the points $z_k = -i\lambda_k$, $k = 1,2,\dots,n$ of the complex plane $\mathbb{C}$. Then (31) yields

$$F\widetilde{w} = f, \tag{33}$$

where $F = GV^{-1}\Lambda$, $f = (2\pi)^{1/2}RV^{-1}g$, $g = (e^{2\lambda_1\pi}, \dots, e^{2\lambda_n\pi})'$. The Kotelnikov theorem implies that each $u(z) \in W_\pi$ can be represented as

$$u(z) = \sum_{k=-\infty}^{\infty} u_k D(\pi z - k\pi), \qquad \sum_{k=-\infty}^{\infty} |u_k|^2 < \infty, \quad (34)$$

where $u_k = u(k) \doteq x_k + iy_k$, $D(z) = z^{-1}\sin z$, $D(0) = 1$. Since

$$\int_{-\infty}^{\infty} \frac{\sin\pi(w-k)}{\pi(w-k)}\,\frac{\sin\pi(w-n)}{\pi(w-n)}\,\mathrm{d}w = \begin{cases} 1, & k = n,\\ 0, & k \neq n,\end{cases}$$

$k, n = 0, \pm 1, \dots$, we get

$$J(u) = \int_{-\infty}^{\infty} |u(w)|^2\,\mathrm{d}w = \sum_{k-\infty}^{\infty} |u_k|^2 = \sum_{k=-\infty}^{\infty} (x_k^2 + y_k^2).$$

Finally, the following optimization problem appears: Minimize the functional

$$J(u) = \sum_{k-\infty}^{\infty} u_k\bar{u}_k \longrightarrow \min_{u_k} \quad (35)$$

in the space W_π, subject to the constraint

$$F\sum_{k-\infty}^{\infty} u_k\hat{D}_k = f, \quad (36)$$

where $\hat{D}_k = (D(\pi z_1 - k\pi), \dots, D(\pi z_n - k\pi))'$, and $\bar{u}_k = x_k - iy_k$ denotes the complex conjugate for u_k. Next, set $D(\pi z_j - \pi k) \doteq a_k(z_j) + ib_k(z_j)$, $j = 1, 2, \dots, n$, where $a_k(z_j)$ and $b_k(z_j)$ are some real numbers. Then the problem (35), (36) can be rewritten as

$$\sum_{k=-\infty}^{\infty} (x_k^2 + y_k^2) \longrightarrow \min, \quad (37)$$

subject to the constraints

$$\begin{cases} \displaystyle\sum_{k=-\infty}^{\infty}\sum_{l=1}^{n} F_{sl}\big(x_k a_k(z_l) - y_k b_k(z_l)\big) = f_s,\\ \displaystyle\sum_{k=-\infty}^{\infty}\sum_{l=1}^{n} F_{sl}\big(y_k a_k(z_l) + x_k b_k(z_l)\big) = 0,\\ \qquad s = 1, 2, \dots, m. \end{cases} \quad (38)$$

The Lagrange function for the problem (37), (38) is

$$\begin{aligned}\Phi(u, \nu, \gamma) &= \frac{1}{2}\sum_{k=-\infty}^{\infty}(x_k^2 + y_k^2)\\ &+ \sum_{s=1}^{m}\nu_s\Big[f_s - \sum_{k=-\infty}^{\infty}\sum_{l=1}^{n} F_{sl}\big(x_k a_k(z_l) - y_k b_k(z_l)\big)\Big]\\ &- \sum_{s=1}^{m}\gamma_s\Big[\sum_{k=-\infty}^{\infty}\sum_{l=1}^{n} F_{sl}\big(y_k a_k(z_l) + x_k b_k(z_l)\big)\Big],\end{aligned}$$

and the stationarity conditions

$$\begin{aligned} x_k &= \sum_{s=1}^{m}\sum_{l=1}^{n} F_{sl}\big(\nu_s a_k(z_l) + \gamma_s b_k(z_l)\big),\\ y_k &= \sum_{s=1}^{m}\sum_{l=1}^{n} F_{sl}\big(\gamma_s a_k(z_l) - \nu_s b_k(z_l)\big),\\ & \qquad k = 0, \pm 1, \pm 2, \dots, \end{aligned}$$

hold. Substituting this into the first equation from (38), we have

$$\begin{aligned}\sum_{l=1}^{n}\sum_{r=1}^{m}\sum_{t=1}^{n} F_{sl}F_{rt}\sum_{k-\infty}^{\infty}\Big[\nu_r\big(a_k(z_t)a_k(z_l) + b_k(z_t)b_k(z_l)\big)\\ +\gamma_r\big(b_k(z_t)a_k(z_l) - a_k(z_t)b_k(z_l)\big)\Big] = f_s, \quad (39)\end{aligned}$$

$s = 1, 2, \dots, m$. Applying (34) to the function $u(z) = D(\pi z - \pi z_t)$ at $z = z_l$ yields

$$D(\pi z_l - \pi z_t) = \sum_{k=-\infty}^{\infty} D(\pi z_t - k\pi)D(\pi z_l - k\pi).$$

Since $\underline{[a_k(z_t) + ib_k(z_t)]}[a_k(z_l) - ib_k(z_l)] = D(\pi z_t - k\pi)\overline{D(\pi z_l - k\pi)}$, from (39) we have

$$\begin{aligned}\sum_{l=1}^{n}\sum_{r=1}^{m}\sum_{t=1}^{n} F_{sl}F_{rt}\Big[\nu_r\mathrm{Re}\left(\overline{D(\pi z_l - \pi z_t)}\right)\\ -\gamma_r\mathrm{Im}\left(\overline{D(\pi z_l - \pi z_t)}\right)\Big] = f_s. \quad (40)\end{aligned}$$

On the analogy with the above calculations, the second equation of (38) leads to

$$\begin{aligned}\sum_{l=1}^{n}\sum_{r=1}^{m}\sum_{t=1}^{n} F_{sl}F_{rt}\Big[\nu_r\mathrm{Im}\left(\overline{D(\pi z_l - \pi z_t)}\right)\\ +\gamma_r\mathrm{Re}\left(\overline{D(\pi z_l - \pi z_t)}\right)\Big] = 0. \quad (41)\end{aligned}$$

Next, set $\beta_r = \nu_r + i\gamma_r$, $r = 1, 2, \dots, m$. Combining (40) and (41) leads to the required relations. Substituting the given values $u_k = x_k + iy_k$ into (34) gives

$$\begin{aligned} u^0(z) &= \sum_{k=-\infty}^{\infty}(x_k + iy_k)D(\pi z - k\pi)\\ &= \sum_{k=-\infty}^{\infty}\sum_{s=1}^{m}\sum_{l=1}^{n} F_{sl}\Big[\nu_s\big(a_k(z_l) + \gamma_s b_k(z_l)\big)\\ &\quad + +i\left(\gamma_s\big(a_k(z_l) - \nu_s b_k(z_l)\big)\right)\Big]D(\pi z - k\pi)\\ &= \sum_{s=1}^{m}\sum_{l=1}^{n} F_{sl}\sum_{k=-\infty}^{\infty}\Big[\nu_s\overline{D(\pi z_l - k\pi)}D(\pi z - k\pi)\\ &\quad + i\gamma_s\overline{D(\pi z_l - k\pi)}D(\pi z - k\pi)\Big]. \end{aligned}$$

Using the representation (34) for $u(z) = \overline{D(\pi z - \pi z_l)}$ yields the required optimal control

$$u^0(z) = \sum_{s=1}^{m}\sum_{l=1}^{n} F_{sl}\left[\nu_s \overline{D(\pi z - \pi z_l)} + i\gamma_s \overline{D(\pi z - \pi z_l)}\right]$$
$$= \sum_{s=1}^{m} \beta_s \sum_{l=1}^{n} F_{sl}\overline{D(\pi z - \pi z_l)},$$

which completes the proof. ■

It is also possible to prove that the Fourier transform of the optimal controls can be represented by the series expansion for the basis $l_k = Q^{-1}(z)(z-k)^{-1}\sin \pi z$, $k = 0, 1, \ldots$, where $Q(z)$ is some polynomial of a finite degree. Therefore the approximate solution can be obtained by the cutting of this power series in much the same way as in the previous section.

4. Example

To illustrate the proposed method we consider the simple optimal control problem

$$\dot{x} = u, \quad t \in [-\pi, \pi], \quad x(-\pi) = x_0,$$
$$x(\pi) = 0, \quad J(u) = \int_{-\pi}^{\pi} u^2(t)\,\mathrm{d}t \to \min.$$

Here $A = 0$, $b = 1$ and $H = 1$. The notation required for this case is as follows:

$$R = -Hx_0 = x_0, \quad G = Hb = 1, \quad V = 1, \quad \Lambda = e^0 = 1,$$
$$F = 1, \quad g = 1, \quad f = -x_0/\sqrt{2\pi}.$$

From Theorem 2 we get $\beta D(0) = -x_0/\sqrt{2\pi}$. Since $D(0) = 1$, we have $\beta = -x_0/\sqrt{2\pi}$ and

$$u^0(z) = -x_0\overline{D(-\pi z)}/\sqrt{2\pi} = -x_0\left(\overline{\frac{\sin \pi z}{\pi z}}\right)/\sqrt{2\pi}.$$

The optimal control function is the Fourier image of the function $u^0(z)$:

$$u^0(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} u^0(\omega)e^{i\omega t}\,\mathrm{d}\omega = -\frac{x_0}{2\pi}\int_{-\infty}^{\infty}\frac{\sin \pi\omega}{\pi\omega}e^{i\omega t}\,\mathrm{d}\omega$$
$$= -\frac{x_0}{2\pi^2}\int_{-\infty}^{\infty}\frac{e^{i\pi\omega} - e^{-i\pi\omega}}{2i\omega}e^{i\omega t}\,\mathrm{d}\omega$$
$$= -\frac{x_0}{4\pi^2 i}\left[\int_{-\infty}^{\infty}\frac{e^{i\omega(\pi+t)}}{\omega}\,\mathrm{d}\omega - \int_{-\infty}^{\infty}\frac{e^{i\omega(t-\pi)}}{\omega}\,\mathrm{d}\omega\right].$$

Applying residual theory to the improper integrals yields

$$\int_{-\infty}^{\infty}\frac{e^{i\omega(\pi+t)}}{\omega}\,\mathrm{d}\omega = \begin{cases} i\pi, & \pi + t > 0,\\ -i\pi, & \pi + t < 0,\end{cases}$$
$$\int_{-\infty}^{\infty}\frac{e^{i\omega(t-\pi)}}{\omega}\,\mathrm{d}\omega = \begin{cases} i\pi, & t - \pi > 0,\\ -i\pi, & t - \pi < 0.\end{cases}$$

Finally,

$$u^0(t) = \begin{cases} -x_0/2\pi, & -\pi \le t \le \pi,\\ 0, & |t| > \pi.\end{cases}$$

On the other hand, the Hamilton-Pontryagin function for the analysed optimal control problem is $H(x,u,\psi,\lambda) = -\lambda u^2 + \psi u$, where the adjoint variable ψ is defined by the following adjoint differential equation:

$$\frac{\mathrm{d}\psi}{\mathrm{d}t} = -\frac{\partial H}{\partial x} = 0, \quad -\pi \le t \le \pi.$$

From the Pontryagin maximum principle (Gabasov and Kirillova, 1988) it follows that the condition $|\lambda| + |\psi(t)| \ne 0$, $\forall\, t \in [-\pi, \pi]$ is fulfilled. If we suppose that $\lambda = 0$, then the maximum of the function $H(x,u,\psi,\lambda)$ on $u \in \mathbb{R}$ is achieved only if $\psi = 0$, which contradicts the above condition. Hence we can set $\lambda = 1$, and then the function $H(x,u,\psi,\lambda)$ reaches an extremum on $u \in \mathbb{R}$ for $u = \psi/2$. Thus we have the following differential problem associated with the maximum principle:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\psi}{2}, \quad \frac{\mathrm{d}\psi}{\mathrm{d}t} = 0, \quad t \in [-\pi, \pi],$$
$$x(-\pi) = x_0, \quad x(\pi) = 0.$$

It is easy to see that the solution to this system is given by the formulae

$$\psi(t) = -\frac{x_0}{\pi}, \quad x(t) = -\frac{x_0}{2\pi}t - \frac{x_0}{2},$$

and hence the optimal control is

$$u^0(t) = \frac{\psi}{2} = -\frac{x_0}{2\pi}, \quad -\pi \le t \le \pi,$$

which coincides with the result obtained above.

5. Conclusion

The theory of complex-valued functions is commonly employed in the solution of various engineering and scientific problems. This paper presented a systematic application of entire function theory to optimization topics. This technique was used for investigating optimal control for linear

$2D$ continuous-discrete systems with mixed constraints (Dymkov, 2001). Attention was restricted to the case of linear dynamics since this is the area where considerable progress can be being done. Work to extend this approach to other models is already made. Results of that will be reported in due course.

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