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CANONICAL FORMS OF SINGULAR 1D AND 2D LINEAR SYSTEMS

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The paper consists of two parts. In the first part, new canonical forms are defined for singular 1D linear systems and a procedure to determine nonsingular matrices transforming matrices of singular systems to their canonical forms is derived. In the second part new canonical forms of matrices of the singular 2D Roesser model are defined and a procedure for determining realisations in canonical forms for a given 2D transfer function is presented. Necessary and sufficient conditions for the existence of a pair of nonsingular block diagonal matrices transforming the matrices of the singular 2D Roesser model to their canonical forms are established. A procedure for computing the pair of nonsingular matrices is presented.

Keywords: canonical form, singular, 2D Roesser model, 1D system, transformation

1. Introduction

A survey of basic results regarding linear singular (descriptor, implicit, generalized) systems can be found in (Cobb, 1984; Dai, 1989; Kaczorek, 1992; Lewis, 1984; 1986; Lewis and Mertzios, 1989; Luenberger, 1967; 1978; Özcaldiran and Lewis, 1989). It is well known (Brunovsky, 1970; Kaczorek, 1992; Luenberger, 1967) that if the pair (A, B) of a standard linear discrete-time system $x_{i+1} = Ax_i + Bu_i$ is reachable, then it can be transformed to its reachable canonical form. Similarly, if the pair (A, C) of the standard system is observable, then it can transformed to its observable canonical form. Similar results can also be obtained for linear time-varying systems (Silverman, 1966). Aplevich (1985) established conditions for minimal representations of singular linear systems.

The most popular models of two-dimensional (2D) systems are those introduced by Roesser (1975), Fornasini and Marchesini (1976; 1978) and Kurek (1985). The models were generalized to singular 2D models (Kaczorek, 1988; 1992; 1995) and positive 2D models (Kaczorek, 1996; Valcher, 1997). The realisation problem for 1D and 2D linear systems was considered in many books and papers (Aplevich, 1985; Dai, 1989; Eising, 1978; Fornasini and Marchesini, 1976; Gałkowski, 1981; 1992; 1997; Hayton *et al.*, 1988; Hinamoto and Fairman, 1984; Kaczorek, 1985; 1987; 1992; 1997a; 1997b; 1997c; 1998; 2000; Żak *et al.*, 1986). An elementary operation approach to state-space realisations of 2D linear systems was developed by Gałkowski (1981; 1992; 1997).

In this paper new canonical forms for singular 1D and 2D linear systems will be defined and a procedure for computing a pair of nonsingular matrices transforming the matrices of singular 1D and 2D systems to their canonical forms will be derived.

The paper is organised as follows. In Section 2 new canonical forms of singular 1D linear systems are introduced. A method of determining realisations of a given 1D transfer function in canonical forms is presented in Section 3. The problem of transforming matrices of a singular 1D linear system to canonical forms is considered in Section 4. Canonical forms of the matrices of a singular 2D Roesser model are defined in Section 5. A method to determine realisations of a given 2D transfer function in canonical forms is developed in Section 6. Conditions on which the matrices of a singular 2D Roesser model can be transformed to their canonical forms are established and a suitable procedure for their transformation is presented in Section 7. Concluding remarks are given in Section 8.

2. Canonical Form of Singular Systems

Let $\mathbb{R}^{n\times m}$ be the set of $n \times m$ matrices with entries from the field of real numbers $\mathbb{R}$ and $\mathbb{R}^n := \mathbb{R}^{n\times 1}$. The set of non-negative integers will be denoted by $\mathbb{Z}_+$ and the set of $p \times m$ rational (proper or improper) matrices in variable z will be denoted by $\mathbb{R}^{p\times m}(z)$. The $n \times n$ identity matrix will be denoted by I_n.

Consider the discrete-time linear system

$$\begin{aligned} Ex_{i+1} &= Ax_i + Bu_i, \\ y_i &= Cx_i, \end{aligned} \tag{1}$$

$i \in \mathbb{Z}_+$, where $x_i \in \mathbb{R}^n$, $u_i \in \mathbb{R}^m$ and $y_i \in \mathbb{R}^p$ are the state, input and output vectors, respectively, and

$$E, A \in \mathbb{R}^{n\times n}, \quad B \in \mathbb{R}^{n\times m}, \quad C \in \mathbb{R}^{p\times n}. \tag{2}$$

It is assumed that $\det E = 0$, but

$$\det[Ez - A] \neq 0 \ \text{ for some } \ z \in \mathbb{C}, \tag{3}$$

where $\mathbb{C}$ is the field of complex numbers.

The transfer matrix of (1) is given by

$$T(z) = C[Ez - A]^{-1}B \in \mathbb{R}^{p\times m}(z). \tag{4}$$

The matrices (2) are called a realisation of a given $T(z) \in \mathbb{R}^{p\times m}(z)$ if they satisfy (4).

Definition 1. The matrices (2) are said to have the *first canonical form* if

$$E = \operatorname{diag}\begin{bmatrix} E_1 & E_2 & \cdots & E_m \end{bmatrix} \in \mathbb{R}^{n\times n},$$

$$E_i = \left[\begin{array}{c:c} I_{q_i} & 0 \\ \hdashline 0 & 0 \end{array}\right] \in \mathbb{R}^{(q_i+1)\times(q_i+1)} \tag{5a}$$

for $n := m + \sum_{i=1}^{m} q_i, \quad i = 1,\dots,m$,

$$A = \operatorname{diag}\begin{bmatrix} A_1 & A_2 & \cdots & A_m \end{bmatrix} \in \mathbb{R}^{n\times n},$$

$$A_i = \left[\begin{array}{c:c} 0 & I_{q_i} \\ \hdashline \multicolumn{2}{c}{a_i} \end{array}\right] \in \mathbb{R}^{(q_i+1)\times(q_i+1)} \tag{5b}$$

where $a_i = \left[a_0^i \dots a_{r_i-1}^i \ \ 1 \ \ 0 \dots 0\right]$,

$$B = \operatorname{diag}\begin{bmatrix} B_1 & B_2 & \cdots & B_m \end{bmatrix} \in \mathbb{R}^{n\times m},$$

$$B_i = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \in \mathbb{R}^{q_i+1} \tag{5c}$$

for $i = 1,\dots,m$, and

$$C = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1m} \\ c_{21} & c_{22} & \cdots & c_{2m} \\ \dots & \dots & \dots & \dots \\ c_{p1} & c_{p2} & \cdots & c_{pm} \end{bmatrix} \in \mathbb{R}^{p\times n},$$

$$c_{ij} = \begin{bmatrix} b_{ij}^0 & b_{ij}^1 & \cdots & b_{ij}^{q_i} \end{bmatrix} \in \mathbb{R}^{1\times(q_i+1)}, \tag{5d}$$

for $i = 1,\dots,p$ and $j = 1,\dots,m$. They have the *second canonical form* if

$$E = \operatorname{diag}\begin{bmatrix} E_1 & E_2 & \cdots & E_p \end{bmatrix} \in \mathbb{R}^{n\times n},$$

$$E_i = \left[\begin{array}{c:c} I_{q_i'} & 0 \\ \hdashline 0 & 0 \end{array}\right] \in \mathbb{R}^{(q_i'+1)\times(q_i'+1)} \tag{5e}$$

for $n := p + \sum_{i=1}^{p} q_i'$,

$$A = \operatorname{diag}\begin{bmatrix} A_1 & A_2 & \cdots & A_p \end{bmatrix} \in \mathbb{R}^{n\times n},$$

$$A_i = \left[\begin{array}{c:c} 0 & \\ \hdashline & a_i^T \\ I_{q_i'} & \end{array}\right] \in \mathbb{R}^{(q_i'+1)\times(q_i'+1)} \tag{5f}$$

for $i = 1,\dots,p$,

$$B = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1m} \\ b_{21} & b_{22} & \cdots & b_{2m} \\ \dots & \dots & \dots & \dots \\ b_{p1} & b_{p2} & \cdots & b_{pm} \end{bmatrix} \in \mathbb{R}^{n\times m},$$

$$b_{ij} = \begin{bmatrix} b_{ij}^0 \\ b_{ij}^1 \\ \vdots \\ b_{ij}^{q_i'} \end{bmatrix} \in \mathbb{R}^{q_i'+1} \tag{5g}$$

for $i = 1,\dots,p$ and $j = 1,\dots,m$.

$$C = \operatorname{diag}\begin{bmatrix} c_1 & c_2 & \cdots & c_p \end{bmatrix} \in \mathbb{R}^{p\times n},$$

$$c_i = \begin{bmatrix} 0 & \cdots & 0 & 1 \end{bmatrix} \in \mathbb{R}^{1\times q_i'+1}. \tag{5h}$$

3. Determination of Realisations in Canonical Forms

Consider the irreducible transfer function

$$T(z) = \frac{b_q z^q + b_{q-1}z^{q-1} + \dots + b_1 z + b_0}{z^r + a_{r-1}z^{r-1} + \dots + a_1 z + a_0}, \quad q > r, \tag{6}$$

where b_i, $i = 0,1,\dots,q$ and a_j, $j = 0,1,\dots,r-1$ are given real coefficients. Defining

$$E := \frac{U}{z^{r-q} + a_{r-1}z^{r-q-1} + \dots + a_1 z^{1-q} + a_0 z^{-q}}, \tag{7}$$

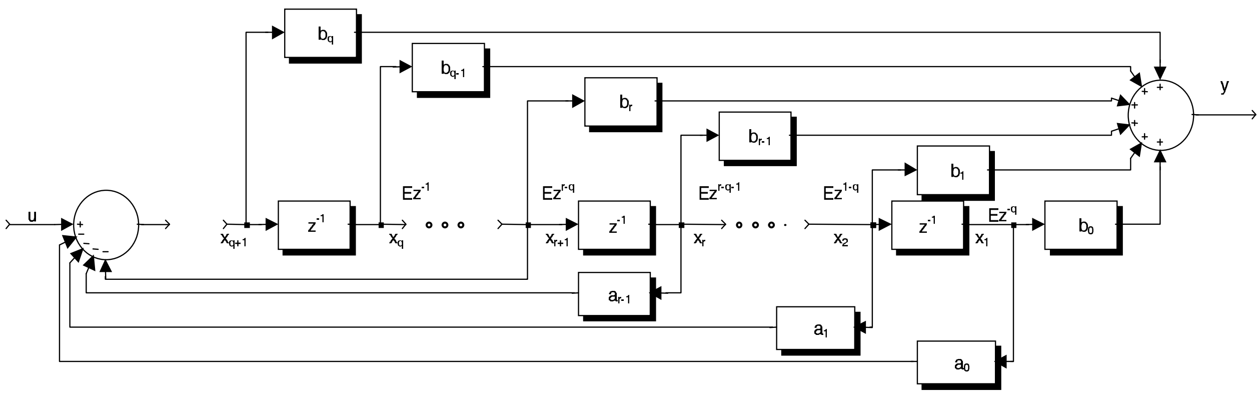

Fig. 1. Block diagram for the transfer function (6).

we can write the equation

$$T(z) = \frac{b_q z^q + b_{q-1} z^{-1} + \cdots + b_1 z^{1-q} + b_0 z^{-q}}{z^{r-q} + a_{r-1} z^{r-q-1} + \cdots + a_1 z^{1-q} + a_0 z^{-q}} = \frac{Y}{U}$$

in the form

$$Y = \left(b_q + b_{q-1} z^{-1} + \cdots + b_1 z^{1-q} + b_0 z^{-q}\right) E. \quad (8)$$

The relation (7) can be rewritten as

$$U - \left(z^{r-q} + a_{r-1} z^{r-q-1} + \cdots + a_1 z^{1-q} + a_0 z^{-q}\right) E = 0. \quad (9)$$

From (8) and (9) the block diagram shown in Fig. 1 follows.

As the state variables $x_1(i), x_2(i), \ldots, x_q(i)$ we choose the outputs of the delay elements. Using Fig. 1, we can write the equations

$$\begin{aligned} x_1(i+1) &= x_2(i), \\ x_2(i+1) &= x_3(i), \\ &\vdots \\ x_{q-1}(i+1) &= x_q(i), \\ x_{q+1}(i+1) &= x_q(i), \\ 0 = -a_0 x_1(i) &- a_1 x_2(i) \\ &- \cdots - a_{r-1} x_r(i) - x_{r+1}(i) + u(i) \end{aligned} \quad (10a)$$

and

$$y(i) = b_0 x_1(i) + b_1 x_2(i) + \cdots + b_q x_{q+1}(i). \quad (10b)$$

Defining

$$x_i := \begin{bmatrix} x_1(i) \\ x_2(i) \\ \vdots \\ x_{q+1}(i) \end{bmatrix},$$

we can write (10) in the form (1), where

$$E_1 = \left[\begin{array}{c:c} I_q & 0 \\ \hdashline 0 & 0 \end{array}\right] \in \mathbb{R}^{(q+1)\times(q+1)},$$

$$A_1 = \left[\begin{array}{c:c} 0 & I_q \\ \hdashline \multicolumn{2}{c}{\bar{a}} \end{array}\right] \in \mathbb{R}^{(q+1)\times(q+1)},$$

$$\bar{a} := [-a_0, -a_1, \ldots, -a_{r-1}, -1, 0, \ldots, 0] \quad (11)$$

$$B_1 = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \in \mathbb{R}^{q+1},$$

$$C_1 = \begin{bmatrix} b_0 & b_1 & \cdots & b_q \end{bmatrix} \in \mathbb{R}^{1\times(q+1)}.$$

The matrices (11) have the desired canonical form (5a)–(5d).

If we choose $x'_k(i) := x_{q-k+2}(i)$ for $k = 1, \ldots, q+1$ then we obtain (1), where

$$E_2 = \left[\begin{array}{c:c} 0 & 0 \\ \hdashline 0 & I_q \end{array}\right] \in \mathbb{R}^{(q+1)\times(q+1)},$$

$$A_2 = \left[\begin{array}{c:c} \multicolumn{2}{c}{\bar{a}'} \\ \hdashline I_q & 0 \end{array}\right] \in \mathbb{R}^{(q+1)\times(q+1)},$$

$$\bar{a}' := [0, \ldots, 0, -1, -a_{r-1}, \ldots, -a_1, -a_0], \quad (12)$$

$$B_2 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \in \mathbb{R}^{q+1},$$

$$C_2 = \begin{bmatrix} b_q & b_{q-1} & \cdots & b_0 \end{bmatrix} \in \mathbb{R}^{1\times(q+1)}.$$

Another method of determining realisations in the canonical form of (6) is presented in (Kaczorek, 2000).

4. Transformation to Canonical Forms

Given the matrices (2) we establish conditions on which they can be transformed to their canonical forms (5) and find two nonsingular matrices $P, Q \in \mathbb{R}^{n\times n}$ such that the matrices

$$\bar{E} = PEQ, \quad \bar{A} = PAQ, \quad \bar{B} = PB, \quad \bar{C} = CQ \qquad (13)$$

have the canonical forms (5). If (3) is satisfied, then

$$[Ez - A]^{-1} = \sum_{i=-\mu}^{\infty} \Phi_i z^{-(i+1)}, \qquad (14)$$

where $\mu \le \operatorname{rank} E - \deg\det[Ez - A] + 1$ is the nilpotence index and the Φ_i's are the fundamental matrices defined by

$$E\Phi_i - A\Phi_{i-1} = \Phi_i E - \Phi_{i-1} A = \begin{cases} 1 & \text{for } i = 0, \\ 0 & \text{for } i \neq 0, \end{cases} \qquad (15)$$

and

$$\Phi_i = 0 \ \text{ for } \ i < -\mu.$$

The solution of (1) is given by

$$x_i = \Phi_i E x_0 + \sum_{j=0}^{i+\mu-1} \Phi_{i-j-1} B u_j, \quad i \in \mathbb{Z}_+. \qquad (16)$$

Definition 2. The system (1) is called n*-step reachable* if for $x_0 = 0$ and any given $x_f \in \mathbb{R}^n$ there exists a sequence $u_i \in \mathbb{R}^m$, $i = 0, 1, \dots, n + \mu - 1$ such that $x_n = x_f$.

Theorem 1. *The system (1) is n-step reachable if and only if*

$$\operatorname{rank} R_n = n, \qquad (17)$$

where

$$R_n := [\Phi_{n-1}B, \dots, \Phi_0 B, \Phi_{-1}B, \dots, \Phi_{-\mu}B]. \qquad (18)$$

Proof. From (16), for $x_0 = 0$ and $i = n$ we have

$$x_f = x_n = \sum_{j=0}^{n+\mu-1} \Phi_{n-j-1} B u_j = R_n u_0^{n+\mu-1}, \qquad (19)$$

where

$$u_0^{n+\mu-1} := \left[u_0^T, \cdots, u_{n-1}^T, u_n^T, \cdots, u_{n+\mu-1}^T\right]^T.$$

From (19) it follows that for any $x_f \in \mathbb{R}^n$ there exists a sequence u_i, $i = 0, 1, \dots, n + \mu - 1$ if and only if (17) holds. ■

Definition 3. The system (1) is called n*-step observable* if for any $x_0 \neq 0$ and given $u_i \in \mathbb{R}^m$ and $y_i \in \mathbb{R}^p$ for $i = -\mu, \dots, n + 1$ it is possible to find the vector Ex_0.

Theorem 2. *The system (1) is n-step observable if and only if*

$$\operatorname{rank} O_n = n, \qquad (20)$$

where

$$O_n := \begin{bmatrix} C\Phi_{-\mu} \\ \vdots \\ C\Phi_{-1} \\ C\Phi_0 \\ \vdots \\ C\Phi_{n-1} \end{bmatrix}. \qquad (21)$$

Proof. From (1) and (16) we have

$$y_i' := y_i - \sum_{j=0}^{i+\mu-1} C\Phi_{i-j-1} B u_j = C\Phi_i E x_0. \qquad (22)$$

Using (22) for $i = -\mu, \dots, -1, 0, \dots, n - 1$ and (21), we obtain

$$\left[{y'}_{-\mu}^T, \dots, {y'}_{-1}^T, {y'}_0^T, \dots, {y'}_{n-1}^T\right]^T = O_n E x_0. \qquad (23)$$

From (23) it follows that it is possible to find the vector Ex_0 if and only if (20) holds. ■

Theorem 3. *Let (2) be any given matrices satisfying (3). Then there exist nonsingular matrices $P, Q \in \mathbb{R}^{n\times n}$ such that the matrices (13) have the canonical form (5) if the system (1) is n-step reachable and n-step observable.*

Proof. Using (13) and (14), we can write

$$\begin{aligned}[\bar{E}z - \bar{A}]^{-1} &= [P(Ez - A)Q]^{-1} = Q^{-1}[Ez - A]^{-1}P^{-1} \\ &= \sum_{i=-\mu}^{\infty} Q^{-1}\Phi_i P^{-1} z^{-(i+1)} \\ &= \sum_{i=-\mu}^{\infty} \bar{\Phi}_i z^{-(i+1)},\end{aligned} \qquad (24)$$

where

$$\bar{\Phi}_i = Q^{-1}\Phi_i P^{-1}, \quad i = -\mu, -\mu+1, \dots . \tag{25}$$

From (18), (25) and $\bar{B} = PB$, we have

$$\begin{aligned} R_n &= [\Phi_{n-1}B, \dots, \Phi_0 B, \Phi_{-1}B, \dots, \Phi_{-\mu}B] \\ &= Q\left[\bar{\Phi}_{n-1}\bar{B}, \dots, \bar{\Phi}_0\bar{B}, \bar{\Phi}_{-1}\bar{B}, \dots, \bar{\Phi}_{-\mu}\bar{B}\right] \\ &= Q\bar{R}_n, \end{aligned} \tag{26}$$

where

$$\bar{R}_n = \left[\bar{\Phi}_{n-1}\bar{B}, \dots, \bar{\Phi}_0\bar{B}, \bar{\Phi}_{-1}\bar{B}, \dots, \bar{\Phi}_{-\mu}\bar{B}\right]. \tag{27}$$

If the system (1) is n-step reachable, then (17) holds and from (26) we obtain

$$Q = \hat{R}_n\tilde{R}_n^{-1}, \tag{28}$$

where $\hat{R}_n$ and $\tilde{R}_n$ are square matrices consisting of n linearly independent corresponding columns of the matrices R_n and $\bar{R}_n$, respectively.

Similarly, from (21), (25) and $\bar{C} = CQ$, we have

$$O_n := \begin{bmatrix} C\Phi_{-\mu} \\ \vdots \\ C\Phi_{-1} \\ C\Phi_0 \\ \vdots \\ C\Phi_{n-1} \end{bmatrix} = \begin{bmatrix} \bar{C}\bar{\Phi}_{-\mu} \\ \vdots \\ \bar{C}\bar{\Phi}_{-1} \\ \bar{C}\bar{\Phi}_0 \\ \vdots \\ \bar{C}\bar{\Phi}_{n-1} \end{bmatrix} P = \bar{O}_n P, \tag{29}$$

where

$$\bar{O}_n := \begin{bmatrix} \bar{C}\bar{\Phi}_{-\mu} \\ \vdots \\ \bar{C}\bar{\Phi}_{-1} \\ \bar{C}\bar{\Phi}_0 \\ \vdots \\ \bar{C}\bar{\Phi}_{n-1} \end{bmatrix}. \tag{30}$$

If the system (1) is n-step observable, then (20) holds and from (29) we obtain

$$P = \tilde{O}_n^{-1}\hat{O}_n, \tag{31}$$

where $\hat{O}_n$ and $\tilde{O}_n$ are square matrices consisting of n linearly independent corresponding rows of the matrices O_n and $\bar{O}_n$, respectively. ■

If the system (1) is n-step reachable and n-step observable, then the matrices $\bar{E}$, $\bar{A}$, $\bar{B}$, $\bar{C}$ in the canonical form (5) can be found using the following procedure:

Procedure 1.

Step 1. Knowing E, A, B, C, find the transfer matrix (4).

Step 2. Using the procedure presented in Section 3, find the realisation of the transfer matrix in the canonical form (5).

Step 3. Using (14) and (24), find the fundamental matrices Φ_i and $\bar{\Phi}_i$ for $i = -\mu, \dots, -1, 0, \dots, n-1$.

Step 4. Using (18), (27) and (21), (30), find R_n, $\bar{R}_n$, O_n and $\bar{O}_n$.

Step 5. Using (28) and (31) find the desired matrices Q and P.

5. Canonical Forms of the Matrices of the Singular 2D Roesser Model

Consider the singular 2D Roesser model

$$E\begin{bmatrix} x^h_{i+1,j} \\ x^v_{i,j+1} \end{bmatrix} = A\begin{bmatrix} x^h_{ij} \\ x^v_{ij} \end{bmatrix} + Bu_{ij}, \tag{32a}$$

$$y_{ij} = C\begin{bmatrix} x^h_{ij} \\ x^v_{ij} \end{bmatrix} \tag{32b}$$

for $i, j \in \mathbb{Z}_+$, where $x^h_{ij} \in \mathbb{R}^{n_1}$ and $x^v_{ij} \in \mathbb{R}^{n_2}$ are respectively the horizontal and vertical state vectors at the point (i, j), $u_{ij} \in \mathbb{R}^m$ is the input vector, $y_{ij} \in \mathbb{R}^p$ is the output vector and

$$\begin{aligned} &E = \begin{bmatrix} E_1 & E_2 \end{bmatrix}, \quad E_1 = \begin{bmatrix} E_{11} \\ E_{21} \end{bmatrix}, \\ &E_2 = \begin{bmatrix} E_{12} \\ E_{22} \end{bmatrix}, \qquad A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \\ &B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \qquad C = \begin{bmatrix} C_1 & C_2 \end{bmatrix}, \\ &E_{kl} \in \mathbb{R}^{n_k \times n_l}, \quad A_{kl} \in \mathbb{R}^{n_k \times n_l}, \quad B_k \in \mathbb{R}^{n_k \times m}, \\ &C_k \in \mathbb{R}^{p \times n_k}, \quad k, l = 1, 2. \end{aligned} \tag{33}$$

It is assumed that $\det E = 0$ and

$$\det\begin{bmatrix} E_{11}z_1 - A_{11}, & E_{12}z_2 - A_{12} \\ E_{21}z_1 - A_{21}, & E_{22}z_2 - A_{22} \end{bmatrix} \neq 0 \tag{34}$$

for some $z_1, z_2 \in \mathbb{C} \times \mathbb{C}$.

The transfer matrix of the system (32) is given by

$$T(z_1,z_2) = C\begin{bmatrix} E_{11}z_1 - A_{11}, & E_{12}z_2 - A_{12} \\ E_{21}z_1 - A_{21}, & E_{22}z_2 - A_{22} \end{bmatrix}^{-1} B = \frac{\sum_{i=0}^{m_1}\sum_{j=0}^{m_2} b_{ij} z_1^{m_1-i} z_2^{m_2-j}}{\sum_{i=0}^{n_1}\sum_{j=0}^{n_2} -a_{ij} z_1^{n_1-i} z_2^{n_2-j}} \tag{35}$$

with $m_1 \geq n_1,\ \ m_2 \geq n_2$.

Definition 4. The matrices (33) are said to have *canonical form* if $\bar{E}_{12}=0,\ \ \bar{E}_{21}=0$,

$$\bar{E}_{11} = \begin{bmatrix} I_{m_1} & 0 \\ 0 & 0 \end{bmatrix} \in \mathbb{R}_+^{(m_1+1)\times(m_1+1)}, \quad \bar{E}_{22} = I_{2m_2},$$

$$\bar{A}_{11} = \begin{bmatrix} 0 & I_{m_1} \\ 0 & 0 \end{bmatrix} \in \mathbb{R}_+^{(m_1+1)\times(m_1+1)},$$

$$\bar{A}_{12} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \end{bmatrix} \in \mathbb{R}_+^{(m_1+1)\times 2m_2},$$

$$\bar{A}_{21} = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \hdashline 0 & 0 & \cdots & 0 & 0 \\ a_{n_1 0} & a_{n_1-1,0} & \cdots & a_{00} & 0 \\ a_{n_1 1} & a_{n_1-1,1} & \cdots & a_{01} & 0 \\ \hdashline a_{n_1 n_2} & a_{n_1-1,n_2} & \cdots & a_{0n_2} & 0 \\ b_{m_1,1} & b_{m_1-1,1} & \cdots & b_{11} & b_{01} \\ \hdashline b_{m_1,m_2-1} & b_{m_1-1,m_2-1} & \cdots & b_{1,m_2-1} & b_{0,m_2-1} \\ b_{m_1 m_2} & b_{m_1-1,m_2} & \cdots & b_{1m_2} & b_{0m_2} \end{bmatrix}$$

$$\in \mathbb{R}_+^{2m_2\times(m_1+1)},$$

$$\bar{A}_{22} = \left[\begin{array}{cc:cc} 0 & I_{m_2-1} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hdashline 0 & 0 & 0 & I_{m_2-1} \\ 0 & 0 & 0 & 0 \end{array}\right] \in \mathbb{R}_+^{2m_2\times 2m_2},$$

$$\bar{B}_1 = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \in \mathbb{R}^{m_1+1}, \quad \bar{B}_2 = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} \in \mathbb{R}^{2m_2},$$

$$\bar{C}_1 = \begin{bmatrix} b_{m_1 0} & b_{m_1-1,0} & \cdots & b_{00} \end{bmatrix} \in \mathbb{R}^{1\times(m_1+1)},$$

$$\bar{C}_2 = \Big[\underbrace{0\cdots 0\ 1}_{m_2+1}\ 0\cdots 0\Big] \in \mathbb{R}^{1\times 2m_2}. \tag{36}$$

Definition 5. The matrices (36) satisfying (35) for a given $T(z_1,z_2)$ are called a *realisation in canonical form of* $T(z_1,z_2)$.

6. Determination of 2D Realisations in Canonical Forms

Given the improper 2D transfer function

$$T(z_1,z_2) = \frac{\sum_{i=0}^{m_1}\sum_{j=0}^{m_2} b_{ij} z_1^{m_1-i} z_2^{m_2-j}}{\sum_{i=0}^{n_1}\sum_{j=0}^{n_2} -a_{ij} z_1^{n_1-i} z_2^{n_2-j}} \tag{37}$$

of the single-input single-output 2D Roesser model (32) with $m_1 \geq n_1$ and $m_2 \geq n_2$, find a realisation in the canonical form (36) of (37). The transfer function (37) can be written as

$$T(z_1,z_2) = \frac{\sum_{i=0}^{m_1}\sum_{j=0}^{m_2} b_{ij} z_1^{-i} z_2^{-j}}{\sum_{i=0}^{n_1}\sum_{j=0}^{n_2} -a_{ij} z_1^{n_1-m_1-i} z_2^{n_2-m_2-j}} = \frac{\sum_{i=0}^{m_1} b_i z_1^{-i}}{\sum_{i=0}^{n_1} -a_i z_1^{n_1-m_1-i}} \tag{38}$$

for $m_1 \geq n_1$ and $m_2 \geq n_2$, where

$$b_i := \sum_{j=0}^{m_2} b_{ij} z_2^{-j}, \quad a_i := \sum_{j=0}^{n_2} a_{ij} z_2^{-n_2-m_2-j}. \tag{39}$$

Taking into account the fact that

$$T(z_1,z_2) = \frac{Y(z_1,z_2)}{U(z_1,z_2)},$$

where $Y(z_1,z_2) = Y$ and $U(z_1,z_2) = U$ are respectively the 2D z-transforms of $y(i,j)$ and $u(i,j)$ (Kaczorek, 1985) , and defining

$$\bar{E} = \frac{U}{\sum_{i=0}^{n_1} -a_i z_1^{n_1-m_1-i}}, \tag{40}$$

from (38) we obtain

$$Y = \sum_{i=0}^{m_1} b_i z_1^{-i} \bar{E}. \tag{41}$$

From (40) we have

$$U + \sum_{i=0}^{n_1} a_i \bar{E} z_1^{n_1-m_1-i} = 0. \tag{42}$$

From (41) and (42), the block diagram shown in Fig. 2 follows for $m_1 = n_1 + 1$.

$$\begin{aligned} x^v_{m_2-n_2}(i,j+1) &= a_{n_1 0} x^h_1(i,j) + a_{n_1-1,0} x^h_2(i,j) \\ &\quad + \cdots + a_{00} x^h_{m_1}(i,j) \\ &\quad + x^v_{m_2-n_2+1}(i,j), \\ x^v_{m_2-n_2+1}(i,j+1) &= a_{n_1,1} x^h_1(i,j) + a_{n_1-1,1} x^h_2(i,j) \\ &\quad + \cdots + a_{01} x^h_{m_1}(i,j) \\ &\quad + x^v_{m_2-n_2+2}(i,j), \end{aligned}$$

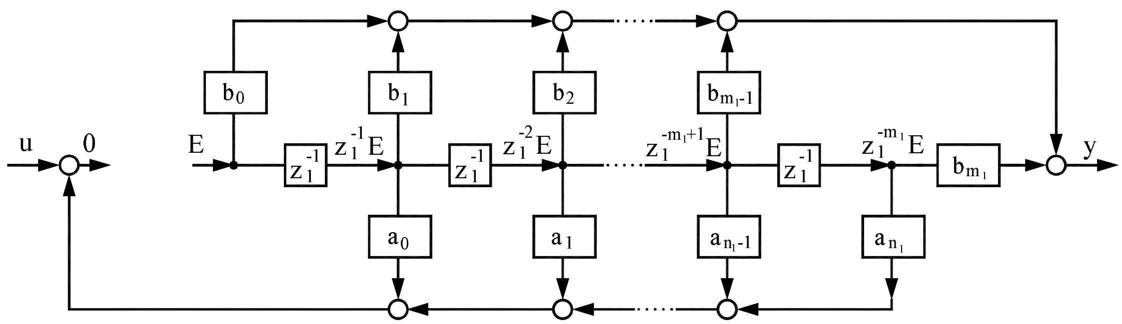

Fig. 2. Block diagram for the transfer function (38).

Note that in addition to m_1 horizontal delay elements (Fig. 2) we need m_2 vertical delay elements to implement the feedback gains $a_i, i = 0, 1, \ldots, n_1$ and m other vertical delay elements to implement the readout gains $b_i, i = 0, 1, \ldots, m_1$. Therefore, the complete block diagram shown in Fig. 2 requires $m_1 + 2m_2$ delay elements (Fig. 3).

As the horizontal state variables $x^h_1(i,j), \ldots, x^h_{m_1}(i,j)$ we choose the output of the horizontal delay elements, and as the vertical state variables $x^v_1(i,j), \ldots, x^v_{2m_2}(i,j)$ we choose the outputs of the vertical delay elements.

Using Fig. 3, we can write the following equations:

$$\begin{aligned} x^h_1(i+1,j) &= x^h_2(i,j), \\ x^h_2(i+1,j) &= x^h_3(i,j), \\ &\vdots \\ x^h_{m_1}(i+1,j) &= x^h_{m_1+1}(i,j), \\ 0 &= x^v_1(i,j) + u(i,j), \\ x^v_1(i,j+1) &= x^v_2(i,j), \\ x^v_2(i,j+1) &= x^v_3(i,j), \\ &\vdots \\ x^v_{m_2-n_2-1}(i,j+1) &= x^v_{m_2-n_2}(i,j), \end{aligned}$$

$$\begin{aligned} &\vdots \\ x^v_{m_2}(i,j+1) &= a_{n_1,n_2} x^h_1(i,j) + a_{n_1-1,n_2} x^h_2(i,j) \\ &\quad + \cdots + a_{0n_2} x^h_{m_1}(i,j), \\ x^v_{m_2+1}(i,j+1) &= b_{m_1,1} x^h_1(i,j) + b_{m_1-1,1} x^h_2(i,j) \\ &\quad + \cdots + b_{21} x^h_{m_1-1}(i,j) + b_{11} x^h_{m_1}(i,j) \\ &\quad + b_{01} x^h_{m_1+1}(i,j) + x^v_{m_2+2}(i,j), \\ &\vdots \\ x^v_{2m_2-1}(i,j+1) &= b_{m_1,m_2-1} x^h_1(i,j) \\ &\quad + b_{m_1-1,m_2-1} x^h_2(i,j) \\ &\quad + \cdots + b_{2,m_2-1} x^h_{m_1-1}(i,j) \\ &\quad + b_{1,m_2-1} x^h_{m_1}(i,j) \\ &\quad + b_{0,m_2-1} x^h_{m_1+1}(i,j) \\ &\quad + x^v_{2m_2}(i,j), \\ x^v_{2m_2}(i,j+1) &= b_{m_1 m_2} x^h_1(i,j) + b_{m_1-1,m_2} x^h_2(i,j) \\ &\quad + \cdots + b_{2m_2} x^h_{m_1-1}(i,j) \\ &\quad + b_{1m_2} x^h_{m_1}(i,j) + b_{0m_2} x^h_{m_1+1}(i,j). \end{aligned} \tag{43}$$

Defining

$$x^h(i,j) = \begin{bmatrix} x^h_1(i,j) & x^h_2(i,j) & \cdots & x^h_{m_1+1}(i,j) \end{bmatrix}^T,$$

$$x^v(i,j) = \begin{bmatrix} x^v_1(i,j) & x^v_2(i,j) & \cdots & x^v_{2m_2}(i,j) \end{bmatrix}^T,$$

from (43) we obtain (32) with matrices E, A, B, and C of the form (36).

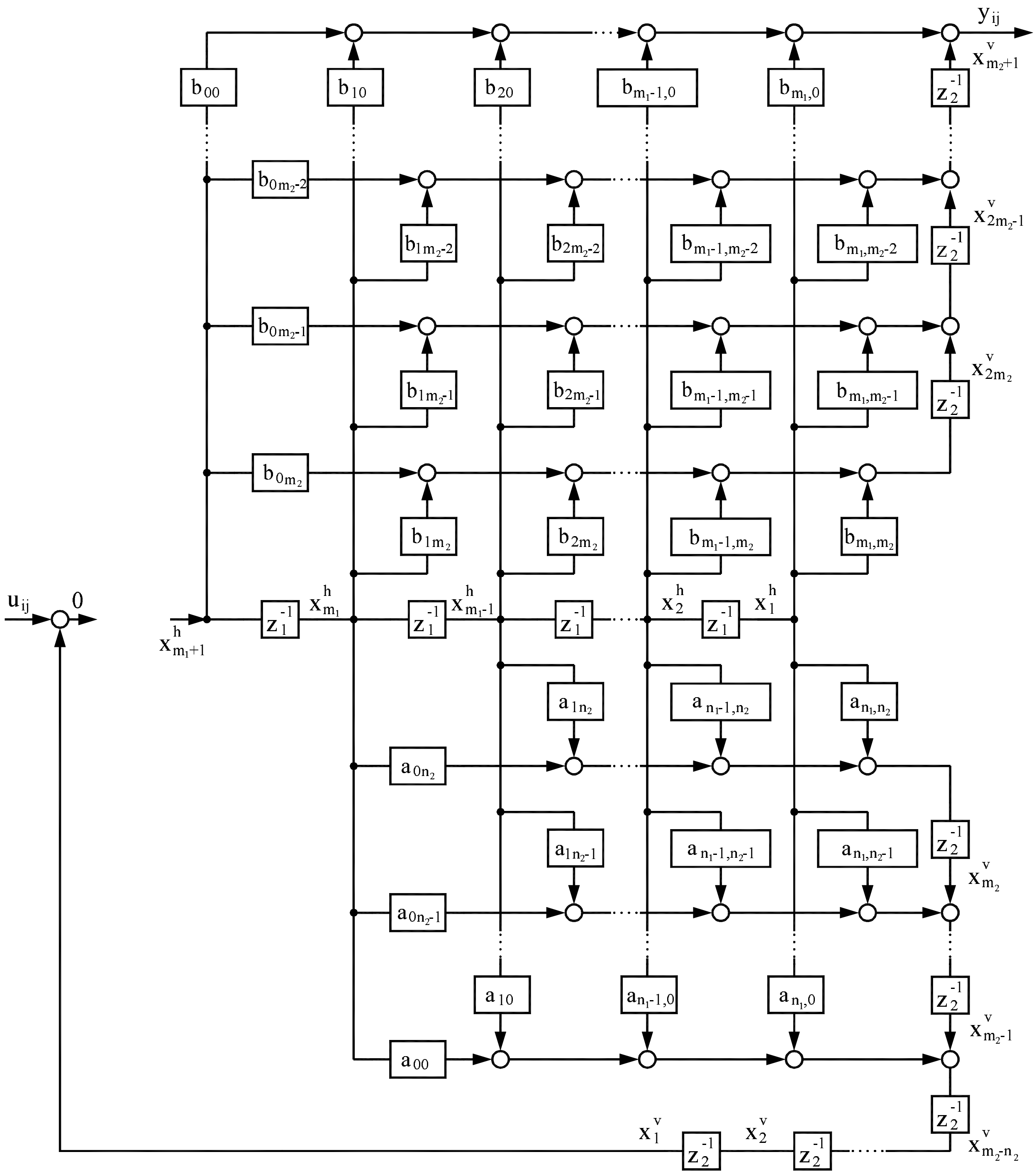

Fig. 3. Complete block diagram for the transfer function (38).

7. Transformation of the Matrices of the Singular Roesser Model to Their Canonical Forms

For the given matrices (33) establish conditions on which they can be transformed to their canonical forms (36), and find nonsingular matrices

$$P = \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix}, \quad Q = \begin{bmatrix} Q_1 & 0 \\ 0 & Q_2 \end{bmatrix}, \qquad (44)$$

$P_k, Q_k \in \mathbb{R}^{n_n \times n_k}$ for $k = 1, 2$, such that the matrices

$$\bar{E} = \begin{bmatrix} \bar{E}_{11} & 0 \\ 0 & \bar{E}_{22} \end{bmatrix} = P \begin{bmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{bmatrix} Q = \begin{bmatrix} P_1E_{11}Q_1 & P_1E_{12}Q_2 \\ P_2E_{21}Q_1 & P_2E_{22}Q_2 \end{bmatrix}$$

$$\bar{A} = \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \\ \bar{A}_{21} & \bar{A}_{22} \end{bmatrix} = P \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} Q = \begin{bmatrix} P_1A_{11}Q_1 & P_1A_{12}Q_2 \\ P_2A_{21}Q_1 & P_2A_{22}Q_2 \end{bmatrix} \qquad (45)$$

$$\bar{B} = \begin{bmatrix} \bar{B}_1 \\ \bar{B}_2 \end{bmatrix} = P \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = \begin{bmatrix} P_1B_1 \\ P_2B_2 \end{bmatrix},$$

$$\bar{C} = \begin{bmatrix} \bar{C}_1 & \bar{C}_2 \end{bmatrix} = \begin{bmatrix} C_1 & C_2 \end{bmatrix} Q = \begin{bmatrix} C_1Q_1 & C_2Q_2 \end{bmatrix}$$

have the canonical forms (36).

Theorem 4. *The matrices (33) can be transformed by the nonsingular matrices (44) to their canonical forms (36) only if*

1. $E_{12} = 0,\ E_{21} = 0,\ \operatorname{rank} E_{11} = m_1,$ $\operatorname{rank} E_{22} = 2m_2.$
2. $\operatorname{rank} A_{11} = m_1,\ \operatorname{rank} A_{12} = 1,$ $\operatorname{rank} A_{22} = 2\,(m_2 - 1),\ B_2 = 0.$

Proof. From (45) we have

$$\bar{E}_{kl} = P_kE_{kl}Q_l, \qquad (46a)$$

$$\bar{A}_{kl} = P_kA_{kl}Q_l, \qquad (46b)$$

$$\bar{B}_k = P_kB_k, \quad \bar{C}_k = C_kQ_k \qquad (46c)$$

for $k, l = 1, 2$. From (46a) it follows that $\bar{E}_{12} = P_1E_{12}Q_2 = 0$, $\bar{E}_{21} = P_2E_{21}Q_1 = 0$ and $E_{12} = 0$, $E_{21} = 0$ since det $P_k \neq 0$ and det $Q_k \neq 0$ for $k = 1, 2$.

Using (46a) and (36), we obtain $\operatorname{rank} E_{11} = \operatorname{rank} P_1E_{11}Q_1 = \operatorname{rank} \bar{E}_{11} = m_1$, $\operatorname{rank} E_{22} = \operatorname{rank} P_2E_{22}Q_2 = \operatorname{rank} \bar{E}_{22} = 2m_2$. In a similar manner, using (46b), (46c) and (36), we can prove the necessity of the conditions of Part 2. ■

If (34) holds, then

$$\begin{bmatrix} E_{11}z_1 - A_{11} & E_{12}z_2 - A_{12} \\ E_{21}z_1 - A_{21} & E_{22}z_2 - A_{22} \end{bmatrix}^{-1} = \sum_{i=-\mu_1}^{\infty} \sum_{j=-\mu_2}^{\infty} T_{ij} z_1^{-(i+1)} z_2^{-(j+1)}, \qquad (47)$$

where the pair (μ_1, μ_2) is the nilpotence index and the T_{ij}'s are the transition matrices defined by

$$\begin{bmatrix} E_1 & 0 \end{bmatrix} T_{i,j-1} + \begin{bmatrix} 0 & E_2 \end{bmatrix} T_{i-1,j} - AT_{i-1,j-1} = \begin{cases} I_n & \text{for } i = j = 0, \\ 0 & \text{for } i \neq 0 \text{ and/or } j \neq 0, \end{cases} \qquad (48)$$

and $T_{ij} = 0$ for $i < -\mu_1$ and/or $j < -\mu_2$.

Let

$$\begin{bmatrix} \bar{E}_{11}z_1 - \bar{A}_{11} & -\bar{A}_{12} \\ -\bar{A}_{21} & \bar{E}_{22}z_2 - \bar{A}_{22} \end{bmatrix}^{-1} = \sum_{i=-\mu_1}^{\infty} \sum_{j=-\mu_2}^{\infty} \bar{T}_{ij} z_1^{-(i+1)} z_2^{-(j+1)}. \qquad (49)$$

Then from (46), (47) and (49) we have

$$T_{ij} = Q\bar{T}_{ij}P \text{ for } i, j \in \mathbb{Z}_+. \qquad (50)$$

The solution x_{ij} of (32a) with the boundary conditions

$$x_{0j}^h, x_{i0}^v \quad \text{for} \quad 0 \le j \le n_2 + \mu_2 - 1 \quad \text{and} \quad 0 \le i \le n_1 + \mu_1 - 1 \qquad (51)$$

is given by

$$x_{ij} = \begin{bmatrix} x_{ij}^h \\ x_{ij}^v \end{bmatrix} = \sum_{k=0}^{i+\mu_1-1} \sum_{l=0}^{j+\mu_2-1} T_{i-k-1,j-l-1}Bu_{kl} + \sum_{l=0}^{j+\mu_2-1} T_{i,j-l-1}E_1x_{0l}^h + \sum_{k=0}^{i+\mu_1-1} T_{i-k-1,j}E_2x_{k0}^v. \qquad (52)$$

Theorem 5. *Let the matrices (33) satisfy the assumption (34) and the conditions of Theorem 4. Then there exist* *nonsingular matrices in (44) such that the matrices (46) have the canonical forms (36) if*

$$\operatorname{rank} R_{n_1n_2} = n \tag{53}$$

and

$$\operatorname{rank} O_{n_1n_2} = n, \tag{54}$$

where

$$R_{n_1n_2} := \big[T_{n_1-1,n_2-1}B, \dots, T_{00}B, T_{-1,0}B, T_{0,-1}B, \dots, T_{-\mu_1,-\mu_2}B\big] \tag{55}$$

and

$$O_{n_1n_2} := \begin{bmatrix} CT_{-\mu_1,-\mu_2} \\ \vdots \\ CT_{00} \\ CT_{-1,0} \\ CT_{0,-1} \\ \vdots \\ CT_{n_1-1,n_2-1} \end{bmatrix}. \tag{56}$$

Proof. From (46), (50) and (55) we have

$$\begin{aligned} R_{n_1n_2} &:= \big[T_{n_1-1,n_2-1}B, \dots, T_{00}B, T_{-1,0}B, T_{0,-1}B, \dots, T_{-\mu_1,-\mu_2}B\big] \\ &= Q\big[\bar{T}_{n_1-1,n_2-1}\bar{B}, \dots, \bar{T}_{00}\bar{B}, \bar{T}_{-1,0}\bar{B}, \bar{T}_{0,-1}\bar{B}, \dots, \bar{T}_{-\mu_1,-\mu_2}\bar{B}\big] \\ &= Q\bar{R}_{n_1n_2}, \end{aligned} \tag{57}$$

where

$$\bar{R}_{n_1n_2} := \big[\bar{T}_{n_1-1,n_2-1}\bar{B}, \dots, \bar{T}_{00}\bar{B}, \bar{T}_{-1,0}\bar{B}, \bar{T}_{0,-1}\bar{B}, \dots, \bar{T}_{-\mu_1,-\mu_2}\bar{B}\big]. \tag{58}$$

If the condition (53) is satisfied, then from (57) we obtain

$$Q = R_n\bar{R}_n^{-1}, \tag{59}$$

where R_n and $\bar{R}_n$ are square matrices consisting of n linearly independent corresponding columns of the matrices $R_{n_1n_2}$ and $\bar{R}_{n_1n_2}$, respectively.

Similarly, from (46), (50) and (56) we have

$$O_{n_1n_2} = \begin{bmatrix} CT_{-\mu_1,-\mu_2} \\ \vdots \\ CT_{00} \\ CT_{-1,0} \\ CT_{0,-1} \\ \vdots \\ CT_{n_1-1,n_2-1} \end{bmatrix} = \begin{bmatrix} \bar{C}\bar{T}_{-\mu_1,-\mu_2} \\ \vdots \\ \bar{C}\bar{T}_{00} \\ \bar{C}\bar{T}_{-1,0} \\ \bar{C}\bar{T}_{0,-1} \\ \vdots \\ \bar{C}\bar{T}_{n_1-1,n_2-1} \end{bmatrix} P = \bar{O}_{n_1n_2}P, \tag{60}$$

where

$$\bar{O}_{n_1n_2} = \begin{bmatrix} \bar{C}\bar{T}_{-\mu_1,-\mu_2} \\ \vdots \\ \bar{C}\bar{T}_{00} \\ \bar{C}\bar{T}_{-1,0} \\ \bar{C}\bar{T}_{0,-1} \\ \vdots \\ \bar{C}\bar{T}_{n_1-1,n_2-1} \end{bmatrix}. \tag{61}$$

If the condition (54) is satisfied, then from (60) we obtain

$$P = \bar{O}_n^{-1}O_n, \tag{62}$$

where O_n and $\bar{O}_n$ are square matrices consisting of n linearly independent corresponding rows of the matrices $O_{n_1n_2}$ and $\bar{O}_{n_1n_2}$, respectively. ∎

Matrices Q and P can be found using the following procedure:

Procedure 2.

Step 1. Knowing E, A, B and C, find the transfer matrix (35).

Step 2. Using the procedure presented in Section 6, find the realization of the transfer matrix in the canonical form (36) .

Step 3. Using (47) and (49), determine the fundamental matrices T_{ij} and $\bar{T}_{ij}$ for $i = -\mu_1, \dots, n_1+1$ and $j = -\mu_2, \dots, n_2+1$.

Step 4. Using (55), (58) and (56), (61), find R_n, $\bar{R}_n$, O_n and $\bar{O}_n$.

Step 5. Using (59) and (62), find the desired matrices Q and P.

8. Concluding Remarks

In the first part of the paper the new canonical forms (5) for multi-input multi-output linear time-invariant systems were introduced. A method of determining realisations of a given 1D transfer function in canonical forms was proposed. Sufficient conditions for the existence of canonical forms for singular linear systems were established (Theorem 3). A procedure for computing a pair of nonsingular matrices P, Q transforming the matrices of singular systems to their canonical forms (5) was presented. The considerations for discrete-time linear systems are also valid for continuous-time linear systems. In the second part, new canonical forms of the matrices of the singular 2D Roesser model were introduced. A method of determining realisations of a given 2D transfer function in canonical forms was proposed. Necessary and sufficient conditions for the existence of a pair of nonsingular block diagonal matrices transforming the matrices of the singular 2D Roesser model to their canonical forms were established. A procedure for computing the pair of nonsingular matrices was presented. The considerations presented for the single-input single-output singular 2D Roesser model can be easily extended to the multi-input multi-output singular 2D Roesser model. An extension for the singular 2D Fornasini-Marchesini-type models (1976; 1978; Kaczorek, 1992) is also possible.

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