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TOWARDS A FRAMEWORK FOR CONTINUOUS AND DISCRETE MULTIDIMENSIONAL SYSTEMS

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Continuous multidimensional systems described by partial differential equations can be represented by discrete systems in a number of ways. However, the relations between the various forms of continuous, semi-continuous, and discrete multidimensional systems do not fit into an established framework like in the case of one-dimensional systems. This paper contributes to the development of such a framework in the case of multidimensional systems. First, different forms of partial differential equations of physics-based systems are presented. Secondly, it is shown how the different forms of continuous multidimensional systems lead to certain discrete models in current use (finite-difference models, multidimensional wave digital filters, transfer function models). The links between these discrete models are established on the basis of the respective continuous descriptions. The presentation is based on three examples of physical systems (heat flow, transmission of electrical signals, acoustic wave propagation).

Keywords: multidimensional systems, partial differential equations, transfer functions, wave digital models

1. Introduction

This paper deals with a fundamental problem in the theory of linear systems: Consider the description of a continuous system given in terms of the basic laws of science. A discrete system approximating the continuous system in a certain sense is sought. This problem arises typically when emulating the behaviour of a real-world system on a digital computer.

For one-dimensional systems, this problem has been extensively studied in various fields, such as control theory, signal processing, and system simulation. Especially for linear and time-invariant systems, a number of different models have been developed. They can be grouped into ordinary differential equation models, transfer function models and graphical descriptions:

- Ordinary differential equations

  Ordinary differential equation models relate the input quantity and a number of their derivatives to the derivatives of the output quantity. The scalar differential equation of such an input-output model can be converted to a first-order equation for a vector of internal variables, the so-called state-space description.

- Transfer functions

  The application of suitable functional transformations for the independent variable (often the time variable) leads to a transfer function description. It is much easier to use than differential equations, since all derivatives are turned into algebraic operations. Widely used functional transformations are the Laplace transformation for initial-value problems and the Fourier transformation for frequency response characterizations.

- Graphical descriptions

  Graphical descriptions are a valuable tool for system modelling, e.g. electrical or duct flow networks or bond graphs. Other graphical descriptions are derived from differential equation models or transfer functions such as signal flow graphs or pole-zero diagrams.

The popularity of all the three families of system models can be attributed to the following reasons:

- These models can be generally applied, since they are independent of the physical nature of the system.

- The links between these models are well established. An experienced user can gain considerable insight into the system behaviour by switching between these alternative descriptions.

- These models for continuous systems are suitable for the derivation of the corresponding discrete models.

The discretization of continuous systems is of special interest in this context. Methods of the conversion of a continuous system into a discrete one include time-domain methods (e.g. impulse-invariant transformation, numerical integration) or frequency-domain methods (e.g. s-to-z plane mappings). In many cases the relations between time-domain and frequency-domain methods are also established (e.g. trapezoidal rule and bilinear transformation). For a survey of these methods see, e.g. (Kowalczuk, 1993). Furthermore, there also exist discretization methods which preserve not only the input-output behaviour of a system, but also internal properties such as passivity (wave digital filters, cf. (Fettweis, 1986)).

For multidimensional (MD) systems the situation has not reached this state of maturity yet. While partial differential equations (PDEs) are a widely applied mathematical model for continuous MD systems, transfer function models or graphical descriptions are not commonly used. An example of a graphical description of continuous MD systems are the MD Kirchhoff circuits presented in (Fettweis and Nitsche, 1991). Transfer functions models for MD systems are described in (Rabenstein, 1998).

Furthermore, there are no general continuous-to-discrete transformations for MD systems. Discretization methods for PDEs from numerical mathematics do not attempt at modelling a discrete system. They rather set up a system of algebraic equations, which can be solved for the solution of the PDE. Other methods are by its nature mixed continuous-discrete (e.g. a linear repetitive process model (Rogers *et al.*, 1997)) or fully discrete (e.g. the Roesser model (Roesser, 1975)). The existence and uniqueness of solutions of partial difference equations, also with respect to boundary conditions, were discussed, e.g., in (Gregor, 1998; Veit, 1996).

The existing framework for the design of discrete MD systems from continuous models is the MD wave digital principle (Fettweis, 1994), although the treatment of initial and boundary conditions is still a subject of research (Fettweis, 1999). Another approach, i.e. the functional transformation method, considers initial and boundary conditions, but it requires the knowledge of eigenfunctions and eigenvalues of the problem (Rabenstein, 1998).

Unfortunately, there is no established framework which links continuous, semi-discrete, and fully discrete models of various kinds and which would allow us to derive one from another. This paper attempts at making a contribution to such a framework. It shows how certain discrete MD models in current use can be derived from a PDE description of the underlying continuous system.

We consider MD systems with a continuous-time, continuous-space description derived from the first principles of physics. The restriction to time and space as independent variables has been made for the clarity of our presentation and does not limit the general applicability. Three examples of such systems are introduced in Section 2. Section 3 presents a unified description of models for different physical phenomena. Section 4 shows that the widespread scalar PDE respresentation can be obtained from this unified description by matrix operations. The links between these various forms of continuous PDE models and the corresponding discrete models are established in Section 5.

2. Physical Models

Throughout the paper, we use three different physical phenomena as examples for the general form of a PDE introduced below. These examples are taken from the fields of heat transfer, electromagnetics and acoustics. The common analysis of these problems consists in setting up relations between a scalar potential quantity (temperature, voltage, pressure) and a vector valued flux quantity (heat flux, current, velocity). These relations involve an energy balance and a potential-flux relationship. They are given below for each of the three examples, using the following notation: time coordinate t, scalar or vector valued space coordinate x or $\mathbf{x}$, potential quantity $u(x,t)$, scalar or vector valued flux quantity $i(x,t)$ or $\mathbf{i}(\mathbf{x},t)$. Time and space derivatives are denoted by $D_t = \partial/\partial t$, $D_x = \partial/\partial x$, grad, and div.

Example 1. The simplest description of the heat flow through a wall is obtained by considering only components in one spatial direction (x). The conservation of energy requires a balance between the temporal changes in the temperature $u(x,t)$ and the scalar heat flux $i(x,t)$:

$$cD_t u(x,t) + D_x i(x,t) = 0, \tag{1}$$

with the heat capacity c. The heat flow in solids is governed by Fourier's law:

$$D_x u(x,t) + r i(x,t) = 0, \tag{2}$$

where r is the inverse of thermal conductivity.

Example 2. A standard model for electrical transmission lines is written in terms of the current $i(x,t)$ and the voltage $u(x,t)$. For long and thin lines, the space variable x is scalar. The electrical parameters are given by the series inductance l, shunt capacitance c, series resistance r, and shunt conductance g. For an infinitesimal section of the transmission line, the voltage drops sum up to zero:

$$lD_t i(x,t) + r i(x,t) + D_x u(x,t) = 0, \tag{3}$$

and the sum of all currents is zero:

$$cD_t u(x,t) + g u(x,t) + D_x i(x,t) = 0. \tag{4}$$

Example 3. With reasonable simplifications, the propagation of sound waves in the air is governed by the equation of motion:

$$\rho_0 \frac{\partial}{\partial t}\mathbf{i}(\mathbf{x},t) + \operatorname{grad} u(\mathbf{x},t) = \mathbf{0}, \tag{5}$$

and the equation of continuity:

$$\frac{1}{\rho_0\, c^2}\frac{\partial}{\partial t}\, u(\mathbf{x},t) + \operatorname{div}\mathbf{i}(\mathbf{x},t) = 0 \tag{6}$$

for the sound pressure u and the particle velocity $\mathbf{i}$. Here $\mathbf{x}$ is the vector of space coordinates $(x,\ y,\ z)$, ρ_0 is the static density of the air, and c is the sound speed.

3. Vector Partial Differential Equations

The foregoing examples show that physical phenomena involving potential and flux quantitites are described by a pair of coupled PDEs. In this section, we combine these PDEs to different forms of one single vector equation, the so-called vector PDE. The first form uses the potential and the flux quantities as unknowns and is called the potential-flux model. The second form applies a matrix operation to normalize the unknowns into a uniform physical dimension. The result is a normalized vector model. In Section 5 we will see that these different forms of a vector PDE give rise to different discretization approaches.

3.1. Potential-Flux Models

Combining a pair of coupled PDEs into a vector PDE leads to the general matrix equation

$$[\mathbf{C}D_t + \mathbf{L}]\,\mathbf{y}(\mathbf{x},t) = \mathbf{0}. \tag{7}$$

$\mathbf{C}$ is a diagonal mass or capacitance matrix and $\mathbf{L}$ is a matrix operator which contains loss terms and spatial derivative operators. The vector of dependent variables $\mathbf{y}(\mathbf{x},t)$ contains potential and flux quantities. A number of technically important PDEs have an operator $\mathbf{L}$ of the form

$$\mathbf{L} = \mathbf{A} + \mathbf{B}\nabla, \tag{8}$$

where $\mathbf{A}$ is diagonal,

$$\mathbf{B} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \mathbf{B}^2 = \mathbf{I}, \quad \mathbf{y}(\mathbf{x},t) = \begin{bmatrix} \mathbf{i}(\mathbf{x},t) \\ u(\mathbf{x},t) \end{bmatrix}. \tag{9}$$

The spatial derivative operator ∇ represents a partial derivative with respect to one space coordinate, a gradient or a divergence operation, depending on the number of spatial dimensions and on whether it is applied to the potential or the flux quantity. All matrix operations for the differential operators have to be performed according to the rules of vector analysis for the operator ∇.

The matrices $\mathbf{A}$, $\mathbf{C}$, and $\mathbf{L}$ for Examples 1–3 are listed below. The equivalence of the resulting vector PDE with the corresponding physical models is verified by inserting (10)–(12) into (7):

Example 1 (continued).

$$\mathbf{A} = \begin{bmatrix} r & 0 \\ 0 & 0 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 0 & 0 \\ 0 & c \end{bmatrix},$$
$$\mathbf{L} = \begin{bmatrix} r & D_x \\ D_x & 0 \end{bmatrix}, \quad \mathbf{y}(x,t) = \begin{bmatrix} i(x,t) \\ u(x,t) \end{bmatrix}. \tag{10}$$

Example 2 (continued).

$$\mathbf{A} = \begin{bmatrix} r & 0 \\ 0 & g \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} l & 0 \\ 0 & c \end{bmatrix},$$
$$\mathbf{L} = \begin{bmatrix} r & D_x \\ D_x & g \end{bmatrix}, \quad \mathbf{y}(x,t) = \begin{bmatrix} i(x,t) \\ u(x,t) \end{bmatrix}. \tag{11}$$

Example 3 (continued).

$$\mathbf{A} = \mathbf{0}, \quad \mathbf{C} = \begin{bmatrix} \rho_0 & 0 \\ 0 & \dfrac{1}{\rho_0\, c^2} \end{bmatrix},$$
$$\mathbf{L} = \begin{bmatrix} 0 & \operatorname{grad} \\ \operatorname{div} & 0 \end{bmatrix}, \quad \mathbf{y}(\mathbf{x},t) = \begin{bmatrix} \mathbf{i}(\mathbf{x},t) \\ u(\mathbf{x},t) \end{bmatrix}. \tag{12}$$

3.2. Normalized Vector Models

The vector of unknowns $\mathbf{y}$ in (7) directly represents the variables of the physical model. However, although physically meaningful, the simultaneous treatment of two dependent variables of different physical natures may be more complex than necessary for model analysis, discretization and algorithm design. To avoid the parallel treatment of flux and potential quantities, these variables can be normalized to the same physical dimension or to a dimensionless representation.

Such a normalization is described by a diagonal matrix $\mathbf{N}$ with suitable reference quantities on the main diagonal. The result is a new vector $\mathbf{y}_n$ of dependent variables:

$$\mathbf{y}_n(\mathbf{x},t) = \mathbf{N}^{-1}\mathbf{y}(\mathbf{x},t), \tag{13}$$

where the elements of $\mathbf{y}_n$ have the same dimension or no dimension at all. Applying the normalization matrix $\mathbf{N}$

also to the differential operator $\mathbf{C}D_t + \mathbf{L}$ in (7) leads to the normalized representation

$$[\mathbf{C}_n D_t + \mathbf{L}_n]\, \mathbf{y}_n(\mathbf{x},t) = \mathbf{0} \tag{14}$$

with $\mathbf{C}_n = \mathbf{NCN}$ and similar for the other matrices. By a suitable choice of the normalizing constants in $\mathbf{N}$ it is possible to obtain a symmetric representation of the matrix of time and space derivatives $[\mathbf{C}_n D_t + \mathbf{B}_n \nabla]$ in (14).

Example 1 (continued). To normalize the dimension of $\mathbf{y}(x,t)$ to unity, we choose a reference heat flux i_0 and a reference temperature u_0, and obtain

$$\mathbf{N} = \begin{bmatrix} i_0 & 0 \\ 0 & u_0 \end{bmatrix}, \quad \mathbf{y}_n(x,t) = \begin{bmatrix} i(x,t)/i_0 \\ u(x,t)/u_0 \end{bmatrix}. \tag{15}$$

After division by $u_0 i_0$, the following normalized vector model is derived:

$$\begin{bmatrix} r/r_0 & D_x \\ D_x & r_0 c\, D_t \end{bmatrix} \begin{bmatrix} i(x,t)/i_0 \\ u(x,t)/u_0 \end{bmatrix} = \mathbf{0} \tag{16}$$

with $r_0 = u_0/i_0$.

Example 2 (continued). To normalize the vector $\mathbf{y}(x,t)$ such that all of its elements are currents, we choose a reference resistance r_0 and obtain

$$\mathbf{N} = \begin{bmatrix} 1 & 0 \\ 0 & r_0 \end{bmatrix}, \quad \mathbf{y}_n(x,t) = \begin{bmatrix} i(x,t) \\ u(x,t)/r_0 \end{bmatrix}, \tag{17}$$

$$\mathbf{C}_n = \begin{bmatrix} l & 0 \\ 0 & r_0^2 c \end{bmatrix}, \quad \mathbf{L}_n = \begin{bmatrix} r & r_0 D_x \\ r_0 D_x & r_0^2 g \end{bmatrix}. \tag{18}$$

For the special choice $r_0^2 = l/c$ we obtain a symmetric representation

$$[\mathbf{C}_n D_t + \mathbf{B}_n \nabla] = \begin{bmatrix} l D_t & r_0 D_x \\ r_0 D_x & l D_t \end{bmatrix}. \tag{19}$$

Example 3 (continued). We apply a normalization similar to that of the previous example also to the potential-flux model of the acoustic wave equation in (12):

$$\mathbf{N} = \begin{bmatrix} 1 & 0 \\ 0 & r_0 \end{bmatrix}, \quad \mathbf{y}_n(x,t) = \begin{bmatrix} \mathbf{i}(\mathbf{x},t) \\ u(\mathbf{x},t)/r_0 \end{bmatrix}, \tag{20}$$

$$\mathbf{C}_n = \begin{bmatrix} \rho_0 & 0 \\ 0 & \dfrac{r_0^2}{\rho_0 c^2} \end{bmatrix}, \quad \mathbf{L}_n = \begin{bmatrix} 0 & r_0 \text{ grad} \\ r_0 \text{ div} & 0 \end{bmatrix}. \tag{21}$$

Again, $\mathbf{C}_n$ can be simplified by a special choice of the normalizing constant:

$$r_0 = \sqrt{3}\,\rho_0 c, \qquad \mathbf{C}_n = \begin{bmatrix} \rho_0 & 0 \\ 0 & 3\rho_0 \end{bmatrix}. \tag{22}$$

Now the normalized vector model for the acoustic wave equation has the form

$$\begin{bmatrix} \rho_0 D_t & r_0 \text{ grad} \\ r_0 \text{ div} & 3\rho_0 D_t \end{bmatrix} \begin{bmatrix} \mathbf{i}(\mathbf{x},t) \\ u(\mathbf{x},t)/r_0 \end{bmatrix} = \mathbf{0}. \tag{23}$$

The symmetric form of this vector PDE is even more obvious when the operators for the gradient and the divergence are broken down into their components:

$$\begin{bmatrix} \rho_0 D_t & 0 & 0 & r_0 D_x \\ 0 & \rho_0 D_t & 0 & r_0 D_y \\ 0 & 0 & \rho_0 D_t & r_0 D_z \\ r_0 D_x & r_0 D_y & r_0 D_z & 3\rho_0 D_t \end{bmatrix} \begin{bmatrix} i_x(\mathbf{x},t) \\ i_y(\mathbf{x},t) \\ i_z(\mathbf{x},t) \\ \dfrac{u(\mathbf{x},t)}{r_0} \end{bmatrix} = \mathbf{0}. \tag{24}$$

4. Scalar Partial Differential Equations

The vector PDEs considered above reflect the physical nature of the relevant continuous MD system, since they contain both potential and flux quantities as dependent variables. If only an input-output model is required, one of the two dependent variables can be eliminated from (7). It is instructive to perform the elimination of either the flux or potential simultaneously. Expressing this elimination process as matrix diagonalization will provide insight into the nature of the differential operator $\mathbf{L}$.

The key to the diagonalization of (7) is the definition of another differential operator $\tilde{\mathbf{L}}$, which is closely associated with $\mathbf{L}$ from (8). The most general definition of $\tilde{\mathbf{L}}$ is given in terms of the Hermitian matrices $\mathbf{A}^H$ and $\mathbf{B}^H$:

$$\tilde{\mathbf{L}} = \mathbf{A}^H - (\mathbf{B}\nabla)^H. \tag{25}$$

We make use of this general definition when the properties of $\tilde{\mathbf{L}}$ are further explored in Section 5.2. For the real and symmetric matrices in the examples below, $\tilde{\mathbf{L}}$ has the form

$$\tilde{\mathbf{L}} = \mathbf{A} - \mathbf{B}\nabla. \tag{26}$$

We show now that the operator $\tilde{\mathbf{L}}$ plays the key role in a very general description of the elimination process mentioned above. To this end, we attempt to diagonalize the differential operator $[\mathbf{C}D_t + \mathbf{L}]$ in (7) by left multiplication with $\mathbf{B}[\mathbf{C}D_t + \tilde{\mathbf{L}}]\mathbf{B}$.

Some matrix algebra gives

$$\mathbf{B}[\mathbf{C}D_t + \tilde{\mathbf{L}}]\mathbf{B}[\mathbf{C}D_t + \mathbf{L}] = \mathbf{M}^2 - \mathbf{I}\nabla^2, \tag{27}$$

where $\mathbf{I}$ is the identity matrix and

$$\mathbf{M} = \mathbf{B}[\mathbf{C}D_t + \mathbf{A}]. \tag{28}$$

If the matrices $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ fulfil the properties given in Section 3.1 ($\mathbf{A}$ and $\mathbf{C}$ diagonal, $\mathbf{B} = \mathbf{I}^2$) then $\mathbf{M}^2$ is diagonal. Thus

$$\left[\mathbf{M}^2 - \mathbf{I}\nabla^2\right] \mathbf{y}(\mathbf{x}, t) = \mathbf{0} \tag{29}$$

is a set of two decoupled PDEs for the flux and for the potential quantity contained in $\mathbf{y}(\mathbf{x}, t)$. This process is shown in detail for the examples considered above.

Example 1 (continued). For $\mathbf{A}$ and $\mathbf{C}$ from (10), $\tilde{\mathbf{L}}$ results in

$$\tilde{\mathbf{L}} = \begin{bmatrix} r & -D_x \\ -D_x & 0 \end{bmatrix}, \tag{30}$$

and $[\mathbf{C}D_t + \tilde{\mathbf{L}}]$ from (27):

$$[\mathbf{C}D_t + \tilde{\mathbf{L}}] = \begin{bmatrix} r & -D_x \\ -D_x & c\,D_t \end{bmatrix}. \tag{31}$$

The matrix M from (28) and its square are respectively given by

$$\mathbf{M} = \begin{bmatrix} 0 & c\,D_t \\ r & 0 \end{bmatrix}, \qquad \mathbf{M}^2 = \mathbf{I}\,rc\,D_t\,. \tag{32}$$

Obviously, $\mathbf{M}^2$ is a diagonal matrix. Then from (27) we obtain

$$\mathbf{M}^2 - \mathbf{I}\nabla^2 = \mathbf{I}\left[rc\,D_t - D_{xx}\right]. \tag{33}$$

Inserting (33) into (29) results in a pair of decoupled equations for the heat flux $i(x, t)$ and the temperature $u(x, t)$, respectively,

$$rcD_t i(x,t) - D_{xx} i(x,t) = 0, \tag{34}$$

$$rcD_t u(x,t) - D_{xx} u(x,t) = 0. \tag{35}$$

Equation (35) is known as the *heat flow equation*.

Example 2 (continued). For $\tilde{\mathbf{L}}$ given by the matrices from (11),

$$\tilde{\mathbf{L}} = \begin{bmatrix} r & -D_x \\ -D_x & g \end{bmatrix}, \tag{36}$$

from (27) it follows that

$$\mathbf{B}[\mathbf{C}D_t + \tilde{\mathbf{L}}]\mathbf{B}[\mathbf{C}D_t + \mathbf{L}] = \mathbf{I}\left[lc\,D_{tt} + (rc + gl)\,D_t + rg - D_{xx}\right]. \tag{37}$$

Thus (29) results in the well-known *telegraph equation* for either the current $i(x, t)$ or the voltage $u(x, t)$:

$$lc\,D_{tt}\,i + (rc + gl)\,D_t\,i + rg\,i - D_{xx}\,i = 0, \tag{38}$$

$$lc\,D_{tt}\,u + (rc + gl)\,D_t\,u + rg\,u - D_{xx}\,u = 0. \tag{39}$$

Example 3 (continued). The operator $\tilde{\mathbf{L}}$ associated with $\mathbf{L}$ from (12) is given by $\tilde{\mathbf{L}} = -\mathbf{B}\nabla$. From (27) it follows that

$$\mathbf{B}[\mathbf{C}D_t + \tilde{\mathbf{L}}]\mathbf{B}[\mathbf{C}D_t + \mathbf{L}] = \mathbf{I}\left[\frac{1}{c^2}\,D_{tt} - \nabla^2\right]. \tag{40}$$

Thus (29) is reduced to two separate PDEs for the velocity $\mathbf{i}(\mathbf{x}, t)$ and the pressure $u(\mathbf{x}, t)$:

$$\frac{1}{c^2}\,D_{tt}\,\mathbf{i}(\mathbf{x},t) - \mathrm{grad}\,\mathrm{div}\,\mathbf{i}(\mathbf{x},t) = \mathbf{0}, \tag{41}$$

$$\frac{1}{c^2}\,D_{tt}\,u(\mathbf{x},t) - \mathrm{div}\,\mathrm{grad}\,u(\mathbf{x},t) = 0. \tag{42}$$

Equation (42) is the *acoustic wave equation* for the sound pressure u.

5. Discrete Multidimensional Models

So far, we have considered representations of the continuous physical models of Section 2. They were vector PDEs in the potential-flux or normalized form, and the more familiar scalar PDEs. Each of these representations is the starting point for a different discretization method in current use.

The most widespread approach is the finite-difference discretization, which is suitable for the scalar PDE model from Section 4. Another discretization method based on transfer function models for MD systems (Rabenstein and Trautmann, 1999) is obtained for the matrix operator $\tilde{\mathbf{L}}$ introduced in (25) for the diagonalization of the vector model. Finally, the normalized vector models lead directly to an MD network model used in the MD wave digital principle (Fettweis, 1994). The connections of the different matrix operator representations with these discretization schemes are presented in this section.

5.1. Finite-Difference Models

The derivation of discrete finite-difference models starts from PDEs in their simplest form, such as (34), (35), or (38), (39), or (41), (42). The discrete models are derived through the replacement of the differential operators with respect to time and space by suitable difference operators. The choice of these operators, their application, and the properties and problems of the resulting finite-difference models are described in standard texts (Tveito and Winther, 1998). As an example, we consider here only the acoustic wave equation from Example 3.

Example 3 (continued). Writing the acoustic wave equation from (42) in Cartesian coordinates gives (arguments $\mathbf{x}$ and t are omitted):

$$\frac{1}{c^2}\,D_{tt}\,u - \left(D_{xx}\,u + D_{yy}\,u + D_{zz}\,u\right) = 0. \tag{43}$$

The sound pressure $u(\mathbf{x}, t)$ is represented at the equidistant points $t = kT$ in time and at the spatial grid $x = lh$, $y = mh$, $z = nh$. T and h are the temporal and spatial step sizes, and k and l, m, n are the discrete time and space coordinates, respectively.

The simplest approximation for the second-order time derivative on this grid is the three-point difference operator

$$D_{tt}\, u \approx \frac{1}{T^2}\Big[u(lh, mh, nh, k(T-1)) - 2u(lh, mh, nh, kT) + u(lh, mh, nh, k(T+1))\Big]. \qquad (44)$$

With a similar approximation for the spatial derivatives, we arrive at a fully discrete approximation of the acoustic wave equation. Again, we note only its simplest form, which is achieved for a special relation between the temporal and spatial step sizes $h = \sqrt{3}\, cT$. Using the abbreviated notation $u(l, m, n, k) = u(lh, mh, nh, kT)$, we obtain

$$\begin{aligned} u(l, m, n, k+1) &= -u(l, m, n, k-1) \\ &+ \frac{1}{3}\Big[u(l+1, m, n, k) + u(l-1, m, n, k) \\ &+ u(l, m+1, n, k) + u(l, m-1, n, k) \\ &+ u(l, m, n+1, k) + u(l, m, n-1, k)\Big]. \end{aligned} \qquad (45)$$

A simpler notation results from using $u(k) = u(l, m, n, k)$:

$$u(k+1) = \frac{1}{3}\sum_{\nu} u_\nu(k) - u(k-1). \qquad (46)$$

Here $u_\nu(k)$ denotes the six-neighbour points (black) of the centre point (grey) shown in Fig. 1. The evaluation of this model computes $u(l, m, n, k+1)$ for all points l, m, n of the spatial grid and all points in time k.

A successfull application of finite-difference methods requires considering of the consistency of the discrete approximations and a careful choice of the step sizes in time and space for the stability of the discrete time-space algorithm in order to achieve convergence to the solution of the continuous model (Lax-Richtmyer theory).

5.2. Transfer Function Models

Transfer function models are standard tools for one-dimensional systems. They allow a system description in the frequency domain and provide a sound basis for time discretization. The extension of this idea to MD systems requires a suitable functional transformation for each

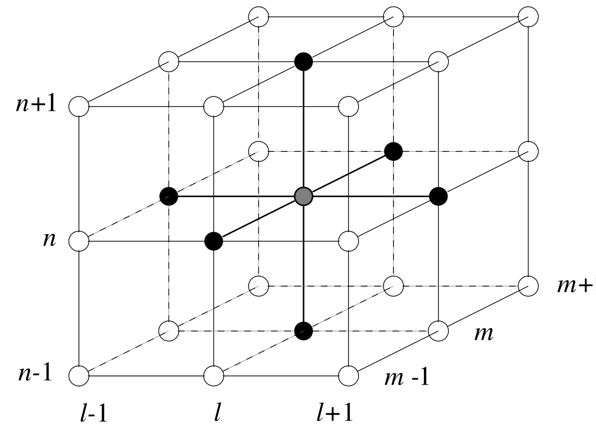

Fig. 1. 3-D finite difference model for the acoustic wave equation: centre point (grey) and neighbour points (black).

independent variable, e.g. time and space. A careful design of the transformation for the space variable allows us to treat boundary-value problems in the same way as the Laplace transformation treats the initial-value problem. The application to PDE models such as (7) requires a two-step procedure. First the Laplace transformation $\mathcal{L}\{\mathbf{y}(\mathbf{x}, t)\} = \mathbf{Y}(\mathbf{x}, s)$ converts (7) into the boundary-value problem

$$[s\mathbf{C} + \mathbf{L}]\mathbf{Y}(\mathbf{x}, s) = \mathbf{C}\mathbf{y}_i(\mathbf{x}), \qquad (47)$$

where $\mathbf{y}_i(\mathbf{x})$ is the initial value of $\mathbf{y}(\mathbf{x}, t)$ for $t = 0$. Second, the spatial transformation converts (47) into an algebraic equation from which the transfer function model is derived. This procedure is described in (Rabenstein and Trautmann, 1999) for general boundary conditions. A short account regarding homogeneous boundary conditions is given here. Since the matrix operators $\mathbf{L}$ and $\tilde{\mathbf{L}}$ play a key role, their properties are briefly investigated.

5.2.1. Properties of the Matrix Operators $\mathbf{L}$ and $\tilde{\mathbf{L}}$

At first, we consider the definitions of $\mathbf{L}$ and $\tilde{\mathbf{L}}$ according to (8), and (25) and two arbitrary vectors $\mathbf{y}_1(\mathbf{x})$ and $\mathbf{y}_2(\mathbf{x})$ compatible with $\mathbf{L}$ and $\tilde{\mathbf{L}}$. By applying of standard rules for differentiation, it is shown that $\mathbf{L}$ and $\tilde{\mathbf{L}}$ satisfy the Lagrange identity

$$\mathbf{y}_1^H[\mathbf{L}\mathbf{y}_2] - [\tilde{\mathbf{L}}\mathbf{y}_1]^H\mathbf{y}_2 = \nabla[\mathbf{y}_1^H\mathbf{B}\mathbf{y}_2]. \qquad (48)$$

Next, we introduce the scalar product

$$\big\langle \mathbf{y}_1(\mathbf{x}), \mathbf{y}_2(\mathbf{x})\big\rangle = \int_V \mathbf{y}_1^H(\mathbf{x})\mathbf{y}_2(\mathbf{x})\,\mathrm{d}V \qquad (49)$$

by integration over the volume V. From the Lagrange identity (48) and the application of the Gauss integral theorem for the right-hand side Green's formula follows:

$$\big\langle \mathbf{y}_1, \mathbf{L}\mathbf{y}_2\big\rangle - \big\langle \tilde{\mathbf{L}}\mathbf{y}_1, \mathbf{y}_2\big\rangle = \int_{\partial V} \mathbf{y}_1(\mathbf{x})\,\mathbf{B}\,\mathbf{y}_2(\mathbf{x})\,\mathrm{d}\mathbf{S}. \qquad (50)$$

It states that the difference of the two scalar products on the left-hand side can be expressed by a surface integration over the boundary ∂V of the volume V with the surface elements $\mathrm{d}\mathbf{S}$. Operator pairs with this property are called adjoint operators (Churchill, 1972; Courant and Hilbert, 1968). Thus $\tilde{\mathbf{L}}$ is the the adjoint operator to $\mathbf{L}$.

The surface integral in (50) is expressed by the boundary conditions of the PDE (7) as described in (Rabenstein and Trautmann, 1999). For homogenous boundary conditions, the property of the adjoint operator takes the familiar form

$$\big\langle \mathbf{y}_1, \mathbf{L}\mathbf{y}_2 \big\rangle = \big\langle \tilde{\mathbf{L}}\mathbf{y}_1, \mathbf{y}_2 \big\rangle. \tag{51}$$

Next, we investigate the properties of the eigenfunctions of these operators and consider the eigenvalue problems for $\mathbf{L}$ and $\tilde{\mathbf{L}}$:

$$\mathbf{L}\mathbf{K}(\mathbf{x}, \beta) = \beta \mathbf{C}\mathbf{K}(\mathbf{x}, \beta), \tag{52}$$

$$\Big[\tilde{\mathbf{L}}\tilde{\mathbf{K}}(\mathbf{x}, \tilde{\beta})\Big]^H = \tilde{\beta}\tilde{\mathbf{K}}^H(\mathbf{x}, \tilde{\beta})\mathbf{C}. \tag{53}$$

Here β and $\tilde{\beta}$ are the eigenvalues of $\mathbf{L}$ and $\tilde{\mathbf{L}}$, respectively, and $\mathbf{K}(\mathbf{x}, \beta)$ and $\tilde{\mathbf{K}}(\mathbf{x}, \beta)$ are the corresponding eigenfunctions. Since the eigenvalues β and $\tilde{\beta}$ are closely related, we write $\tilde{\mathbf{K}}(\mathbf{x}, \beta)$ in terms of the eigenvalues β of $\mathbf{K}(\mathbf{x}, \beta)$. These eigenvalue problems are a generalization of the Sturm-Liouville-type problems (Churchill, 1972; Körner, 1988). Their eigenvalues and eigenfunctions have the following properties:

- The eigenvalues β and $\tilde{\beta}$ are discrete. They are denoted by β_μ and $\tilde{\beta}_\mu$, respectively, with integer μ.
- β_μ and $\tilde{\beta}_\mu$ are related by complex conjugation $\tilde{\beta}_\mu = \beta_\mu^*$. Since each eigenvalue $\tilde{\beta}_\mu$ has its conjugate counterpart, we can write the transformation kernel of the forward transformation in terms of the eigenvalues β_μ.
- $\mathbf{K}(\mathbf{x}, \beta_\mu)$ and $\tilde{\mathbf{K}}(\mathbf{x}, \beta_\mu)$ are biorthogonal functions with respect to the weighting matrix $\mathbf{C}$ (N_μ is a normalization factor and $\delta_{\mu\nu}$ is the Kronecker symbol):

$$\big\langle \tilde{\mathbf{K}}(\mathbf{x}, \beta_\mu), \mathbf{C}\mathbf{K}(\mathbf{x}, eta_\nu) \big\rangle = N_\mu \delta_{\mu\nu}. \tag{54}$$

5.2.2. Definition of the Spatial Transformation

The spatial transformation is given by a scalar product of $\mathbf{Y}(\mathbf{x}, s)$ with the eigenfunctions $\tilde{\mathbf{K}}(\mathbf{x}, \beta_\mu)$ of $\tilde{\mathbf{L}}$ with respect to the weighting matrix $\mathbf{C}$:

$$\mathcal{T}\{\mathbf{Y}(\mathbf{x}, s)\} = \bar{Y}(\beta_\mu, s) = \big\langle \tilde{\mathbf{K}}(\mathbf{x}, \beta_\mu), \mathbf{C}\mathbf{Y}(\mathbf{x}, s) \big\rangle. \tag{55}$$

The application of $\mathcal{T}$ to (47) results in

$$s\bar{Y}(\beta_\mu, s) + \big\langle \tilde{\mathbf{K}}(\mathbf{x}, \beta_\mu), \mathbf{L}\mathbf{Y}(\mathbf{x}, s) \big\rangle = \bar{y}_i(\beta_\mu). \tag{56}$$

To express $\big\langle \tilde{\mathbf{K}}(\mathbf{x}, \beta_\mu), \mathbf{L}\mathbf{Y}(\mathbf{x}, s) \big\rangle$ by $\bar{Y}(\beta_\mu, s)$, we use Green's formula with homogeneous boundary conditions (51) for $\mathbf{y}_1 = \tilde{\mathbf{K}}$ and $\mathbf{y}_2 = \mathbf{Y}$, the eigenvalue problem (53), and the definition of the scalar product (49), and obtain (some arguments are omitted);

$$\big\langle \tilde{\mathbf{K}}, \mathbf{L}\mathbf{Y} \big\rangle = \big\langle \tilde{\mathbf{L}}\tilde{\mathbf{K}}, \mathbf{Y} \big\rangle = \tilde{\beta}_\mu \big\langle \tilde{\mathbf{K}}, \mathbf{C}\mathbf{Y} \big\rangle = \tilde{\beta}_\mu \bar{Y}(\beta_\mu, s). \tag{57}$$

The result

$$\big\langle \tilde{\mathbf{K}}, \mathbf{L}\mathbf{Y} \big\rangle = \tilde{\beta}_\mu \bar{Y}(\beta_\mu, s) \tag{58}$$

formally agrees with the differentiation theorem of the Laplace transformation, since the differentiation operator $\mathbf{L}$ on the left-hand side is expressed by the transform of $\mathbf{Y}(\mathbf{x})$. Thus the relation (58) is the key property of the transformation $\mathcal{T}$, since it allows the replacement of the spatial differentiation in (47) by an algebraic expression.

The inverse spatial transformation is given by

$$\begin{aligned} \mathbf{Y}(\mathbf{x}, s) &= \mathcal{T}^{-1}\big\{\bar{Y}(\beta_\mu, s)\big\} \\ &= \sum_{\mu=-\infty}^{\infty} \frac{1}{N_\mu} \bar{Y}(\beta_\mu, s)\mathbf{K}(\mathbf{x}, \beta_\mu). \end{aligned} \tag{59}$$

Since we have discrete eigenvalues β_μ, we do not have to integrate over the spatial frequency domain, but we only have to sum up the discrete spatial frequencies. N_μ can be evaluated by (54).

5.2.3. Transfer Function

The application of the Laplace transformation and the spatial transformation (55) with the spatial differentiation theorem (58) on the initial-boundary value problem (7) leads to the algebraic expresssion

$$s\bar{Y}(\beta_\mu, s) + \tilde{\beta}_\mu \bar{Y}(\beta, s) = \bar{y}_i(\beta_\mu). \tag{60}$$

Solving (60) for $\bar{Y}(\beta_\mu, s)$ results in the transfer function model

$$\bar{Y}(\beta_\mu, s) = \bar{G}(\beta_\mu, s)\, \bar{y}_i(\beta_\mu) \tag{61}$$

with the multidimensional transfer function

$$\bar{G}(\beta_\mu, s) = \frac{1}{s + \tilde{\beta}_\mu}. \tag{62}$$

It can be seen in (62) that we obtain a scalar transfer function, $\bar{G}(\beta_\mu, s)$, of two independent frequency variables, s and β_μ. The corresponding vector PDE (7) depends on two independent coordinates, i.e. time and space. The obvious reduction from two or more outputs in $\mathbf{y}(\mathbf{x}, t)$ (e.g. a potential and a flux quantity) in the vector PDE (7) to only one scalar output $\bar{Y}(\tilde{\beta}_\mu, s)$ in the transfer function (61) can be explained by the fact that the outputs in $\mathbf{y}(\mathbf{x}, t)$ depend on each other. They are connected by the relation described in the vector PDE (7). This relation is used for the construction of the spatial transformation. The information about the dependencies between the outputs can be found in the vectors of the transformation kernels, $\tilde{\mathbf{K}}$ and $\mathbf{K}$.

The matrix operator $\tilde{\mathbf{L}}$, initially introduced for the diagonalization of the vector PDE model, now serves as the key element for the development of an MD transfer function model. For the treatment of inhomogeneous boundary conditions, see, e.g. (Rabenstein, 1998; Rabenstein and Trautmann, 1999).

5.2.4. Discretization

The continuous solution (61) of the initial-boundary value problem can be discretized by standard discretization schemes. This process is shown here for the impulse invariant transformation for the temporal frequency domain. With $t = kT$, containing the discretization interval T and the integer variable k, the impulse invariant transformation of (61), (62) results with the discrete temporal frequency variable z in

$$\bar{Y}^d(\beta_\mu, z) = \bar{y}_i(\beta_\mu)\bar{G}^d(\beta_\mu, z) \tag{63}$$

with

$$\bar{G}^d(\beta_\mu, z) = \frac{z}{z - e^{-\tilde{\beta}_\mu T}}. \tag{64}$$

For the spatial frequency domain there is no need to apply any discretization scheme since the spatial frequency variable β_μ is already discrete with $\mu \in \mathbb{Z}$.

To obtain a solution in the time and space domain, we have to apply the corresponding inverse transformations. They include the inverse z-transformation for the temporal frequency domain and the inverse spatial transformation (59) for the spatial frequency domain. The application of the shifting theorem for the inverse z-transformation results in a recursive system of first order for each spatial frequency β_μ:

$$\bar{y}^d(\beta_\mu, k) = \bar{y}_i(\beta_\mu) + e^{-\tilde{\beta}_\mu T}\bar{y}^d(\beta_\mu, k-1) \tag{65}$$

This solution in the discrete time and discrete spatial frequency domain can be transformed back into the space domain by applying the inverse spatial transformation (59) on (65) and evaluating the sum at arbitrary spatial points $\mathbf{x}_a \in V$. It results in parallel recursive systems of the first order, weighted at the outputs with the normalized transformation kernels $\frac{1}{N_\mu}\mathbf{K}(\mathbf{x}_a, \beta_\mu)$. These weighted outputs are then summed up to the output vector at the given spatial point $\mathbf{x}_a$ and the time steps $t = kT$. The infinite sum given in (59), for the continuous-time variable must be truncated in the discrete-time domain to a finite length N to avoid aliasing.

The resulting system can be implemented directly using a computer. The output of this discretized model at the sampled time steps is exactly the same as the output of the continuous model described by the PDE and initial and boundary conditions. The only assumption that must be made is that the inputs of the continuous model can be modelled by weighted impulses due to the impulse invariant transformation.

Example 2 (continued). The transformation kernels of the forward and inverse spatial transformation with homogeneous boundary conditions and the length $d = x_1 - x_0$ of the electrical transmission line result in

$$\tilde{\mathbf{K}}(x, \beta_\mu) = \begin{bmatrix} (r - \tilde{\beta}_\mu l) d \dfrac{\sin(\mu\pi x/d)}{\mu\pi} \\ \cos(\mu\pi x/d) \end{bmatrix} \tilde{K}_2(x_0), \tag{66}$$

$$\mathbf{K}(x, \beta_\mu) = \begin{bmatrix} -(r - \beta_\mu l) d \dfrac{\sin(\mu\pi x/d)}{\mu\pi} \\ \cos(\mu\pi x/d) \end{bmatrix} K_2(x_0). \tag{67}$$

The discrete values of the spatial frequency variable β_μ are solutions of the quadratic equation

$$(r - \beta_\mu l)(g - \beta_\mu c) + \left(\frac{\mu\pi}{d}\right)^2 = 0. \tag{68}$$

The norm factor N_μ follows from (54) with $\tilde{K}_2(x_0) = K_2(x_0) = 1$ as

$$N_\mu = \frac{2\beta_\mu lc - rc - gl}{2\beta_\mu c - 2g} d. \tag{69}$$

5.3. MD Wave Digital Models

The third familiy of discrete MD models considered here are MD wave digital models. They follow from the normalized vector models introduced in Section 3.2. We discuss this approach for the acoustic wave equation of Example 3.

Performing the normalization process according to (20)–(24) for an inhomogeneous model (not considered in (5)–(6)) results in

$$\mathbf{Z}_n \mathbf{i}_n(\mathbf{x}, t) = \mathbf{e}_n(\mathbf{x}, t), \tag{70}$$

where $\mathbf{Z}_n = \mathbf{C}_n D_t + \mathbf{L}_n$ is the matrix operator in (24) and $\mathbf{i}_n(\mathbf{x}, t) = \mathbf{y}_n(\mathbf{x}, t)$ is the vector of unkowns with all variables normalized to the dimension of flux quantities. It is a four-dimensional vector containing the three spatial components of $\mathbf{i}(\mathbf{x}, t)$ as well as the normalized potential $u(\mathbf{x}, t)/r_0$ from (20). Here $\mathbf{e}_n(\mathbf{x}, t)$ is a vector of source functions with all variables normalized to the dimension of potentials. Note that $\mathbf{e}_n(\mathbf{x}, t) = \mathbf{0}$ for the homogeneous case in (24).

The normalized representation of the MD system in (70) can be converted into a discrete-time discrete-space model by two related approaches. The first one is the classical MD wave digital approach, which is based on the theory of wave digital filters for one-dimensional systems (Fettweis, 1986). It was extended to the multidimensional case in (Fettweis, 1994;1999; Fettweis and Nitsche, 1991) and related publications. A detailed description of the modelling and discretization process according to the wave digital principle requires in-depth knowledge of classical analog circuit theory and its formal extension to MD systems.

The second approach is closely related to the wave digital principle. However, in contrast to the literature cited above, only basic knowledge of numerical mathematics, linear algebra, and multidimensional system theory is required. This approach converts the PDE (70) directly into a state-space description of the resulting discrete-time discrete-space model.

5.3.1. Classical MD Wave Digital Approach

The classical approach according to the MD wave digital principle considers (70) as a matrix formulation of Ohm's law. It states that potential and flux quantities are linked by an impedance matrix $\mathbf{Z}_n$. Taking the symmetric form of $\mathbf{Z}_n$ into account according to (24), the matrix equation (70) formally agrees with the mesh current description of an MD network. Thus the normalized PDE model (14) can be mapped onto an MD network which can serve as the starting point of a discrete model according to the MD wave digital principle. This process ensures that the physical structure of the problem is preserved in the discrete model.

As an illustration, the MD network model for the acoustic wave equation according to (24) is shown in Fig. 2. The Laplace transformation with respect to time

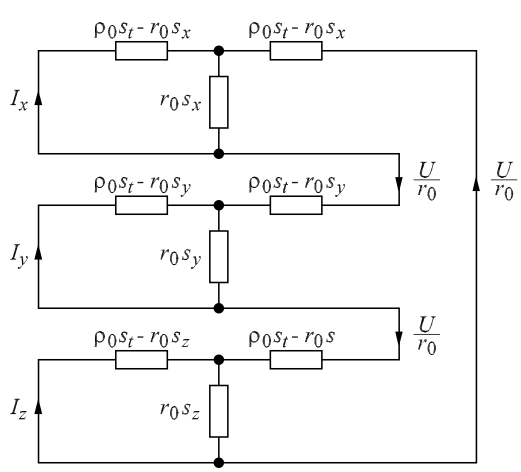

Fig. 2. Network description of the acoustic wave equation.

and space variables represents the differentiation operators D_n by the frequency variables $s_n,\ n = x, y, z, t$. Boundary conditions are neglected at this point. The first three rows in (24) correspond to the three meshes on the left-hand side in Fig. 2. They are connected by the mesh on the right-hand side, which represents the continuity equation. $I_n,\ n = x, y, z$ and U/r_0 are the complex amplitudes of the three spatial components of $i_n,\ n = x, y, z$ and of u/r_0 in (24), respectively.

The transition from the complex potential and flux quantities $I_n,\ n = x, y, z$, and U to the wave quantities A_n and B_n according to

$$\begin{bmatrix} A_n \\ B_n \end{bmatrix} = \begin{bmatrix} 1 & R_0 \\ 1 & -R_0 \end{bmatrix} \begin{bmatrix} U \\ I_n \end{bmatrix} \qquad (71)$$

introduces a causal relationship between incident and reflected waves A_n and B_n. R_0 is the so-called port resistance. After some changes in the network structure, a discrete-time, discrete-space model is introduced by an s-to-z mapping according to the bilinear transformation. A suitable choice of the port resistance R_0 adjusts the parameters of the bilinear transformation to the required spatial and temporal step sizes. The resulting structure of the discrete system is shown in Fig. 3.

5.3.2. MD State-Space Approach

The derivation of the discrete system according to the state-space approach starts again from the normalized model (70). After a series of intermediate steps, the state-space representation of a discrete-time and discrete-space algorithm is obtained. These steps are:

- Separation into spatial components

  The PDE description (70) is separated into three different spatial components. Each component contains derivatives with respect to time and only one of the spatial directions x, y or z.

- Numerical solution for each component

  A numerical integration is carried out for each of the spatial components. The discretization is performed by the trapezoidal rule in two dimensions (one time dimension and one space dimension).

- Combination of the spatial components

  The discrete-time, discrete-space approximations for each of the three spatial components are combined into a full four-dimensional representation (one time dimension and three space dimensions) of the PDE description (70).

- State-space formulation

  A suitable choice of internal states allows us to formulate the discrete model in the state-space context.

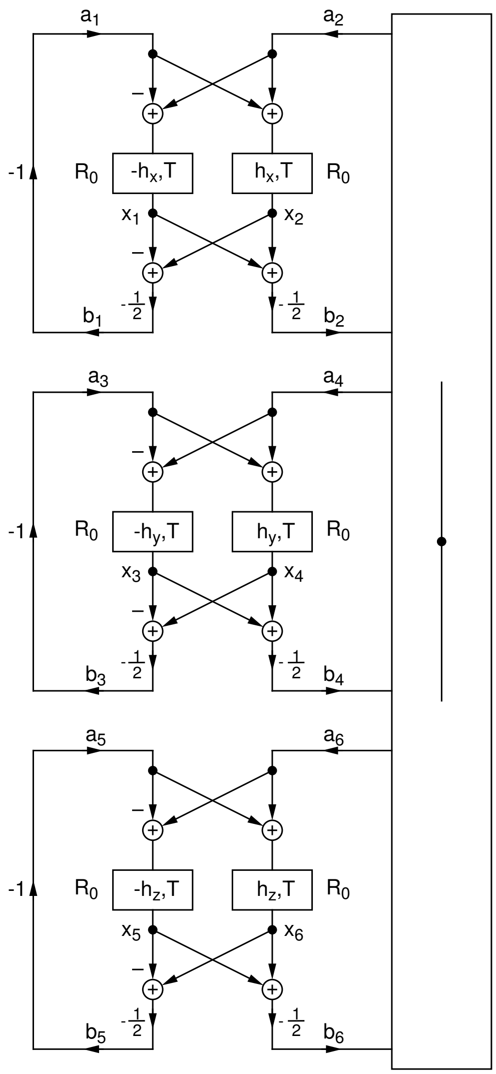

Fig. 3. Multidimensional wave digital model for the acoustic wave equation. Parameters: spatial shifts h_x, h_x, h_x, time delay T, discrete wave quantities a_n, b_n, state variables x_n, port resistance R_0. See (Schetelig and Rabenstein, 1998) for details.

The following subsections cover these steps in detail.

Separation into spatial components. An inspection of (24) shows that this PDE for three spatial components can be reduced to three PDEs with only one spatial component each. For example, the PDE for the x-component has the form

$$\bar{\mathbf{Z}}_1\,\bar{\mathbf{i}}_1 = \bar{\mathbf{u}}_1 \tag{72}$$

with

$$\bar{\mathbf{Z}}_1 = \begin{bmatrix} \rho_0 D_t & r_0 D_x \\ r_0 D_x & \rho_0 D_t \end{bmatrix}, \quad \bar{\mathbf{i}}_1 = \begin{bmatrix} i_1 \\ i_4 \end{bmatrix},$$

$$\bar{\mathbf{u}}_1 = \begin{bmatrix} u_{11} \\ u_{14} \end{bmatrix}. \tag{73}$$

The matrix $\bar{\mathbf{Z}}_1$ is obtained from $\mathbf{Z}$ by eliminating of the second and the third row and column, which contain only y and z components. Furthermore, the element $3\rho_0 D_t$ in $\mathbf{Z}$ contributes equally to all spatial components and is represented in $\bar{\mathbf{Z}}_1$ by one third of its value. Similarly, the first element in $\bar{\mathbf{u}}_1$ is equal to the first element in $\mathbf{e}$. The sum of u_{14} and the corresponding elements for the other spatial components is equal to the fourth element in $\mathbf{e}$.

Numerical solution for each component. The numerical solution of the separate spatial components is derived through a series of steps which are shown in Fig. 4. The procedure is explained for the spatial direction x. The starting point is the partial differential operator $\bar{\mathbf{Z}}_1$ from (73). Since it involves both time and space differentiation, it is not very suitable for direct numerical integration. A decoupled form with simple differentiation operators would be more desirable. This is achieved by two measures: First, a variable transformation, which decouples the differentiation operators or, in other words, a diagonalization of the operator matrix $\bar{\mathbf{Z}}_1$. Second, a coordinate transformation, such that each entry in the diagonal operator matrix contains only differentiation with respect to a single coordinate.

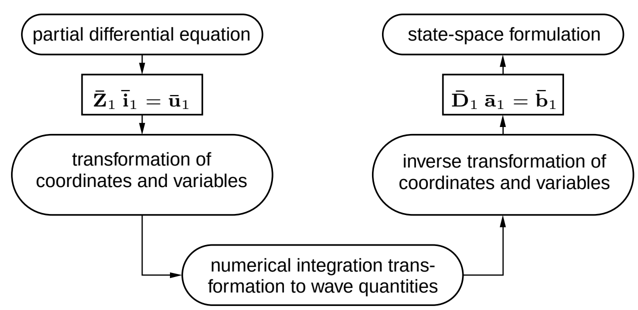

Fig. 4. Transformation of the partial differential operator for one spatial component into a state-space formulation.

The resulting decoupled form allows numerical integration by the trapezoidal rule. Care has to be taken to avoid delay-free loops, which would call for an iterative solution. This problem is circumvented by another transformation of the variables. It leads to the so-called wave quantities $\bar{\mathbf{a}}_1$ and $\bar{\mathbf{b}}_1$ (Fettweis, 1994). Now, the spatial components can be integrated numerically, though in transformed coordinates and variables. Therefore, the decoupling steps have to be reversed, to arrive at a discrete-time, discrete-space wave quantity formulation of the original problem. A more detailed description of the coordinate and variable transformations shown in Fig. 4 is given in (Rabenstein and Zayati, 2000).

The matrix $\bar{\mathbf{D}}_1$ in Fig. 4 contains shift operators in the x-direction and delay operators in the time direction. As a difference operator matrix it corresponds to the differential operator matrix, $\bar{\mathbf{Z}}_1$.

State-space formulation. The final form of the algorithm can be formulated in terms of a state-space description. The state $\mathbf{z}$ is associated with one of the wave quantities (see (Rabenstein and Zayati, 2000) for details). The state-space model consists of the state equation and the output equation:

$$\mathbf{z} = \mathcal{D}\left[\mathcal{A}\mathbf{z} + \mathcal{B}\mathbf{e}\right], \tag{74}$$

$$\mathbf{i} = \mathcal{C}\mathbf{z} + \mathcal{F}\mathbf{e}. \tag{75}$$

The state equation (74) results from condensing the procedure outlined in Fig. 4 into one matrix equation. The output equation (75) represents the conversion of the wave quantities back to acoustic variables: pressure and velocity. The concise matrix formulation of the discrete model as a state-space description allows a direct implementation of (74), (75) in a computer code.

Boundary conditions. The operator matrix $\mathcal{D}$ in the state equation (74) contains shifts in both directions of each spatial dimension. This requires the knowledge of the previous states at all adjacent points. However, if a point lies on the boundary of the spatial domain, e.g. on the wall of an enclosure, then one or more of the adjacent points are beyond the boundary, where the PDE is no more valid. In this case, the state of these points has to be determined from boundary conditions rather than from the PDE (5), (6), see also (Krauß, 1996). A detailed presentation of the incorporation of various types of boundary conditions is beyond the scope of this paper. Only a short outline of the general approach is given here.

The idea is to split the state vector $\mathbf{z}$ into two components: the interior states $\mathbf{z}_i$ and the boundary states $\mathbf{z}_b$. The interior states follow from a state equation similar to (74). The boundary states follow from the interior states and the boundary conditions. The state-space representation has to consider both types of states appropriately. Its general form is given by

$$\mathbf{z}_i = \left(\mathbf{T}_i^T\mathcal{D}\right)\left[\mathcal{A}\mathbf{z} + \mathcal{B}\mathbf{e}\right], \tag{76}$$

$$\mathbf{z}_b = \mathcal{A}_b\mathbf{z}_i + \mathcal{B}_b\mathbf{e}, \tag{77}$$

$$\mathbf{z} = \mathbf{T}_i\mathbf{z}_i + \mathbf{T}_b\mathbf{z}_b, \tag{78}$$

$$\mathbf{i} = \mathcal{C}\mathbf{z} + \mathcal{F}\mathbf{e}. \tag{79}$$

The matrices $\mathbf{T}_i$ and $\mathbf{T}_b$ contain only ones and zeros. They depend on the geometry and describe whether a state is an interior state $\mathbf{z}_i$ or a boundary state $\mathbf{z}_b$. Equation (76) is very similar to the state equation (74), except that it delivers only the interior states. The boundary states are computed in (77) from the interior states and the boundary conditions which determine $\mathcal{A}_b$ and $\mathcal{B}_b$. Both interior and boundary states are merged into the complete state vector $\mathbf{z}$ in (78). It is used to provide the output quantities in (79) and to update the interior states in (76).

Example 4. An example of the application of the MD state-space model is given in Fig. 5. It shows the dynamical sound field produced by a horn loudspeaker. The plot was obtained from a simulation of the acoustic wave equation in one time and three space coordinates. It shows the sound pressure in a horizontal plane at a fixed point in time.

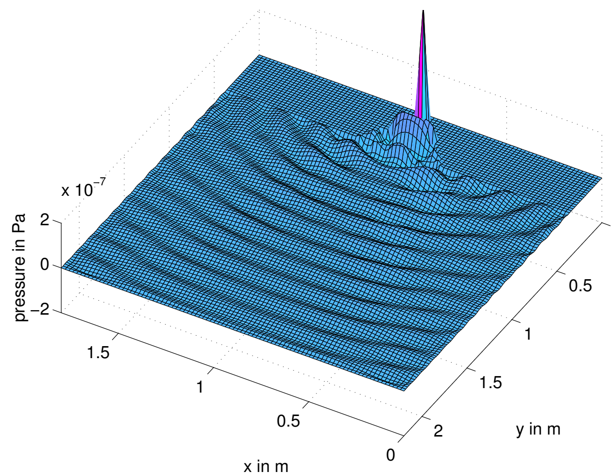

Fig. 5. Wave propagation from a horn loudspeaker built into a reflecting wall (top).

6. Conclusions

As a contribution to a unifying framework for continuous and discrete MD models, we investigated MD systems based on physical models. Three different physical phenomena served as examples for the formulation of mathematical models from the first principles of physics: heat flow in one spatial dimension, voltage and current distribution on an electrical transmission line, and the propagation of sound waves in the air in three spatial dimensions. All approaches share the formulation of the mathematical description in terms of potential and flux quantities. These are temperature and heat flux for the heat flow example, voltage and electrical current for the transmission line, and pressure and particle velocity for the propagation of sound waves. The resulting physical models follow from conservation laws and material characterizations according to the special physical phenomenon under consideration.

To abstract from the physical effect and to formulate a more general model of a multidimensional system, conversions of these physical models into different standardized forms are required. As a starting point the flux and potential quantities are combined to a single vector of unknowns. This approach leads immediately to potential-flux models. They have the advantage that their variables are directly related to the quantities of the physical model. On the other hand, potential-flux models require a careful observation of the physical dimensions of their variables. This can be avoided by normalizing the model equations to the same physical dimension or by making the variables totally dimensionless. This results in normalized vector models. Their differential operators have a more uniform structure but less physical meaning than potential-flux models. Another form of abstraction is obtained by converting the vector model into a scalar form. The advantage is a much simpler, i.e. scalar, form of the model. The price to pay is the elimination of one of the initial variables, usually the flux variable. As a result, the formulation of boundary or interface conditions in terms of impedance laws is no longer possible or has to be circumvented by the introduction of additional derivative terms. It is interesting to note that this elimination process can be formulated concisely as diagonalization. It is performed by multiplication with the adjoint matrix differential operator of the differential operator from the vector model.

In conclusion, we derived three different continuous models from the initial physical description: two vector representations, the potential-flux model and the normalized vector model, and one scalar representation. These three models are closely connected by matrix operations, performing either normalization or diagonalization.

Furthermore, we showed that each of these continuous representations results in a different family of discrete models in current use: finite-difference models, transfer functions models and MD wave digital models. Finite-difference models are easily obtained from scalar representations by standard methods of the discretization of differential operators. From the potential-flux model, the corresponding transfer function model can be derived by suitable transformations with respect to the time and space variables. The transformation for the time variable is the well-known Laplace transformation. The transformation for the space variable has to be adapted to the spatial domain and the given boundary conditions. It follows from the spatial differential operator of the physical model and its adjoint operator. The latter was shown to be identical to the matrix differential operator already used in the above diagonalization process. Finally, discrete multidimensional systems based on the wave digital principle are directly based on the normalized vector model. Two different approaches were considered: the classical derivation by means of a multidimensional network description and the multidimensional state-space approach.

All the discrete multidimensional models just considered have appeared before in the literature. However, little has been known about the relations between them and how to transform one discrete representation into another. Here it was shown that they are linked together by the close connections between their corresponding continuous models. However, the framework is not complete. Topics for future research are a unifying treatment of initial and boundary conditions and the inclusion of the finite-element method through a variational formulation of the physical model.

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