Int. J. Appl. Math. Comput. Sci., 2006, Vol. 16, No. 1, 115–127

NONLINEAR DIAGNOSTIC FILTER DESIGN: ALGEBRAIC AND GEOMETRIC POINTS OF VIEW

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The problem of diagnostic filter design is studied. Algebraic and geometric approaches to solving this problem are investigated. Some relations between these approaches are established. New definitions of fault detectability and isolability are formulated. On the basis of these definitions, a procedure for diagnostic filter design is given in both algebraic and geometric terms.

Keywords: diagnostic filter, nonlinear systems, geometric approach, algebraic approach, fault detection and isolation, observers

1. Introduction

An existing approach to maintaining fault tolerance and safety for critical purpose systems consists in timely detection and isolation of faults followed by system accommodation. Therefore, the design of such systems necessitates on-line fault detection and isolation (FDI) methods.

Numerous methods of FDI have been proposed within the scope of the analytical redundancy concept (Chow and Willsky, 1984). According to this idea, FDI includes residual generation as a result of a mismatch between the system behaviour and its reference model behaviour and, then, decision making based on the evaluation of the residual. This paper is concentrated only on first stage, i.e. on the residual generation.

The methods used for residual generation are based on closed-loop (diagnostic observers or filters) and open-loop (parity relations) techniques. This paper deals with the problem of diagnostic filter design for nonlinear dynamic systems. By definition (Alcorta-Garcia and Frank, 1997; Frank, 1990; 1996), the diagnostic filter is an observer (or a bank of observers) whose output (residual) is structured according to faults arising in the system under monitoring. Up to now, several approaches to diagnostic filter design have been developed; in this paper our attention will be concentrated on geometric and algebraic approaches.

In the framework of the geometric approach, a solution to the diagnostic filter design problem was first proposed by Massoumnia (1986) and Massoumnia *et al.* (1989) for linear systems. Later, this solution was developed for nonlinear systems by De Persis and Isidori (2001) and, then, by Join *et al.* (2002a; 2002b). Also, in (Edelmayer *et al.*, 2004), nonlinear system inversion techniques were considered for diagnostic filter design within the scope of the geometric approach.

Using the Lie algebra, a solution to the diagnostic filter design problem was obtained by Frank and Ding (1997) in another manner: the result involves the so-called unknown input observer approach.

For linear systems, an algebraic approach based on the Kronecker canonical form of the system under consideration was developed by Mironovskii (1980) and Frank (1990). In the nonlinear case, the algebraic approach considered in the present paper is based on the algebra of functions, which is an extension of the pare algebra proposed by Hartmanis and Stearns (1996) for finite automata. In contrast to the former algebra, whose constructions are determined on the set of partitions, the algebra of functions uses the constructions determined on the set of vector functions. The main feature of this algebra is the possibility to obtain algorithms that are similar for both discrete- and continuous-time nonlinear systems (Zhirabok and Shumsky, 1993a; 1993b).

The algebra of functions was first proposed for fault detection in nonlinear systems by Zhirabok and Shumsky (1987). Then, this algebra was developed for solving various diagnostic tasks (Shumsky, 1988; 1991; Zhirabok, 1997) and for nonlinear systems (Zhirabok and Shumsky, 1993a; 1993b).

Independently of the approach in use, fault diagnostic filter design involves a full decoupling problem. The solution of this problem in differential geometric terms can be reduced to finding some controllability conditioned invariant (or (h, f) invariant) distributions (De Persis and Isidori, 2001). On the other hand, in the framework of the algebraic approach, one concentrates on finding "special" vector functions (Shumsky, 1991), which play the same role as these distributions (it will be shown below that the annihilator of the controllability (h, f) invariant distribution is spanned by the exact differential of the above "special" vector function).

The aim of this paper is to consider the connection that exists between the algebraic and geometric approaches and, then, to propose a design algorithm for affine systems involving new conditions of fault detectability and isolability. A conference version of this paper is (Shumsky and Zhirabok, 2005).

The paper is organised as follows: Section 2 describes the problem in detail. It starts with the specification of the nonlinear dynamic system under diagnosis. Then, definitions of strong/weak detectability and isolability are introduced and a new form for the matrix specifying the structure of the residual (the so-called fault syndrome matrix) is proposed. After this, using Petrov's two-channel principle as a starting point, an approach to diagnostic filter design is formulated and some defining equations are given. Section 3 is devoted to the algebraic approach. First, a brief description of algebraic tools in use is given. Then, involving these tools, the solution of the full decoupling problem is considered and the way of constructing the fault syndrome matrix is discussed. In Section 4, a geometric interpretation of the algebraic approach is given. At the beginning of this section, the connection existing between algebraic and geometric tools is investigated for affine systems. After this, the ultimate designing procedure for diagnostic filter is formulated in geometric terms. An example is considered in Section 5. Section 6 concludes the paper.

2. Problem Description and Preliminary Results

Consider the system

$$\dot{x}(t) = f\big(x(t), u(t), \vartheta(t)\big), \tag{1}$$

$$y(t) = h\big(x(t)\big), \tag{2}$$

where $x(t) \in X \subseteq \mathbb{R}^n$ is the state vector, $u(t) \in U \subseteq \mathbb{R}^m$ is the control vector, $y(t) \in Y \subseteq \mathbb{R}^l$ is the measurable output vector, $\vartheta(t) \in \mathbb{R}^s$ is the vector of parameters, f and h are nonlinear vector functions assumed to be smooth for $x(t)$ and $\vartheta(t)$. It is also assumed that f is such that a solution of (1) exists for every initial state $x(t_0)$, and for a faultless system $\vartheta(t) = \vartheta^0$ holds for every t, where ϑ^0 is a given nominal value of the parameter vector.

The set of faults considered for the design of the diagnostic filter is specified by a list of faults $\{\rho_1, \rho_2, \ldots, \rho_d\}$, $d \geq s$. Single and multiple faults are distinguished. It is assumed that every single fault ρ_i, $i = 1, 2, \ldots, s$, results in unknown time behaviour of the appropriate parameter $\vartheta_i(t)$ such that $\vartheta_i(t) \neq \vartheta_i^0$. A multiple fault is considered as a collection of single faults occurring simultaneously. Notice that this representation of faults corresponds not only to actuator or plant faults, but also to sensor faults, considered as pseudoactuator faults, see e.g. (Massoumnia *et al.*, 1989; Park *et al.*, 1994).

To detect and isolate faults in the system (1), (2), a diagnostic filter in the form of a bank of reduced-order nonlinear observers is involved. Every observer generates the appropriate subvector of the residuals $r^{(j)}$, $j = 1, 2, \ldots, q$, and the residual vector r is composed of these subvectors.

Usually, see e.g. (Gertler and Kunwer, 1993), the structure properties of the residual vector are characterized by the binary matrix S of fault syndromes (FS) with the elements $S_{ji} = 1$ if the subvector $r^{(j)}$ is sensitive to the single fault ρ_i, otherwise (if $r^{(j)}$ is insensitive to ρ_i) $S_{ji} = 0$, $j = 1, 2, \ldots, q$ and $i = 1, 2, \ldots, s$. Various ways of choosing the FS matrix were discussed (Chen and Patton, 1994; Gertler and Kunwer, 1993). In the case of a square matrix $\boldsymbol{S}$ ($q = s$), it was shown that a diagonal structure of this matrix guarantees the isolation of multiple faults but imposes strong demands on the system. Also, the matrix with zeros only on its diagonal allows isolating only single faults, but gives more possibilities for the design.

The notion of sensitivity to a given fault looks trivial and means that if the fault distorts the state vector at some instant of time, then an appropriate residual subvector or, at least, some of its time derivatives are nonzero at the same instant of time. But as soon as the state vector is directly unobservable, while the output vector is directly measurable, it is reasonable to reformulate the above notion in the following manner: Let t_0 be an instant of time when the fault ρ_i results in a distortion of the system output. The subvector $r^{(j)}$ is called sensitive to the fault ρ_i if for some $\tau \geq 0$ we have $r^{(j)}(t_0 + \tau) \neq 0$. For nonlinear systems, the delay τ between the first distortion of the system output due to the fault ρ_i and the instant of time when the subvector $r^{(j)}$ takes a nonzero value depends on control and may be significant (or even infinite), which prevents making the decision on time. As a result, in the nonlinear case, characteristics of the residual structure become more exhaustive when using, instead of the term "sensitivity", the term "detectability" of the fault via the residual subvector, drawing a distinction between weak and strong detectabilities.

Definition 1. The fault ρ_i is called *weakly detectable* via the residual $r^{(j)}$ if there exist a state $x(t_0)$, a finite time interval $T = [t_0, t]$, $t_0 < t$, and control $u(\tau) \in U$, $\tau \in [t_0, t]$, such that $r^{(j)}(t) \neq 0$.

Definition 2. The fault ρ_i is called *strongly detectable* via the residual $r^{(j)}$ if $r^{(j)}(t_0) \neq 0$.

As soon as the notions of weak and strong detectabilities are introduced, the elements of the FS matrix take three values: $S_{ji} = 1$ if the fault ρ_i is strongly detectable via the residual $r^{(j)}$, $S_{ji} = 0$ if $r^{(j)}$ is insensitive to the fault ρ_i, and $S_{ji} = z$ if the fault ρ_i is weakly detectable via the residual $r^{(j)}$. Now, the FS matrix is constructed not only for single faults, but for multiple ones, too. It is also worth introducing the following definitions of weak and strong fault distinguishability and isolability.

Definition 3. The faults ρ_i and ρ_j are called *weakly (strongly) distinguishable* if the corresponding columns of the FS matrix do not coincide under $z = 1$ $(z = 0)$.

Definition 4. The faults $\rho_1, \rho_2, \ldots, \rho_d$ are called *weakly (strongly) isolable* if no two columns of the FS matrix coincide under $z = 1$ $(z = 0)$.

Weak distinguishability (isolability) of faults means that these faults (all faults) are distinguishable (isolable) under some "favourable" control. In contrast to this, strong distinguishability (isolability) means that these faults are distinguishable (isolable) under arbitrary control.

The key problem of finding the FS matrix for a given system and the set of faults is related to solving two tasks: (i) fully decoupling effects of faults in the output space of a diagnostic filter and (ii) analysing fault detectability via subvectors of the residual.

The idea of full decoupling is based on the compensation of fault effects in the output space of the observer. If no assumption is made about the time behaviour of the system parameters affected by faults, such a compensation is possible only if there exist at least two different ways (channels) of fault effect propagation (Petrov's two channels principle). According to Fig. 4 in the survey (Frank, 1990), the first channel is the actual system; the second is the unfaulty model with the feedback gain matrix or a residual observer-based generator.

To illustrate the way of implementing this principle in the framework of the problem under consideration, consider the structure interpretation given in Fig. 1 (Shumsky, 1991). Notice that this interpretation is considered in the survey (Alcorta-Garcia and Frank, 1997) as one of the ways to solve the full decoupling problem.

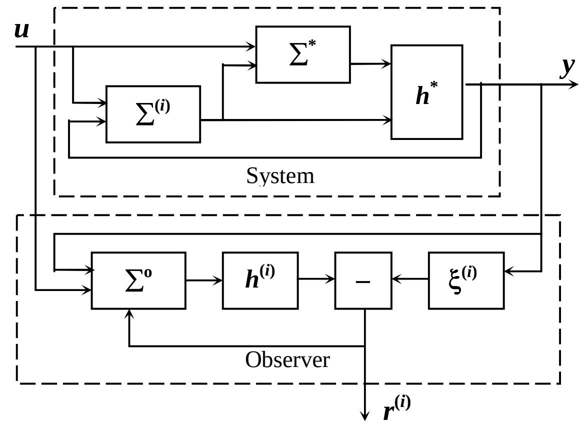

Fig. 1. Structure interpretation of observer-based residual generation involving the two-channel principle.

In Fig. 1, the system (1), (2) is decomposed into two subsystems $\Sigma^{(i)}$, Σ^* and the function h^*, which are specified as follows:

$$\Sigma^{(i)}: \quad \dot{x}^{(i)}(t) = f^{(i)}\big(x^{(i)}(t),\, y(t),\, u(t),\, \vartheta^{(i)}(t)\big), \tag{3}$$

$$\Sigma^{*}: \quad \dot{x}^{*}(t) = f^{*}\big(x^{*}(t),\, x^{(i)}(t),\, u(t), \vartheta(t)\big), \tag{4}$$

$$h^{*}: \quad h^{*}\big(x^{*}(t), x^{(i)}(t)\big) = h\big(x(t)\big), \tag{5}$$

where $\vartheta^{(i)}$ is some subvector of ϑ unaffected by the fault ρ_i. The observer in Fig. 1 has the following description:

$$\Sigma^{o}: \quad \dot{x}^{(o)}(t) = f^{(i)}(x^{(o)}(t), y(t),\, u(t),\, \vartheta^{(i,0)}) + G\big(x^{(o)}(t), y(t),\, u(t)\big) r^{(i)}(t), \tag{6}$$

$$r^{(i)}: \quad r^{(i)}(t) = h^{(i)}\big(x^{(o)}(t)\big) - \xi^{(i)}\big(y(t)\big), \tag{7}$$

with the functions $\xi^{(i)}$, $h^{(i)}$ satisfying the equality

$$\xi^{(i)}\big(h^{*}\big(x^{*}(t), x^{(i)}(t)\big)\big) = h^{(i)}\big(x^{(i)}(t)\big). \tag{8}$$

In (6), $\vartheta^{(i,0)}$ denotes a nominal value of the subvector $\vartheta^{(i)}$, and G is the gain matrix function.

In Fig. 1, the subsystems $\Sigma^{(i)}$, Σ^* and the functions h^*, $\xi^{(i)}$ pertain to the first channel, while the subsystem Σ^o and the function $h^{(i)}$ belong to the second channel. To explain this scheme, let $x^{(o)}(0) = x^{(i)}(0)$. Consider first the fault-free case when $\vartheta(t) = \vartheta^0$ for all t. Notice that from (2), (5), (7), and (8) it follows that $r^{(i)}(0) = 0$. Since $\vartheta^{(i)}(t) = \vartheta^{(i,0)}$ for all t, descriptions of the channels coincide, which automatically results in $r^{(i)}(t) = 0$ for all t. Then, because $\vartheta^{(i)}$ is unaffected by the fault ρ_i (i.e. $\vartheta^{(i)}(t) = \vartheta^{(i,0)}$ for all t), the equality $r^{(i)}(t) = 0$ also holds in the presence of this fault.

Now let $x^{(o)}(0) \neq x^{(i)}(0)$, and assume that there is no fault in the system. The design of an asymptotically stable observer with the property that $t \to \infty$ implies $x^{(o)}(t) - x^{(i)}(t) \to 0$ $(r^{(i)}(t) \to 0)$ involves the appropriate choice of the gain matrix function. The above problem has been extensively studied (see, e.g., the survey (Misawa and Hedrick, 1989) and the papers by Birk and Zeitz (1988), Ding and Frank (1990), Gauthier and Kupca (2000)). This is the reason for concentrating below only on the problem of finding the functions $f^{(i)}$, $h^{(i)}$, $\xi^{(i)}$, $i = 1, 2, \dots, q$ assuming that $x^{(o)}(0) = x^{(i)}(0)$.

According to Shumsky (1991), the solution to the above problem is based on the following assumption: there exists a global coordinate transformation given by a smooth vector function $\alpha^{(i)}$ such that for the faultless system and every t we have

$$x^{(i)}(t) = \alpha^{(i)}\big(x(t)\big), \quad x(t) \in X. \tag{9}$$

Using (1), (3), and (9), we obtain the defining equation for $f^{(i)}$:

$$f^{(i)}\big(\alpha^{(i)}(x), h(x), u, \vartheta^{(i)}\big) = \frac{\partial \alpha^{(i)}}{\partial x} f(x, u, \vartheta), \tag{10}$$

where $\partial\alpha/\partial x$ is the functional (Jacobi) matrix:

$$\frac{\partial \alpha}{\partial x} = \begin{bmatrix} \partial\alpha_1/\partial x_1 & \partial\alpha_1/\partial x_2 & \dots & \partial\alpha_1/\partial x_n \\ \partial\alpha_2/\partial x_1 & \partial\alpha_2/\partial x_2 & \dots & \partial\alpha_2/\partial x_n \\ \vdots & \vdots & \vdots & \vdots \\ \partial\alpha_p/\partial x_1 & \partial\alpha_p/\partial x_2 & \dots & \partial\alpha_p/\partial x_n \end{bmatrix},$$

p is a number of the components of the vector function α.

Because $\vartheta^{(i)}$ is unaffected by the fault ρ_i, from (10) it follows that

$$\begin{aligned} &\frac{\partial f^{(i)}}{\partial \vartheta_k}\big(\alpha^{(i)}(x), h(x), u, \vartheta^{(i)}\big) \\ &\quad= \frac{\partial}{\partial \vartheta_k}\Big(\frac{\partial \alpha^{(i)}}{\partial x} f(x, u, \vartheta)\Big) \\ &\quad= \frac{\partial \alpha^{(i)}}{\partial x}\Big(\frac{\partial f}{\partial \vartheta_k}(x, u, \vartheta)\Big) = 0 \end{aligned} \tag{11}$$

for every ϑ_k subjected to the distortion by this fault. Vice versa, if (11) holds, then $\vartheta^{(i)}$ is unaffected by the fault ρ_i. Then, from (2), (5), (8), and (9) we also obtain the defining equation for $h^{(i)}$:

$$h^{(i)}\big(\alpha^{(i)}(x)\big) = \xi^{(i)}\big(h(x)\big). \tag{12}$$

Thus, the functions $f^{(i)}$ and $h^{(i)}$ are found from (10) with $\vartheta^{(i)} = \vartheta^{(i,0)}$, $\vartheta = \vartheta^0$, and (12), respectively, under the known functions $\alpha^{(i)}$ and $\xi^{(i)}$. This is why in the next section attention is focused on finding the functions $\alpha^{(i)}$, $\xi^{(i)}$ and studying their properties, taking into account both the solvability condition for (10)–(12) and the demands imposed on a structure of the FS matrix by the set of faults.

3. Algebraic Approach

In this section, the algebra of functions is used for solving the general problem of finding $\alpha^{(i)}$ and $\xi^{(i)}$ for every $i = 1, 2, \dots, q$, and determining the FS matrix.

3.1. Algebraic Tools

Denote by $\Im_S$ the set of smooth vector functions with the domain S. For $\alpha, \beta \in \Im_S$, the partial preordering relation $\leq$ is defined as follows: $\alpha \leq \beta$ if and only if there exists some differentiable function γ, determined on the set of values of α such that $\beta = \gamma \circ \alpha$, where '$\circ$' is the symbol of composition. To verify if $\alpha \leq \beta$, one can check the equality of ranks for the functional matrices $J_\alpha(s) = \partial\alpha(s)/\partial s$ and $J_{\alpha\times\beta}(s) = \partial(\alpha(s) \times \beta(s))/\partial s$:

$$\alpha \leq \beta \Leftrightarrow \operatorname{rank} J_\alpha(s) = \operatorname{rank} J_{\alpha\times\beta}(s), \quad \forall s \in S,$$

where the symbol '$\times$' is used to simplify (but not only) the notation for the composite vector function, namely, $\alpha \times \beta = (\alpha^{\mathsf T}, \beta^{\mathsf T})^{\mathsf T}$, and '$\mathsf T$' is the symbol of transposition. If $\alpha \leq \beta$ and $\beta \leq \alpha$, then α and β are called equivalent: $\alpha \sim \beta$. Thus, the relation $\sim$ splits the set $\Im_S$ into equivalent function classes.

Every function $\alpha \in \Im_S$ specifies the equivalence E_α on S: $(s^1, s^2) \in E_\alpha \Leftrightarrow \alpha(s^1) = \alpha(s^2)$. The relation E_α defines the appropriate partition of S. One can easily see that equivalent functions give the same partitions of S. Moreover, if E_α and E_β represent equivalence corresponding to the functions α and β, then

$$[\alpha \leq \beta] \Leftrightarrow \big[(s^1, s^2) \in E_\alpha \Rightarrow (s^1, s^2) \in E_\beta,\ \forall s^1, s^2 \in S\big].$$

Therefore, the set of equivalent function classes corresponds to the partial ordering set of partitions of S, and the first is a grid with zero, given by an arbitrary one-to-one function (in particular, the identity function $i(s) = s$, $\forall s \in S$), and unity, given by arbitrary constant function ($c(s) = \mathrm{Const}$, $\forall s \in S$).

The operations $\times$ and $\oplus$ are defined as follows:

$$\begin{aligned} &\big[\alpha \times \beta \in \Im_S\big]\&\big[\alpha \times \beta \leq \alpha,\ \alpha \times \beta \leq \beta\big] \\ &\qquad\qquad \&\big[\gamma \leq \alpha,\ \gamma \leq \beta \Rightarrow \gamma \leq \alpha \times \beta\big], \end{aligned}$$

$$\begin{aligned} &\big[\alpha \oplus \beta \in \Im_S\big]\&\big[\alpha \leq \alpha \oplus \beta,\ \beta \leq \alpha \oplus \beta\big] \\ &\qquad\qquad \&\big[\alpha \leq \gamma,\ \beta \leq \gamma \Rightarrow \alpha \oplus \beta \leq \gamma\big]. \end{aligned}$$

From these definitions it follows that the function $\alpha\times\beta$ is a maximum bottom of the functions α and β, and $\alpha\oplus\beta$ is a minimal top of those. Therefore, the operations $\times$ and $\oplus$ defined on the set $\Im_S$ correspond to the product and the sum of the partitions of S specified by the functions α and β, respectively.

For a healthy system (1), the relation $\Delta\subset\Im_X\times\Im_X$ is introduced as follows:

$$\big[(\alpha,\beta)\in\Delta\big]\Leftrightarrow\big[\pi_u\times\alpha\circ\pi_x\leq J_\beta f\big],$$

where $\pi_u, \pi_u(x,u)=u$ and $\pi_x, \pi_x(x,u)=x$ are projections.

Some useful properties of the partial preordering relation $\leq$, the relation Δ, and the operations $\times$ and $\oplus$ are given below:

(i) $\alpha\leq\beta\Rightarrow\alpha\times\gamma\leq\beta\times\gamma$,

(ii) $\alpha\leq\beta\Rightarrow\alpha\oplus\gamma\leq\beta\oplus\gamma$,

(iii) $\alpha\leq\beta\Leftrightarrow\alpha\oplus\beta\sim\beta$,

(iv) $\alpha\sim\beta\Rightarrow\alpha\times\gamma\sim\beta\times\gamma$,

(v) $\big[(\alpha,\,\beta)\in\Delta\ \&\ \gamma\leq\alpha\big]\Rightarrow(\gamma,\,\beta)\in\Delta$.

Notice that the properties (i)–(iv) follow immediately from grid theory, see, e.g. (Hartmanis and Stearns, 1966): the property (v) was proved in (Zhirabok and Shumsky, 1987).

3.2. Fault Decoupling

Our task is now to describe the properties of the functions $\alpha^{(i)}$ and $\xi^{(i)}$, $i=1,2,\ldots,q$, with a language of the algebra of functions and to give a procedure for finding them. To solve this task, introduce the vector function $\alpha^{(i,0)}$ such that

$$\frac{\partial\alpha^{(i,0)}}{\partial x}\Big(\frac{\partial f}{\partial\vartheta_k}(x,u,\vartheta)\Big)=0 \tag{13}$$

for every ϑ_k subjected to a distortion by the fault ρ_i and $\alpha^{(i,0)}\leq\alpha^{(i)}$ for every function $\alpha^{(i)}$ satisfying (11). Thus, $\alpha^{(i,0)}$ forms a basis for the functions satisfying the condition (11). The solvability condition for (10)–(12) is given by the following theorem (Shumsky, 1991):

Theorem 1. *Equations (10)–(12) are solvable if and only if*

$$\big(h\times\alpha^{(i)},\alpha^{(i)}\big)\in\Delta,\quad \alpha^{(i,0)}\leq\alpha^{(i)}, \tag{14}$$

$$\alpha^{(i)}\leq\xi^{(i)}\circ h. \tag{15}$$

Proof. (Sufficiency) Let $(h\times\alpha^{(i)},\alpha^{(i)})\in\Delta$. According to the definition of the relation Δ, the inequality $\pi_u\times(h\times\alpha^{(i)})\circ\pi_x\leq J_{\alpha^{(i)}}f$ holds. From this functional inequality, according to the definition of the partial preordering relation, one can find some function, denoted by $f^{(i)}$, such that (10) holds. Then, from the functional inequality $\alpha^{(i,0)}\leq\alpha^{(i)}$ one can write $\alpha^{(i)}=\gamma\circ\alpha^{(i,0)}$ for some vector function γ defined on the set of $\alpha^{(i,0)}$ values. Differentiating both the sides of the above equality for t and ϑ_k, from (13) it follows that

$$\frac{\partial\alpha^{(i)}}{\partial x}\Big(\frac{\partial f}{\partial\vartheta_k}(x,\,u,\,\vartheta)\Big) = \frac{\partial\gamma}{\partial\alpha^{(i,0)}}\frac{\partial\alpha^{(i,0)}}{\partial x}\frac{\partial f}{\partial\vartheta_k}(x,u,\vartheta)=0.$$

Hence, (11) also holds. Concluding the proof of sufficiency, consider the functional inequality (15). According to the definition of partial preordering, one can find some function, denoted by $h^{(i)}$, such that (12) holds.

(Necessity) For a given function $\alpha^{(i)}$, let (10) and (12) be solvable. In this case, the functional inequality (15) follows immediately from (12). Then (10) results in the functional inequality $\pi_u\times(h\times\alpha)\circ\pi_x\leq J_\alpha f$, and the inclusion $(h\times\alpha^{(i)},\alpha^{(i)})\in\Delta$ follows immediately. The functional inequality $\alpha^{(i,0)}\leq\alpha^{(i)}$ is a result of the above assumption about $\alpha^{(i,0)}$. ■

The next theorem gives a regular rule for finding the minimal function $\alpha^{(i)}$ satisfying (14), and constitutes a modified version of the theorem proposed by Shumsky (1991). Note that this function corresponds to the subsystem $\Sigma^{(i)}$ of a maximal dimension, and this subsystem is free from the fault ρ_i.

Theorem 2. *Let $\alpha^{(i,j)}$, $j=0,1,2,\ldots$, be a sequence of functions satisfying the conditions*

(i) $\big(\alpha^{(i,j)}\leq\alpha^{(i,j+1)}\big)\&\big(h\times\alpha^{(i,j)},\ \alpha^{(i,j+1)}\big)\in\Delta$,

(ii) $\big(\alpha^{(i,j)}\leq\beta\big)\&\big(h\times\alpha^{(i,j)},\beta\big)\in\Delta\Rightarrow\alpha^{(i,j+1)}\leq\beta$,

and suppose that there exists a natural number k such that $\alpha^{(i,k+1)}\sim\alpha^{(i,k)}$. Then the function $\alpha^{(i,k)}$ satisfies (14), and for every function $\alpha^{(i)}$ satisfying (14) we have

$$\alpha^{(i,k)}\leq\alpha^{(i)}. \tag{16}$$

Proof. To show that the function $\alpha^{(i,k)}$ satisfies (14), it is sufficient to write $(h\times\alpha^{(i,k)},\ \alpha^{(i,k+1)})\in\Delta$ and substitute $\alpha^{(i,k)}$ for $\alpha^{(i,k+1)}$ on the right-hand side of this relation using the properties (iv) and (v) of the relation Δ. Let the function $\alpha^{(i)}$ satisfy (14). Observe that $\alpha^{(i,0)}\leq\alpha^{(i)}$. From the property (v) of the relation Δ and the implication (ii), it follows that $(h\times\alpha^{(i,0)},\ \alpha^{(i)})\in\Delta$ and $\alpha^{(i,1)}\leq\alpha^{(i)}$. By analogy, $\alpha^{(i,2)}\leq\alpha^{(i)},\ldots,\alpha^{(i,k)}\leq\alpha^{(i)}$. ■

So, one can let $\alpha^{(i,k)} = \alpha^{(i)}$ for the minimal function $\alpha^{(i)}$, $i = 1, 2, \ldots, q$. The above theorem implies the following result:

Corollary 1. *There holds*

$$\alpha^{(i,k)} \oplus h \leq \xi^{(i)} \circ h. \tag{17}$$

Indeed, from (15) and (16) we have that $\alpha^{(i,k)} \leq \xi^{(i)} \circ h$. Since $h \leq \xi^{(i)} \circ h$, from the properties (ii) and (iii) of the operation $\oplus$ it follows that $\alpha^{(i,k)} \oplus h \leq (\xi^{(i)} \circ h) \oplus h \sim \xi^{(i)} \circ h$.

Theorems 1 and 2 result in the following algorithm for finding the functions $\alpha^{(i)}$ and $\xi^{(i)}$ such that the residual subvector $r^{(i)}$ is insensitive to the fault ρ_i:

Algorithm 1.

1. From Eqn. (13) find the function $\alpha^{(i,0)}$ with a maximum number of functionally independent components .
2. Find the function $\alpha^{(i,k)}$ using the rule of Theorem 2 and let $\alpha^{(i,k)} = \alpha^{(i)}$.
3. Find the function $\xi^{(i)}$ satisfying the relation

$$\xi^{(i)} \circ h \sim \alpha^{(i)} \oplus h. \tag{18}$$

Remark 1. The relation (18) characterizes the minimal function $\xi^{(i)}$ satisfying (17). Indeed, if some function β satisfies (17), i.e., $\alpha^{(i,k)} \oplus h \leq \beta \circ h$, then $\xi^{(i)} \circ h \leq \beta \circ h$ due to the relation (18), or $\gamma \circ \xi^{(i)} \circ h = \beta \circ h$ for some function γ. As a rule, the function h is onto Y and, therefore, the last equality implies $\gamma \circ \xi^{(i)} = \beta$, or $\xi^{(i)} \leq \beta$.

The implementation of Algorithm 1 needs performing the operation $\oplus$ and a special algebraic operator. If necessary, we can define rules for their calculation in (Zhirabok and Shumsky, 1993a). In Section 4 the appropriate rules are given in geometric terms for affine systems.

So, using Algorithm 1, we obtain the functions $\alpha^{(i)}$ and $\xi^{(i)}$ for every fault ρ_i, $i = 1, 2, \ldots, d$.

3.3. Detectability Analysis and FS Matrix Construction

Let the inequality $\alpha^{(i)} \leq \xi^{(j)} \circ h$ be true. According to Theorem 1 this means that the residual subvector $r^{(j)}$ is insensitive to the fault ρ_i. But as soon as (15) holds, we can write the implication $\xi^{(i)} \leq \xi^{(j)} \Rightarrow \alpha^{(i)} \leq \xi^{(j)} \circ h$. Therefore, the inequality

$$\xi^{(i)} \leq \xi^{(j)} \tag{19}$$

constitutes the sufficient condition for residual subvector $r^{(j)}$ insensitivity to the fault ρ_i.

If (19) does not hold, then $r^{(j)}$ is sensitive to the fault ρ_i. Thus, a violation of (19) is a condition of weak or strong detectability of ρ_i via the residual subvector $r^{(j)}$.

Remark 2. To check if (19) is violated, it is sufficient to prove the following rank condition for some $y \in Y$:

$$\operatorname{rank} \frac{\partial(\xi^{(i)} \times \xi^{(j)})}{\partial y} > \operatorname{rank} \frac{\partial \xi^{(i)}}{\partial y}. \tag{20}$$

Theorem 3. *The fault ρ_i is strongly detectable via the residual $r^{(j)}$, $j \neq i$, if*

$$\xi^{(i)} \times \xi^{(j)} i_Y, \tag{21}$$

where i_Y is the identity function with the domain Y.

Proof. Let $y^{(i)}(t)$ denote an output of the system with the fault ρ_i and t_0 be an instant of the first output distortion, i.e. $y^{(i)}(t_0) \neq y(t_0)$. Consider first the observer insensitive to the fault ρ_i. As soon as the vector $y(t)$ appears on the right-hand side of the equation for the derivatives $\dot{x}^{(i)}$ and $\dot{x}^{(o)}$ (see (3) and (6)), only these derivatives are distorted by the fault ρ_i at $t = t_0$ whereas the variables $x^{(i)}$ and $x^{(o)}$ are unaffected by this fault at $t = t_0$. Therefore, $\xi^{(i)}(y^{(i)}(t_0)) = \xi^{(i)}(y(t_0))$ and from (21) we have $\xi^{(j)}(y^{(i)}(t_0)) \neq \xi^{(j)}(y(t_0))$. Consider now the observer insensitive to the fault ρ_j. Because its state vector $x^{(o)}(t)$ is also unaffected by the fault ρ_j at $t = t_0$ and, as a result, $h^{(j)}(x^{(o)}(t_0)) = \xi^{(j)}(y(t_0))$, from the above we get $r^{(j)}(t_0) = h^{(j)}(x^{(o)}(t_0)) - \xi^{(j)}(y^{(i)}(t_0)) \neq 0$. ∎

Remark 3. To check if (21) holds, it is sufficient to prove the following rank condition:

$$\operatorname{rank} \frac{\partial(\xi^{(i)} \times \xi^{(j)})}{\partial y} = l, \quad \forall y \in Y. \tag{22}$$

The primary FS $d \times d$ matrix is constructed as follows: The diagonal elements of this matrix are zero because the residual subvector $r^{(i)}$ is insensitive to the fault ρ_i by definition. Then, applying the conditions (20) and (22), we write the (j, i)-nondiagonal elements of this matrix: it is unity whenever (22) is true, "z" when only (20) holds, and zero if (20) fails (or, which is the same, (19) holds). Involving the primary FS matrix, fault isolability is investigated. The final FS matrix is obtained by excluding the redundant rows (i.e. rows whose excluding does not influence fault isolability).

4. Geometric Interpretation

In this section, the connection between algebraic and geometric tools is investigated for nonlinear systems, whose dynamics are affine in the control and the fault action:

$$\dot{x}(t) = f\big(x(t)\big) + g\big(x(t)\big)u(t) + w\big(x(t)\big)\vartheta(t), \quad (23)$$

where $g(x)$ and $w(x)$ are smooth matrix functions of appropriate dimensions.

4.1. Functions, Codistributions and Distributions

For the vector function $\alpha \in \Im_S$, the codistribution Ω_α is introduced as follows: $\Omega_\alpha = \operatorname{span}\{J_{\alpha_i}(s), 1 \le i \le p_\alpha\}$, where $J_{\alpha_i}(s)$ is the i-th row of the Jacobian matrix $J_\alpha(s)$, and p_α is the dimension of the vector function α.

Let $\alpha, \beta \in \Im_S$. It is easy to see that $\alpha \le \beta$ iff $\Omega_\alpha \supseteq \Omega_\beta$. Indeed, if $\alpha \le \beta$, then $\beta = \gamma \circ \alpha$ for some differentiable function β and

$$J_{\beta_i} = \frac{\partial \beta_i}{\partial x} = \frac{\partial \gamma}{\partial \alpha} J_\alpha = \sum_{i=1}^{p_\alpha} \frac{\partial \gamma}{\partial \alpha_i} J_{\alpha_i}, \quad 1 \le i \le p_\beta,$$

which implies

$$\begin{aligned}\Omega_\beta &= \operatorname{span}\big\{J_{\beta_i}(s),\ 1 \le i \le p_\beta\big\} \subseteq \Omega_\alpha \\ &= \operatorname{span}\big\{J_{\alpha_i}(s),\ 1 \le i \le p_\alpha\big\}.\end{aligned}$$

Conversely, if $\Omega_\alpha \supseteq \Omega_\beta$, then $J_\beta(s) = C(s)J_\alpha(s)$, where $C(s)$ is an appropriate matrix function. Therefore,

$$\begin{aligned}\operatorname{rank} J_{\alpha\times\beta}(s) &= \operatorname{rank}\begin{bmatrix} J_\alpha(s) \\ J_\beta(s)\end{bmatrix} \\ &= \operatorname{rank}\begin{bmatrix} J_\alpha(s) \\ C(s)J_\alpha(s)\end{bmatrix} = \operatorname{rank} J_\alpha(s),\end{aligned}$$

which implies $\alpha \le \beta$. It also follows that $\alpha \sim \beta$ iff $\Omega_\alpha = \Omega_\beta$.

By the definition of the function $\alpha \times \beta$, the inequalities $\alpha \times \beta \le \alpha$ and $\alpha \times \beta \le \beta$ hold. Therefore, $\Omega_{\alpha\times\beta} \supseteq \Omega_\alpha$ and $\Omega_{\alpha\times\beta} \supseteq \Omega_\beta$. Since $\alpha \times \beta$ is a maximum bottom, the codistribution $\Omega_{\alpha\times\beta}$ is the minimal one that contains both codistributions Ω_α and Ω_β, i.e. $\Omega_{\alpha\times\beta} = \Omega_\alpha + \Omega_\beta$. By analogy, $\alpha \le \alpha \oplus \beta$ and $\beta \le \alpha \oplus \beta$ and, therefore, $\Omega_\alpha \supseteq \Omega_{\alpha\oplus\beta}$ and $\Omega_\beta \supseteq \Omega_{\alpha\oplus\beta}$. Since $\alpha \oplus \beta$ is a minimal top, the codistribution $\Omega_{\alpha\oplus\beta}$ is the maximum one included in the intersection of the codistributions Ω_α and Ω_β, i.e. $\Omega_{\alpha\oplus\beta} \subseteq \Omega_\alpha \cap \Omega_\beta$.

At a given point s, the intersection $\Omega_\alpha \cap \Omega_\beta$ can be found by solving the homogeneous equation

$$\sum_{i=1}^{p_\alpha} a_i(s) J_{\alpha_i}(s)^{\mathsf{T}} - \sum_{i=1}^{p_\beta} b_i(s) J_{\beta_i}(s)^{\mathsf{T}} = 0 \quad (24)$$

for the unknown functions $a_i(s)$, $1 \le i \le p_\alpha$, and $b_i(s)$, $1 \le i \le p_\beta$ (Isidori, 1989, p. 18). Because the codistribution $\Omega_{\alpha\oplus\beta}$ is spanned by the exact differential of the function $\alpha \oplus \beta$, the coefficient matrix $(a_1(s), a_2(s), \ldots, a_{p_\alpha}(s), \quad b_1(s), \quad b_2(s), \ldots, b_{p_\beta}(s))$ must additionally satisfy (Korn and Korn, 1961, p. 300 [1]):

$$\frac{\partial\big(a_i(s)J_{\alpha_i}(s)^T\big)}{\partial s_j} = \frac{\partial\big(a_j(s)J_{\alpha_j}(s)^T\big)}{\partial s_i}. \quad (25)$$

(A similar equation can be written for the coefficients $b_i(s)$, $1 \le i \le p_\beta$, and the function β.) The set of independent solutions of (24), (25) forms the basis for the codistribution $\Omega_{\alpha\oplus\beta}$.

In the geometric approach, codistributions are introduced as dual objects for distributions. In the case of affine systems, the description of the computational procedure for the functions $\alpha^{(i,j)}$, $j = 1, 2, \ldots, k$, from Theorem 2 in terms of distributions looks more effective than in algebraic terms.

Let Ω_α be a codistribution specified by $\Omega_\alpha = \Lambda_\alpha^\perp$, where Λ_α is some distribution and the symbol '$\perp$' is used for the annihilator. The basis for the codistribution Ω_α is formed by the exact differential J_α of the function α satisfying the condition

$$J_\alpha \lambda_\alpha = 0, \quad \forall \lambda_\alpha \in \Lambda_\alpha. \quad (26)$$

The procedure of finding the function α from (26) is known as the procedure of integrating the distribution Λ_α. Necessary and sufficient conditions for the integration of Λ_α (as well as the procedure of integration) are given by the Frobenius theorem (Isidori, 1989, p. 23). According to the Frobenius theorem, the distribution Λ_α is integrable iff it is involutive with respect to the operation of differentiating the Lie brackets.

4.2. Realization of the Geometric Approach

For the beginning, consider the first step of Algorithm 1, which deals with computing the function $\alpha^{(i,0)}$ according to (13). Denote by $w^{(i)}$ the matrix composed of the columns of the matrix w which correspond to the parameters affected by the fault ρ_i. According to (13), the function $\alpha^{(i,0)}$ is such that $(\partial\alpha^{(i,0)}/\partial x)w^{(i)} = 0$. Let $\Lambda_{\alpha^{(i,0)}}$ be the minimal involutive distribution spanned by $w^{(i)}$. In this case, the function $\alpha^{(i,0)}$ is found by the integration of $\Lambda_{\alpha^{(i,0)}}$.

To formulate the second step of Algorithm 1 in geometric terms, consider the construction $(h \times \alpha^{(i,j)}, \alpha^{(i,j+1)}) \in \Delta$ of Theorem 2. From the definition of the relation Δ it follows that $\pi_u \times (\alpha^{(i,j)} \times h) \circ \pi_x \le J_{\alpha^{(i,j+1)}} f$ for the system (1). For the system (23),

[1] The number of this page corresponds to the Russian edition.

the last inequality implies the inequalities $\alpha^{(i,j)} \times h \le J_{\alpha_k^{(i,j+1)}} f$ and $\alpha^{(i,j)} \times h \le J_{\alpha_k^{(i,j+1)}} g_v$ for every k, $1 \le k \le p_{\alpha^{(i,j+1)}}$, and v, $1 \le v \le m$, or, which is the same, $L_\varphi J_{\alpha_k^{(i,j+1)}} \subseteq \Omega_{\alpha^{(i,j)}} + \Omega_h$, $\varphi \in \{f, g_1, \dots, g_m\}$, where $L_\varphi J_{\alpha_k^{(i,j+1)}}$ denotes the Lie derivative of the covector field $J_{\beta_k^{(i,j+1)}}$ along the vector field φ.

Let $\Lambda_{\alpha^{(i,j)}}$ be a distribution such that $\Omega_{\alpha^{(i,j)}} = \Lambda^\perp_{\alpha^{(i,j)}}$. Let also $\omega^{(j)} \in \Lambda_{\alpha^{(i,j)}} \cap \ker J_h$. Clearly, $\langle L_\varphi J_{\alpha_k^{(i,j+1)}},\ \omega^{(j)}\rangle = 0$, where the symbol $\langle\cdot,\cdot\rangle$ denotes the inner product. As soon as $\alpha^{(i,j)} \le \alpha^{(i,j+1)}$, the inclusions $\Omega_{\alpha^{(i,j)}} \supseteq \Omega_{\alpha^{(i,j+1)}}$ and $\Omega^\perp_{\alpha^{(i,j)}} = \Lambda_{\alpha^{(i,j)}} \subseteq \Omega^\perp_{\alpha^{(i,j+1)}} = \Lambda_{\alpha^{(i,j+1)}}$ hold. As a result, if $\omega^{(j)} \in \Lambda_{\alpha^{(i,j)}}$, then $\omega^{(j)} \in \Lambda_{\alpha^{(i,j+1)}}$ and $\langle J_{\alpha_k^{(i,j+1)}}, \omega^{(j)}\rangle = 0$, $1 \le k \le p_{\alpha^{(i,j+1)}}$. Taking into account the well-known identity (Isidori, 1989, p. 10)

$$L_\varphi\langle J_{\alpha_k^{(i,j+1)}}, \omega^{(j)}\rangle = \langle L_\varphi J_{\alpha_k^{(i,j+1)}}, \omega^{(j)}\rangle + \langle J_{\alpha_k^{(i,j+1)}}, [\varphi, \omega^{(j)}]\rangle,$$

where $[\cdot,\cdot]$ denotes the Lie brackets, we obtains $\langle J_{\alpha_k^{(i,j+1)}}, [\varphi,\ \omega^{(j)}]\rangle = 0$.

Now, if $\Lambda_{\alpha^{(i,j+1)}}$ is a minimal involutive distribution containing $\Lambda_{\alpha^{(i,j)}} + \operatorname{span}\{[\varphi, \omega^{(j)}],\ \varphi \in \{f, g_1, \dots, g_m\},\ \omega^{(j)} \in \Lambda_{\alpha^{(i,j)}} \cap \ker J_h\}$, then the function $\alpha^{(i,j+1)}$ is found by the integration of $\Lambda_{\alpha^{(i,j+1)}}$. Indeed, because of $\Lambda_{\alpha^{(i,j+1)}} \supseteq \Lambda_{\alpha^{(i,j)}}$, the inequality $\alpha^{(i,j)} \le \alpha^{(i,j+1)}$ holds. Then the inclusion $\omega^{(j)} \in \Lambda_{\alpha^{(i,j)}} \cap \ker J_h$ implies $\langle J_{\alpha_k^{(i,j+1)}}, \omega^{(j)}\rangle = 0$ and $\langle J_{\alpha_k^{(i,j+1)}}, [\varphi, \omega^{(j)}]\rangle = 0$. From the above identity we obtain $\langle L_\varphi J_{\alpha_k^{(i,j+1)}}, \omega^{(j)}\rangle = 0$, which means $L_\varphi J_{\alpha_k^{(i,j+1)}} \subseteq \Omega_{\alpha^{(i,j)}} + \Omega_h$, or, which is the same, $(h \times \alpha^{(i,j)},\ \alpha^{(i,j+1)}) \in \Delta$. This proves the property (i) of Theorem 2.

The property (ii) of Theorem 2 follows immediately from constructing $\Lambda_{\alpha^{(i,j+1)}}$ as a minimal involutive distribution containing

$$\Lambda_{\alpha^{(i,j)}} + \operatorname{span}\Big\{[\varphi, \omega^{(j)}],\ \varphi \in \{f, g_1, \dots, g_m\},\ \omega^{(j)} \in \Lambda_{\alpha^{(i,j)}} \cap \ker J_h\Big\}.$$

Finally, if for some k one has $\Lambda_{\alpha^{(i,k+1)}} = \Lambda_{\alpha^{(i,k)}}$, it means that $\alpha^{(i,k+1)} \sim \alpha^{(i,k)}$.

Consequently, the computation of the function $\alpha^{(i)}$ needs the integration of the distribution $\Lambda_{\alpha^{(i,k)}}$ obtained by the application of the following recursive formula:

$$\Lambda_{\alpha^{(i,j+1)}} \supseteq \Lambda_{\alpha^{(i,j)}} + \operatorname{span}\Big\{[\varphi, \omega^{(j)}], \varphi \in \{f, g_1, \dots, g_m\}, \omega^{(j)} \in \Lambda_{\alpha^{(i,j)}} \cap \ker J_h\Big\},\quad 0 \le j \le k, \tag{27}$$

where $\Lambda_{\alpha^{(i,j+1)}}$ is a minimal involutive distribution. Note, that the same formula was used in (De Persis and Isidori, 2001) and it results in the so-called controllability (h, f) invariant distribution. An advantage of (27) (in spite of the appropriate calculating constructions of the algebraic approach) is the possibility of using only the integration of the distribution $\Lambda_{\alpha^{(i,k)}}$ to find the function $\alpha^{(i,k)}$, while the appropriate computational constructions of the algebraic approach (Zhirabok and Shumsky, 1993a) need solving nonhomogeneous partial differential equations at every step of the procedure given by Theorem 2 [2].

The third step of Algorithm 1 assumes calculating the functions $\xi^{(i)}$, $i = 1, 2, \dots, q$. A geometric realization of this step involves finding the codistributions $\Omega_{\xi^{(i)} \circ h}$, and is based on the solution of Eqns. (24) and (25) for the functions $\alpha^{(i,k)}$ and h to obtain the codistribution $\Omega_{\alpha^{(i,k)} \oplus h} = \Omega_{\xi^{(i)} \circ h}$.

To conclude this section, the detectability conditions are reformulated in geometric terms. From Remark 2 (cf. (20)) it follows that the fault ρ_i is weakly detectable via the residual subvector $r^{(j)}$ if

$$\operatorname{rank}\left(\Omega_{\xi^{(i)} \circ h} + \Omega_{\xi^{(j)} \circ h}\right) > \operatorname{rank} \Omega_{\xi^{(i)} \circ h}. \tag{28}$$

Note that we use the codistribution $\Omega_{\xi \circ h}$ instead of Ω_ξ to avoid calculation of Ω_ξ (this does not alter the rank of the codistribution). It also follows from Remark 3 (cf. (22)) that the fault ρ_i is strongly detectable via the residual subvector $r^{(j)}$ if

$$\operatorname{rank}\left(\Omega_{\xi^{(i)} \circ h} + \Omega_{\xi^{(j)} \circ h}\right) = l. \tag{29}$$

5. Example

Consider the system described by (23) and (2) with the matrix functions

$$f(x) = \begin{bmatrix} x_1 x_4 \\ x_3(1 - x_3) \\ 0 \\ 0 \end{bmatrix}, \quad h(x) = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \qquad g(x) = w(x) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & x_1 \\ 1 & 0 \end{bmatrix}. \tag{30}$$

The system (30) assumes only actuator faults. The structure of the system (30) is given in Fig. 2.

[2] To justify the algebraic approach, it is worth noticing that computational constructions of this approach allow us to solve the problem of finding the function not only for continuous-time systems which do not belong to the class of affine systems, but for discrete-time systems, too, see (Zhirabok and Shumsky, 1993a).

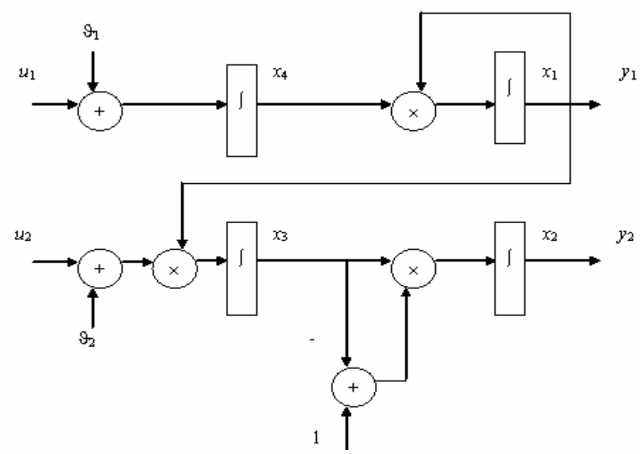

Fig. 2. Structure of the system (30).

Firstly, consider calculating the functions $\alpha^{(i)}$, $\xi^{(i)}$, $1 \le i \le 3$, and constructing the FS matrix. For the single faults we have $\Lambda_{\alpha^{(1,0)}} = \mathrm{span}\,\{w^{(1)}\}$, $\Lambda_{\alpha^{(2,0)}} = \mathrm{span}\,\{w^{(2)}\}$, and for the multiple fault $\Lambda_{\alpha^{(3,0)}} = \mathrm{span}\,\{w^{(1)},\ w^{(2)}\}$, where $w^{(i)}$ is the appropriate column of the matrix w. Making necessary calculations, from (27) we get

$$\Lambda_{\alpha^{(1,1)}}(x) = \Lambda_{\alpha^{(1,2)}}(x) = \mathrm{span}\left\{\begin{bmatrix} 0 & x_1 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}\right\},$$

$$\begin{aligned}\Lambda_{\alpha^{(2,1)}}(x) &= \Lambda_{\alpha^{(2,2)}}(x) \\ &= \mathrm{span}\left\{\begin{bmatrix} 0 & 0 \\ 0 & x_1(1-2x_3) \\ x_1 & -x_1x_4 \\ 0 & 0 \end{bmatrix}\right\},\end{aligned}$$

$$\begin{aligned}\Lambda_{\alpha^{(3,1)}}(x) &= \Lambda_{\alpha^{(3,2)}}(x) = \Lambda_{\alpha^{(1,1)}}(x) + \Lambda_{\alpha^{(2,1)}}(x) \\ &= \mathrm{span}\left\{\begin{bmatrix} 0 & x_1 & 0 & 0 \\ 0 & 0 & 0 & x_1(1-2x_3) \\ 0 & 0 & x_1 & -x_1x_4 \\ 1 & 0 & 0 & 0 \end{bmatrix}\right\}.\end{aligned}$$

Then, according to (24) and (25), we also obtain

$$\begin{aligned}\Omega_{\xi^{(1)}\circ h}(x) &= \Lambda^{\perp}_{\alpha^{(1,1)}}(x) \cap \mathrm{span}\,\{J_{h_i}(x),\ i = 1,2\} \\ &= \mathrm{span}\,\{0,\ 1,\ 0,\ 0\}, \\ \Omega_{\xi^{(2)}\circ h}(x) &= \Lambda^{\perp}_{\alpha^{(2,1)}}(x) \cap \mathrm{span}\,\{J_{h_i}(x),\ i = 1,2\} \\ &= \mathrm{span}\,\{1,\ 0,\ 0,\ 0\}, \\ \Omega_{\xi^{(3)}\circ h}(x) &= \Lambda^{\perp}_{\alpha^{(3,1)}}(x) \cap \mathrm{span}\,\{J_{h_i}(x),\ i = 1,2\} = 0.\end{aligned}$$

The functions $\alpha^{(i)}$, $1 \le i \le 3$ are obtained by integrating the distributions $\Lambda_{\alpha^{(i,1)}}(x)$:

$$\alpha^{(1)}(x) = \begin{bmatrix} x_2 \\ x_3 \end{bmatrix}, \quad \alpha^{(2)}(x) = \begin{bmatrix} x_1 \\ x_4 \end{bmatrix},$$

$$\alpha^{(3)}(x) = \mathrm{const.}$$

Finally, the functions $\xi^{(i)}$, $1 \le i \le 3$ are obtained from the codistributions $\Omega_{\xi^{(i)}\circ h}(x)$ in the following form:

$$\xi^{(1)}(y) = y_2, \quad \xi^{(2)}(y) = y_1, \quad \xi^{(3)}(y) = \mathrm{const.}$$

We apply the above results to the isolability analysis. Because of fulfilling (22) (or, which is the same, (29)) for $i = 1$, $j = 2$ and $i = 2$, $j = 1$, we conclude that the fault ρ_2 is strongly detectable via the residual subvector $r^{(1)}$ and, respectively, ρ_1 is strongly detectable via the residual subvector $r^{(2)}$. In contrast to this, for $i = 1$, $j = 3$ and $i = 2$, $j = 3$ only (20) (or (28)) holds, i.e. the fault ρ_3 is only weakly detectable via the residuals $r^{(1)}$ and $r^{(2)}$. Then, because (19) holds for $i = 3$, $j = 1$ and $i = 3$, $j = 2$, the residual subvector $r^{(3)}$ is insensitive to both faults ρ_1 and ρ_2. The primary FS matrix is given in Table 1.

Table 1. Primary FS matrix for the system (30).

Residual	Faults		
	ρ_1	ρ_2	ρ_3
$r^{(1)}$	0	1	z
$r^{(2)}$	1	0	z
$r^{(3)}$	0	0	0

An analysis of this matrix shows that the single faults ρ_1 and ρ_2 are strongly distinguishable whereas every single fault and the multiple fault ρ_3 are only weakly distinguishable. To explain this, consider the situation when the fault ρ_1 affects the system output such that at some instant of time t^* it takes $y_1(t^*) = 0$[3]. Under this assumption, $y_2(t)$ becomes insensitive to the fault ρ_2 for every $t \ge t^*$ and the arbitrary control $u(\tau) \in U$, $\tau \in [t^*, t]$. Thus, in this situation, the system with the single fault ρ_1 shows the same behaviour as the system with the multiple fault ρ_3 at $t \ge t^*$.

Similarly, if the fault ρ_2 distorts the output y_2 at t_0 and $y_1(t_0+\tau) = 0$ is true, then the system with the single fault ρ_2 shows the same behaviour as the system with the multiple fault ρ_3 arising at $t \ge t_0 + \tau$. Also observe that

[3] To understand the reasoning given below, it is sufficient to look at the structure of the system (30) given in Fig. 2.

if for some t^* we have $x_1(t^*) = 0$ and the system (30) is healthy at t^*, then each fault $\rho_i,\ 1 \le i \le 3$, occurring at some $t \ge t^*$, will never be detected, because it will never affect the system output.

Clearly, the final FS matrix is obtained by excluding the third row. So, a two-component residual generator (including the subsystems $\Sigma^{(i)}$ and the functions $\xi^{(i)}$, $h^{(i)},\ i = 1,\ 2$, corresponding to the rows of the final FS matrix) should be designed. Taking the following coordinate transformation specified by the functions $\alpha^{(1)}(x)$ and $\alpha^{(2)}(x)$:

$$x^{(1)} = \begin{bmatrix} x_2 \\ x_3 \end{bmatrix}, \quad x^{(2)} = \begin{bmatrix} x_1 \\ x_4 \end{bmatrix}$$

and applying the relations (10) and (12) to finding $f^{(i)}$ and $h^{(i)}$, respectively, we obtain

$$\begin{aligned} \Sigma^{(1)}: \qquad \dot{x}^{(1)}(t) &= \begin{bmatrix} x_2^{(1)}(t)\big(1 - x_2^{(1)}(t)\big) \\ 0 \end{bmatrix} \\ &\quad + \begin{bmatrix} 0 & 0 \\ 0 & y_1(t) \end{bmatrix} u(t), \qquad (31) \\ r^{(1)}: \qquad r^{(1)}(t) &= x_1^{(1)}(t) - y_2(t) \end{aligned}$$

for the first component, and

$$\begin{aligned} \Sigma^{(2)}: \qquad \dot{x}^{(2)}(t) &= \begin{bmatrix} x_1^{(2)}(t) x_2^{(2)}(t) \\ 0 \end{bmatrix} \\ &\quad + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} u(t), \qquad (32) \\ r^{(2)}: \qquad r^{(2)}(t) &= x_1^{(2)}(t) - y_1(t) \end{aligned}$$

for the second one.

Some of the above results are proved by simulation. In the simulations, the initial coordinates of the system (30) (as well as the initial coordinates of the observers (31) and (32)) are assumed to be equal to zero. The control values are $u_1 = 0.1\sin(t)$ and $u_2 = 0.1\cos(t)$. Figures 3 and 4 illustrate the residual behaviour under a single fault, while Figs. 5 and 6 are given for the multiple one. Then a nonzero value of ϑ_1 emerges at $t = 5$, while a nonzero value of ϑ_2 is observed at $t = 7$. It is clearly seen from Figs. 5 and 6 that the isolability of the multiple fault depends on the fault mode. It is also worth noting that if we take another value of the control u_1, e.g. $u_1 = 0.5$, the multiple fault in the mode $\vartheta_1 = -5$, $\vartheta_2 = 0.05$ becomes isolable, see Fig. 7.

6. Conclusion

In this paper, former results obtained by the authors in the framework of the algebraic approach were extended for nonlinear diagnostic filter design. The relations existing between the algebraic and differential geometric approaches were investigated. For systems that are affine in control and fault actions, it was shown that the procedure developed for finding "special" vector functions results in a known procedure proposed by De Persis and Isidori (2001). Taking into account the fact that the algebraic approach was developed for both continuous- and discrete-time nonlinear systems, while the geometric one was proposed only for continuous time systems, this result can be considered as a step towards extending the geometric approach to the case of discrete-time nonlinear systems.

As nominal control is assumed for the fault diagnosis, due to the nonlinear character of the systems under consideration, the next feature of this paper follows immediately from an obvious idea: a fault should be detected and isolated only if it distorts the system output. This allowed us to introduce new definitions of weak/strong fault detectability/isolability. Relations were proposed for checking these conditions. These relations deal with the system behaviour in the output space. It looks simpler and more reasonable in spite of the known relations given in papers by De Persis and Isidori (2001), Join *et al.* (2002b), which assume the analysis of the system behaviour in the state space.

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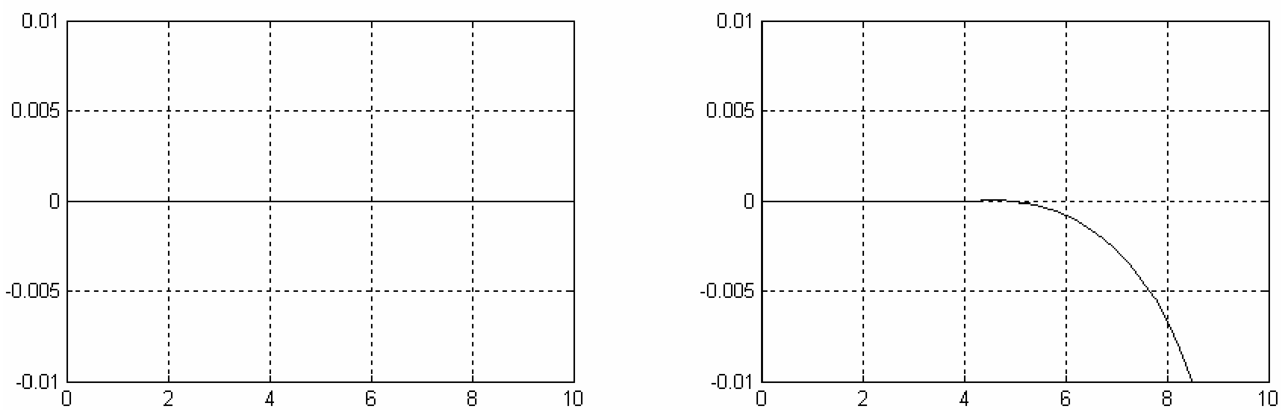
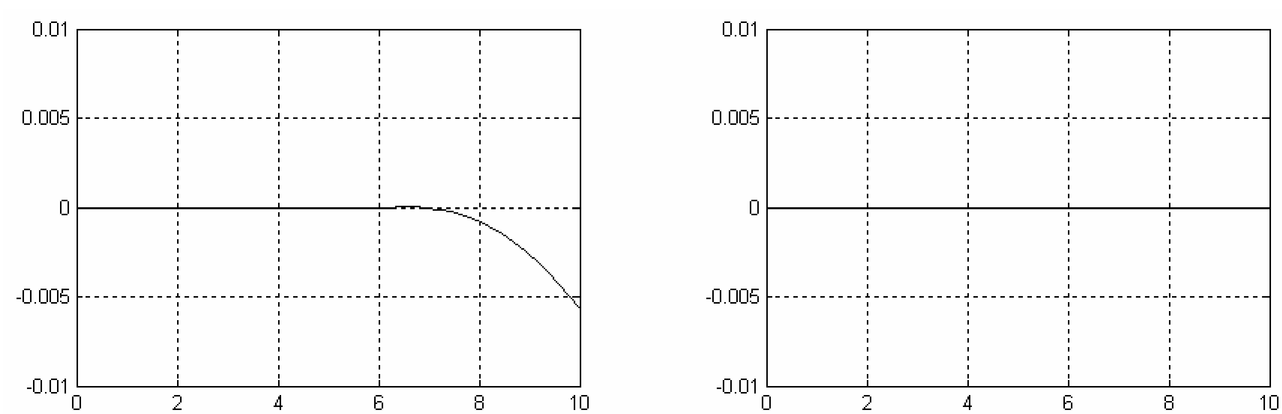


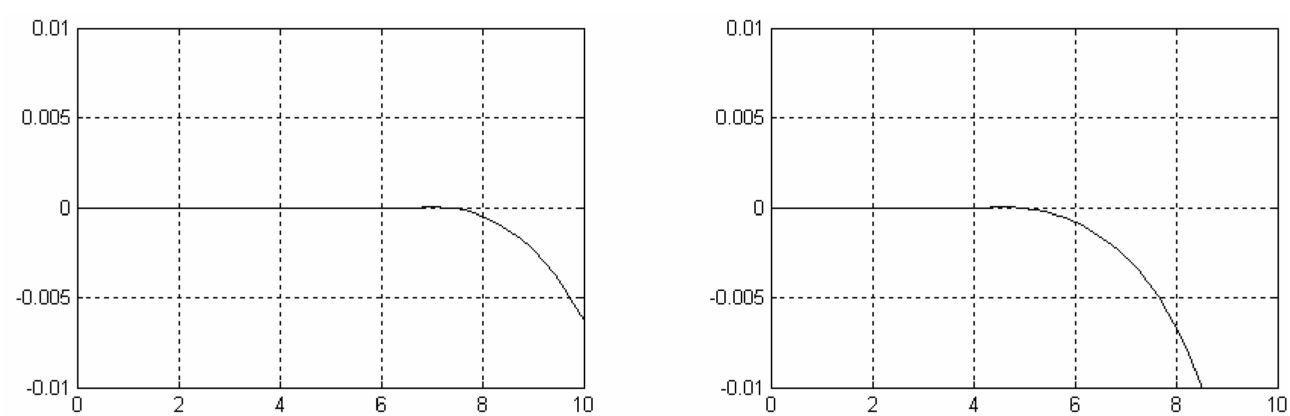
$\left(r^{(1)}\right)$ $\left(r^{(2)}\right)$

Fig. 3. Residual behaviour under the single fault ρ_1 $(\vartheta_1 = 0.05,\ \vartheta_2 = 0)$.



$\left(r^{(1)}\right)$ $\left(r^{(2)}\right)$

Fig. 4. Residual behaviour under the single fault ρ_2 $(\vartheta_1 = 0,\ \vartheta_2 = 0.05)$.



$\left(r^{(1)}\right)$ $\left(r^{(2)}\right)$

Fig. 5. Residual behaviour under the multiple fault ρ_3 $(\vartheta_1 = 0.05,\ \vartheta_2 = 0.05)$ and the control values $u_1 = 0.1\sin(t),\ u_2 = 0.1\cos(t)$.

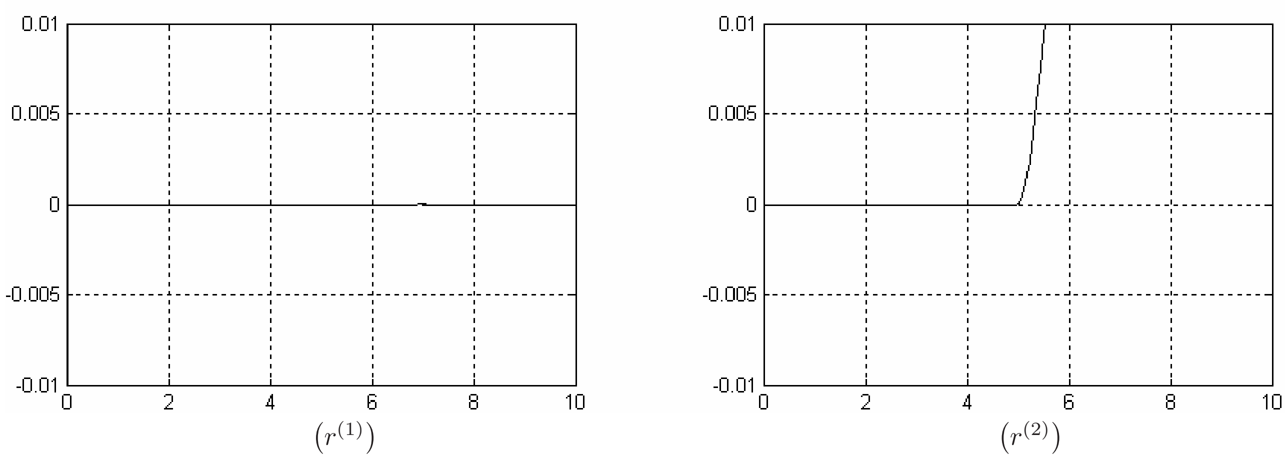


Fig. 6. Residual behaviour under the multiple fault ρ_3 $(\vartheta_1 = -5,\ \vartheta_2 = 0.05)$ and the control values $u_1 = 0.1\sin(t),\ u_2 = 0.1\cos(t)$.

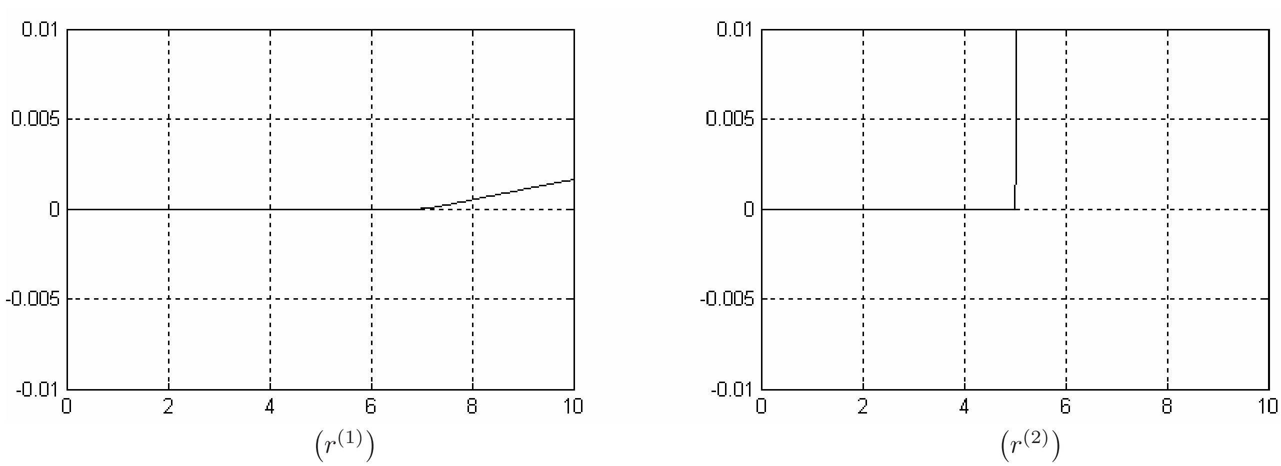


Fig. 7. Residual behaviour under the multiple fault ρ_3 $(\vartheta_1 = -5,\ \vartheta_2 = 0.05)$ and the control values $u_1 = 0.5,\ u_2 = 0.1\cos(t)$.

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Received: 29 September 2005