Int. J. Appl. Math. Comput. Sci., 2004, Vol. 14, No. 1, 19–23

INFINITE EIGENVALUE ASSIGNMENT BY AN OUTPUT FEEDBACK FOR SINGULAR SYSTEMS

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The problem of an infinite eigenvalue assignment by an output feedback is considered. Necessary and sufficient conditions for the existence of a solution are established. A procedure for the computation of the output-feedback gain matrix is given and illustrated with a numerical example.

Keywords: infinite eigenvalue assignment, feedback, singular system

1. Introduction

It is well known (Dai, 1989; Kaliath, 1980; Wonham, 1979; Kaczorek, 1993; Kučera, 1981) that if the pair (A, B) of a standard linear system $\dot{x} = Ax + Bu$ is controllable then there exists a state-feedback gain matrix K such that $\det[I_n s - A + BK] = p(s)$, where $p(s) = s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0$ is a given arbitrary n-th order polynomial. By changing K we may modify arbitrarily only the coefficients $a_0, a_1, \ldots, a_{n-1}$ but we are not able to change the degree n of the polynomial which is determined by the matrix $I_n s$. In singular linear systems we are also able to change the degree of the closed-loop characteristic polynomials by a suitable choice of the state-feedback matrix K. The problem of finding a state-feedback matrix K such that $\det[Es - A + BK] = \alpha \neq 0$ (α is independent of s) was considered in (Kaczorek, 2003; Chu and Ho, 1999). The infinite eigenvalue assignment problem by a feedback is very important in the design of perfect observers (Kaczorek, 2000; 2002; 2003).

In this paper the problem of an infinite eigenvalue assignment by an output feedback is formulated and solved. This is an extension of the method given in (Kaczorek, 2003) for an output feedback case. Necessary and sufficient conditions for the existence of a solution to the problem will be established and a procedure for the computation of an output-feedback gain matrix will be presented.

2. Problem Formulation

Let $\mathbb{R}^{n\times m}$ be the set of real $n \times m$ matrices and $\mathbb{R}^n := \mathbb{R}^{n\times 1}$. Consider the continuous-time linear system

$$E\dot{x} = Ax + Bu, \quad y = Cx, \tag{1}$$

where $\dot{x} = \mathrm{d}x/\mathrm{d}t$ and $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ and $y \in \mathbb{R}^p$ are respectively the semistate, input and output vectors. Moreover, $E, A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$. The system (1) is called singular if $\det E = 0$ and it is called standard when $\det E \neq 0$.

It is assumed that $\operatorname{rank} E = r < n$, $\operatorname{rank} B = m$, $\operatorname{rank} C = p$ and the pair (E, A) is regular, i.e.

$$\det[Es - A] \neq 0 \tag{2}$$

for some $s \in \mathbb{C}$ (the field of complex numbers). Let us consider the system (1) with the output feedback

$$u = v - Fy, \tag{3}$$

where $v \in \mathbb{R}^m$ is a new input and $F \in \mathbb{R}^{m\times p}$ is a gain matrix. From (1) and (3) we have

$$E\dot{x} = (A - BFC)x + Bv. \tag{4}$$

Problem 1. Given matrices E, A, B, C of (1) and a nonzero scalar α (independent of s), find an $F \in \mathbb{R}^{m\times p}$ such that

$$\det[Es - A + BFC] = \alpha. \tag{5}$$

In this paper necessary and sufficient conditions for the existence of a solution to Problem 1 will be established and a procedure for the computation of F will be proposed.

3. Problem Solution

From the equality

$$Es - A + BFC = [Es - A, B]\begin{bmatrix} I_n \\ FC \end{bmatrix} = [I_n, BF]\begin{bmatrix} Es - A \\ C \end{bmatrix} \tag{6}$$

and (5) it follows that Problem 1 has a solution only if

$$\operatorname{rank}[Es - A, B] = n \tag{7}$$

and

$$\operatorname{rank}\begin{bmatrix} Es - A \\ C \end{bmatrix} = n \tag{8}$$

for all finite $s \in \mathbb{C}$. The problem will be solved using the following two-step procedure:

Step 1. (Subproblem 1). Given E, A, B of (1) and a scalar α, find a matrix $K = FC$ such that

$$\det[Es - A + BK] = \alpha. \tag{9}$$

Step 2. (Subproblem 2). Given C and K depending on some free parameters $k_1, k_2, \dots, k_l$ (found in Step 1), find a matrix F satisfying the equation

$$K = FC. \tag{10}$$

The solution of Subproblem 1 is based on the following lemma (Chu and Ho, 1999; Kaczorek, 2003):

Lemma 1. *If the condition (2) is satisfied, then there exist orthogonal matrices U and V such that*

$$U[Es - A]V = \begin{bmatrix} E_1 s - A_1 & * \\ 0 & E_0 s - A_0 \end{bmatrix},$$

$$UB = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}, \tag{11a}$$

where $E_1, A_1 \in \mathbb{R}^{n_1 \times n_1}$, $E_0, A_0 \in \mathbb{R}^{n_0 \times n_0}$, $B_1 \in \mathbb{R}^{n_1 \times m}$, the subsystem (E_1, A_1, B_1) is completely controllable, the pair (E_0, A_0) is regular, E_1 is upper triangular and '$*$' denotes an unimportant matrix. Moreover, the matrices E_1, A_1 and B_1 are of the form

$$E_1 s - A_1 = \begin{bmatrix} E_{11}s - A_{11} & E_{12}s - A_{12} & \cdots & E_{1,k-1}s - A_{1,k-1} & E_{1k}s - A_{1k} \\ -A_{21} & E_{22}s - A_{22} & \cdots & E_{2,k-1}s - A_{2,k-1} & E_{2k}s - A_{2k} \\ 0 & -A_{32} & \cdots & E_{3,k-1}s - A_{3,k-1} & E_{3k}s - A_{3k} \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \cdots & -A_{k,k-1} & E_{kk}s - A_{kk} \end{bmatrix},$$

$$B_1 = \begin{bmatrix} B_{11} \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \tag{11b}$$

where $E_{ij}, A_{ij} \in \mathbb{R}^{\bar{n}_i \times \bar{n}_j}$, $i, j = 1, \dots, k$, $B_{11} \in \mathbb{R}^{\bar{n}_i \times m}$, $\sum_{i=1}^{n} \bar{n}_i = n_1$, with $B_{11}, A_{21}, \dots, A_{k,k-1}$ of full row rank and $E_{22}, \dots, E_{kk}$ nonsingular.

Remark 1. The matrix $\bar{C} = CV$ has no special form.

Theorem 1. *Let (2) and (7) be satisfied and let the matrices E, A, B of (1) be transformed into the forms (11). A matrix K satisfying (9) exists if and only if*

(i) *the subsystem (E_1, A_1, B_1) is singular, i.e.*

$$\det E_1 = 0, \tag{12a}$$

(ii) *if $n_0 > 0$, then the degree of the polynomial $\det[E_0 s - A_0]$ is zero, i.e.*

$$\deg \det[E_0 s - A_0] = 0 \text{ for } n_0 > 0. \tag{12b}$$

Proof. (Necessity) From (9) and (11a) we have

$$\begin{aligned} &\det[Es - A + BK] \\ &= \det U^{-1} \det V^{-1} \det[E_1 s - A_1 + B_1 \bar{K}] \\ &\quad \times \det[E_0 s - A_0] = \alpha, \end{aligned} \tag{13}$$

where $\bar{K} = KV \in \mathbb{R}^{m \times n}$ and $\det[E_0 s - A_0] = 1$ if $n_0 = 0$. From (13) it follows that the condition (9) holds only if the conditions (12) are satisfied.

(Sufficiency) First consider the single-input $(m = 1)$ case. In this case we have

$$E_1 = \begin{bmatrix} e_{11} & e_{12} & \cdots & e_{1n_1} \\ 0 & e_{22} & \cdots & e_{2n_1} \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \cdots & e_{n_1 n_1} \end{bmatrix},$$

$$A_1 = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1,n_1-1} & a_{1n_1} \\ a_{21} & a_{22} & \cdots & a_{2,n_1-1} & a_{2n_1} \\ 0 & a_{31} & \cdots & a_{3,n_1-1} & a_{3n_{1a}} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & a_{n_1,n_1-1} & a_{n_1n_1} \end{bmatrix},$$

$$B_1 = b_1 = \begin{bmatrix} b_{11} \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \tag{14}$$

where $e_{ii} \neq 0$, $a_{i,i-1} \neq 0$ for $i = 2, \ldots, n_1$ and $b_{11} \neq 0$.

The condition (12a) implies that $e_{11} = 0$. Premultiplying the matrix $[E_1 s - A_1, b_1]$ by a matrix of orthogonal row operations P_1 it is possible to make the entries $e_{12}, e_{13}, \ldots, e_{1n_1}$ of E_1 zero since $e_{ii} \neq 0$, $i = 2, \ldots, n_1$. By this reduction only the entries of the first row of A_1 will be modified,

$$\bar{E}_1 = P_1 E_1 = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & e_{22} & \cdots & e_{2n_1} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & e_{n_1n_1} \end{bmatrix},$$

$$\bar{A}_1 = P_1 A_1 = \begin{bmatrix} \bar{a}_{11} & \bar{a}_{12} & \cdots & \bar{a}_{1,n_1-1} & \bar{a}_{1n_1} \\ a_{21} & a_{22} & \cdots & a_{2,n_1-1} & a_{2n_1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & a_{31} & \cdots & a_{3,n_1-1} & a_{3n_{1a}} \\ 0 & 0 & \cdots & a_{n_1,n_1-1} & a_{n_1n_1} \end{bmatrix},$$

$$\bar{b}_1 = P_1 b_1 = b_1. \tag{15}$$

Let

$$\bar{k}_1 = \frac{1}{b_{11}}[-\bar{a}_{11}, -\bar{a}_{12}, \ldots, -\bar{a}_{1,n_1-1}, 1 - \bar{a}_{1n_1}]. \tag{16}$$

Using (13), (15) and (16), we obtain

$$\det[\bar{E}_1 s - \bar{A}_1 + \bar{b}_1 \bar{k}_1]$$

$$= \begin{vmatrix} 0 & 0 & \cdots & 0 & 1 \\ -a_{21} & e_{22}s - a_{22} & \cdots & e_{2,n_1-1}s - a_{2,n_1-1} & e_{2n_1}s - a_{2n_1} \\ 0 & -a_{31} & \cdots & e_{3,n_1-1}s - a_{3,n_1-1} & e_{3n_{1a}}s - a_{3n_{1a}} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & -a_{n_1,n_1-1} & e_{n_1n_1}s - a_{n_1n_1} \end{vmatrix}$$

$$= a_{21}a_{31} \cdots a_{n_1,n_1-1} = \bar{\alpha}, \tag{17}$$

where $\bar{\alpha} = \alpha \det U \det V \det P_1 \det[E_0 s - A_0]^{-1}$.

The deliberations can be easily extended to multi-input systems, $m > 1$. In this case the matrix of orthogonal row operations P_1 is chosen so that all the entries of the first row of $\bar{E}_1 = P_1 E_1$ are zero. By this reduction, only the entries of A_{1i}, $i = 1, \ldots, k$ and B_{11} will be modified. The modified matrices will be denoted by $\bar{A}_{1i}$, $i = 1, \ldots, k$ and $\bar{B}_{11}$, respectively.

Let

$$\bar{K} = \bar{B}_1^{-1} \left\{ \left[\bar{A}_{11}, \bar{A}_{12}, \ldots, \bar{A}_{1k} \right] + G \right\}. \tag{18}$$

The matrix $G \in \mathbb{R}^{m \times n}$ in (18) is chosen so that

$$\bar{E}_1 s - \bar{A}_1 + \bar{B}_1 \bar{K} = \begin{bmatrix} 0 & 0 & \cdots & 0 & (-1)^{l+1}h \\ \bar{a}_{21} & * & \cdots & * & * \\ 0 & \bar{a}_{32} & \cdots & * & * \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \bar{a}_{l,l-1} & * \end{bmatrix}, \tag{19}$$

where '$*$' denotes unimportant entries,

$$h = \frac{\alpha(-1)^{l+1}}{\bar{a}_{21}\bar{a}_{32} \ldots \bar{a}_{l,l-1} c},$$

$$c = \det U^{-1} \det V^{-1} \det P_1^{-1} \det[E_0 s - A_0].$$

Using (13), (18) and (19), it is easy to verify that

$$\det[Es - A + BK] = c \det[\bar{E}_1 s - \bar{A}_1 + \bar{B}_1 \bar{K}] = \alpha. \tag{20}$$

■

Remark 2. Note that for $m > 1$ some entries of the matrix G in (18) can be chosen arbitrarily. Therefore, the matrix $K = \bar{K} V^{-1}$ has a number of free parameters denoted by $k_1, k_2, \ldots, k_l$. The free parameters will be chosen so that (10) has a solution F for given C and K.

It is well known that (10) has a solution if and only if

$$\operatorname{rank} C = \operatorname{rank} \begin{bmatrix} C \\ K \end{bmatrix} \tag{21a}$$

or, equivalently,

$$\operatorname{Im} K^T \subset \operatorname{Im} C^T. \tag{21b}$$

The free parameters $k_1, k_2, \ldots, k_l$ are chosen so that (21) holds. Therefore, the following theorem has been proved:

Theorem 2. *Let the conditions (2), (7), (8) and (12) be satisfied. Problem 1 has a solution, i.e. there exists an F satisfying (5) if and only if the free parameters $k_1, k_2, \ldots, k_l$ of K can be chosen so that (10) has a solution F for given C and K.*

From the condition (21) and (16) we have the following result:

Corollary 1. *For* $m = 1$ *Problem 1 has a solution if and only if the row* $[\bar{a}_{11}, \bar{a}_{12}, \dots, \bar{a}_{1n_1-1}\bar{a}_{1n_1} - 1]$ *is proportional to the matrix* C.

Remark 3. If the system order is not high, say $n \le 5$, elementary row and column operations can be used instead of the orthogonal operations.

4. Example

For the singular system (1) with

$$E = \begin{bmatrix} 0 & 2 & 1 & 0 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$A = \begin{bmatrix} 1 & -1 & 0 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & -1 & 1 & -1 \\ 0 & 0 & 2 & 1 \end{bmatrix},$$

$$B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0.5 & 1 & 3 & -2 \\ 2.5 & 3 & 4 & -1 \end{bmatrix} \tag{22}$$

we wish to find a gain matrix $F \in \mathbb{R}^{2\times 2}$ such that the condition (5) is satisfied for $\alpha = 1$.

In this case the pair (E, A) is regular since

$$\det[Es - A] = \begin{vmatrix} -1 & 2s+1 & s & -1 \\ 0 & s-1 & -s-2 & 2s \\ 0 & 1 & s-1 & 1-s \\ 0 & 0 & -2 & s-1 \end{vmatrix}$$

$$= (3-s)(s-1)^2 - (s+2)(s-1) + 4s.$$

The matrices (22) have already the desired forms (11) with $A_0 = 0$, $B_0 = 0$, $E_1 = E, A_1 = A$, $B_1 = B$, $n_1 = n = 4$, $\bar{n}_1 = 2$, $\bar{n}_2 = \bar{n}_3 = 1$, $m = 2$ and

$$E_{11} = \begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix}, \quad E_{12} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad E_{13} = \begin{bmatrix} 0 \\ 2 \end{bmatrix},$$

$$E_{22} = [1], \quad E_{23} = [-1], \quad E_{33} = [1]$$

$$A_{11} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}, \quad A_{12} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \quad A_{13} = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

$$A_{21} = [0 \ \ -1], \quad A_{22} = [1], \quad A_{23} = [-1], \quad A_{32} = [2],$$

$$A_{33} = [1], \quad B_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Using elementary row operations (Kaczorek, 1993; Kaczorek, 2003), we obtain

$$P_1 = \begin{bmatrix} 1 & -2 & -3 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and

$$[\bar{E}_1 s - \bar{A}_1, \bar{B}_1] = P_1[Es - A, B]$$

$$= \begin{bmatrix} -1 & 0 & 5 & -5 & 1 & -2 \\ 0 & s & -1 & 2 & 0 & 1 \\ 0 & 1 & s-1 & 1-s & 0 & 0 \\ 0 & 0 & -2 & s-1 & 0 & 0 \end{bmatrix}.$$

Taking into account that in this case

$$[\bar{A}_{11}, \bar{A}_{12}, \bar{A}_{13}] = \begin{bmatrix} 1 & 0 & -5 & 5 \\ 0 & 0 & 1 & -2 \end{bmatrix},$$

$$\bar{B}_1 = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}, \quad G = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0.5 & k_1 & k_2 & k_3 \end{bmatrix}$$

and using (18), we obtain

$$K = \bar{K} = \bar{B}_1^{-1}\left\{[\bar{A}_{11}, \bar{A}_{12}, \bar{A}_{13}] + G\right\}$$

$$= \begin{bmatrix} 2 & 2k_1 & 2k_2 - 3 & 1 + 2k_3 \\ 0.5 & k_1 & k_2 + 1 & k_3 - 2 \end{bmatrix},$$

where k_1, k_2, k_3 are free parameters.

The free parameters are chosen so that the condition

$$\operatorname{rank} \begin{bmatrix} 0.5 & 1 & 3 & -2 \\ 2.5 & 3 & 4 & -1 \end{bmatrix}$$

$$= \operatorname{rank} \begin{bmatrix} 0.5 & 1 & 3 & -2 \\ 2.5 & 3 & 4 & -1 \\ 2 & 2k_1 & 2k_2 - 3 & 1 + 2k_3 \\ 0.5 & k_1 & k_2 + 1 & k_3 - 2 \end{bmatrix} \tag{23}$$

is satisfied, which implies $k_1 = 1$, $k_2 = 2$, $k_3 = 0$. The equation

$$F \begin{bmatrix} 0.5 & 1 & 3 & -2 \\ 2.5 & 3 & 4 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 1 & 1 \\ 0.5 & 1 & 3 & -2 \end{bmatrix}$$

has the solution

$$F = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}.$$

It is easy to check that

$$\begin{aligned} &\det[Es - A + BK] \\ &\quad = \det P_1^{-1} \det[\bar{E}s - \bar{A} + \bar{B}K] \\ &\quad = \begin{vmatrix} 0 & 0 & 0 & 1 \\ 0.5 & s+1 & 2 & 0 \\ 0 & 1 & s-1 & 1-s \\ 0 & 0 & -2 & s-1 \end{vmatrix} = 1. \end{aligned}$$

5. Concluding Remarks

The problem of an infinite eigenvalue assignment by output feedbacks has been formulated and solved. Necessary and sufficient conditions for the existence of a solution to the problem were established. A two-step procedure for the computation of the output-feedback gain matrix was derived and illustrated with a numerical example. With slight modifications the deliberations can be extended to singular discrete-time linear systems. An extension to two-dimensional linear systems (Kaczorek, 1993) is also possible, but it is not trivial.

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Received: 25 May 2003
Revised: 8 July 2003