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COMPUTATION OF REALIZATIONS COMPOSED OF DYNAMIC AND STATIC PARTS OF IMPROPER TRANSFER MATRICES

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The problem of computing minimal realizations of a singular system decomposed into a standard dynamical system and a static system of a given improper transfer matrix is formulated and solved. A new notion of the minimal dynamical-static realization is introduced. It is shown that there always exists a minimal dynamical-static realization of a given improper transfer matrix. A procedure for the computation of a minimal dynamical-static realization for a given improper transfer matrix is proposed and illustrated by a numerical example.

Keywords: minimal realization, decomposition, improper transfer matrix, singular linear system

1. Introduction

The computation of a minimal realization for a given transfer matrix is one of the classical problems in control theory. There exist many well-known methods for the computation of minimal realizations for given proper and improper transfer matrices (Christodoulou and Mertzios, 1985; Kaczorek, 1992; Kailath, 1980; Roman and Bullock, 1975; Sinha Naresk, 1975; Wolovich and Guidorsi, 1977). It is also well known that a singular linear system described by static equations can be decomposed into two subsystems, a standard dynamical subsystem and a static subsystem (Kaczorek, 1992). The main purpose of this paper is to propose a method for the computation of minimal realizations of a singular system decomposed into a standard dynamical system and a static system of a given improper transfer matrix. A new notion of the minimal dynamical-static realization will be introduced. It will be shown that there always exists a minimal dynamical-static realization of a given improper transfer matrix. A procedure for the computation of a minimal dynamical-static realization of a given improper transfer matrix will be proposed.

To the best of the author's knowledge, the problem of computing a minimal dynamical-static realization for a given improper transfer matrix has not been considered yet.

2. Preliminaries and problem formulation

Let $\mathbb{R}^{n\times m}$ be the set of $n \times m$ real matrices and $\mathbb{R}^n := \mathbb{R}^{n\times 1}$. Consider the singular continuous-time linear system

$$E\dot{x} = Ax + Bu, \tag{1a}$$
$$y = Cx, \tag{1b}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^p$ are respectively the state vector, the input vector and the output vector, and $E, A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$. It is assumed that $\det E = 0$ and

$$\det[Es - A] \neq 0 \tag{2}$$

for some $s \in \mathbb{C}$ (the field of complex numbers).

It is well known (Kaczorek, 1992) that the singular system (1) can be decomposed into the standard dynamical system

$$\dot{x}_1 = A_1x_1 + B_1u, \tag{3a}$$
$$y_1 = C_1x_1, \tag{3b}$$

and the static system

$$x_2 = A_{21}x_1 + B_{20}u + B_{21}\dot{u} + +B_{2r}u^{(r)}, \tag{4a}$$
$$y_2 = C_2x_2, \tag{4b}$$

such that

$$y = y_1 + y_2, \quad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = Qx, \quad \det Q \neq 0 \tag{5}$$

(often $Q = I$), where $x_1 \in \mathbb{R}^{n_1}$, $x_2 \in \mathbb{R}^{n_2}$, $n_1 + n_2 = n$, $A_1 \in \mathbb{R}^{n_1\times n_1}$, $B_1 \in \mathbb{R}^{n_1\times m}$, $C_1 \in \mathbb{R}^{p\times n_1}$, $A_{21} \in \mathbb{R}^{n_2\times n_1}$, $B_{2k} \in \mathbb{R}^{n_2\times m}$ for $k = 0, 1, \dots, r$ and $u^{(r)} = \mathrm{d}^r u/\mathrm{d}t^r$.

The decomposition can be obtained using the modified shuffle algorithm (Kaczorek, 1992).

Lemma 1. *The transfer matrix of the singular system decomposed into the standard dynamical system (3) and the static system (4) is given by*

$$\begin{aligned} T(s) &= (C_1 + C_2A_{21})[I_{n_1}s - A_1]^{-1}B_1 \\ &\quad + C_2(B_{20} + B_{21}s + \cdots + B_{2r}s^r). \end{aligned} \tag{6}$$

Proof. From (3a) and (4a) we have

$$\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} [I_{n_1}s - A_1] & 0 \\ -A_{21} & I_{n_2} \end{bmatrix}^{-1} \times \begin{bmatrix} B_1 \\ B_{20} + B_{21}s + \cdots + B_{2r}s^r \end{bmatrix} U, \tag{7}$$

where $X_k = X_k(s) = L[x_k(t)]$, $U = U(s) = L[u(t)]$ are the Laplace transforms of x_k and u, respectively.

Taking into account that

$$\begin{bmatrix} [I_{n_1}s - A_1] & 0 \\ -A_{21} & I_{n_2} \end{bmatrix}^{-1} = \begin{bmatrix} [I_{n_1}s - A_1]^{-1} & 0 \\ A_{21}[I_{n_1}s - A_1]^{-1} & I_{n_2} \end{bmatrix},$$

from (3b), (4b) and (5) we obtain for the Laplace transform of y,

$$\begin{aligned} Y &= \begin{bmatrix} C_1 & C_2 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \\ &= \begin{bmatrix} C_1 & C_2 \end{bmatrix} \begin{bmatrix} [I_{n_1}s - A_1]^{-1} & 0 \\ A_{21}[I_{n_1}s - A_1]^{-1} & I_{n_2} \end{bmatrix} \\ &\quad \times \begin{bmatrix} B_1 \\ B_{20} + B_{21}s + \cdots + B_{2r}s^r \end{bmatrix} U \\ &= \big[(C_1 + C_2A_{21})[I_{n_1}s - A_1]^{-1}B_1 \\ &\quad + C_2(B_{20} + B_{21}s + \cdots + B_{2r}s^r)\big]U. \end{aligned} \tag{8}$$

Formula (6) follows from (8). ■

Definition 1. The matrices A_1, A_{21}, B_1, B_{20}, $B_{21}, \dots, B_{2r}$, C_1, C_2 constitute a *dynamical-static realization* of an improper transfer matrix $T(s)$ if they satisfy (6). A realization is called *minimal* if the matrices A_1 and A_{21} have minimal dimensions among all realizations of $T(s)$.

The realization problem can be stated as follows: Given an improper transfer matrix $T(s) \in \mathbb{R}^{p\times m}(s)$ (the set of $p\times m$ rational matrices in s), find a dynamical-static realization of a given improper transfer matrix $T(s)$.

In what follows, a procedure for the computation of a minimal dynamical-static realization of a given improper transfer matrix will be proposed.

3. Problem Solution

Any given improper transfer matrix $T(s) \in \mathbb{R}^{p\times m}(s)$ can be decomposed into the polynomial part

$$P(s) = P_0 + P_1s + \cdots + P_rs^r \tag{9}$$

and the strictly proper part $T_{sp}(s)$, i.e.,

$$T(s) = P(s) + T_{sp}(s). \tag{10}$$

From the comparison of (6) and (10), we have

$$\begin{aligned} P(s) &= P_0 + P_1s + \cdots + P_rs^r \\ &= C_2(B_{20} + B_{21}s + \cdots + B_{2r}s^r) \end{aligned} \tag{11}$$

and

$$T_{sp}(s) = (C_1 + C_2A_{21})[I_{n_1}s - A_1]^{-1}B_1. \tag{12}$$

Using one of the well-known methods (Christodoulou and Mertzios, 1985; Kaczorek, 1992; Kailath, 1980; Roman and Bullock, 1975; Sinha Naresk, 1975; Wolovich and Guidorsi, 1977), we can determine a minimal realization $A_1, B_1, \bar{C}_1$ of $T_{sp}(s)$ satisfying

$$\bar{C}_1[I_{n_1}s - A_1]^{-1}B_1 = T_{sp}(s). \tag{13}$$

Given the matrices $P_k,\ k = 0, 1, \dots, r$ and $A_1, B_1, \bar{C}_1$, in order to solve the realization problem, we have to find the matrices $A_1, A_{21}, B_1, B_{2k},\ k = 0, 1, \dots, r$ and C_1 and C_2 satisfying

$$C_1 + C_2A_{21} = \bar{C}_1, \quad C_2B_{2k} = P_k \tag{14}$$

for $k = 0, 1, \dots, r$.

Note that there exist many matrices A_{21}, C_1, C_2 and $B_{2k},\ k = 0, 1, \dots, r$ satisfying (14) for given $\bar{C}_1$ and $P_k,\ k = 0, 1, \dots, r$. One way to find the desired matrices is to choose first C_2 and A_{21} (or C_1 and C_2) and compute C_1 (or A_{21}) and $B_{2k},\ k = 0, 1, \dots, r$ from (14). Therefore, we can compute a minimal dynamical-static realization of a given improper transfer matrix $T(s) \in \mathbb{R}^{p\times m}(s)$ using the following procedure:

Procedure 1.

Step 1. Decompose a given transfer matrix $T(s)$ into the polynomial part (9) and the strictly proper part $T_{sp}(s)$.

Step 2. Using one of the well-known methods compute a minimal realization $A_1, B_1, \bar{C}_1$ of $T_{sp}(s)$.

Step 3. Choose the matrices C_2, A_{21} (or C_1 and C_2) and, using (14), compute the matrices $B_{2k},\ k = 0, 1, \dots, r$ and C_1 (or A_{21}).

Remark 1. The dimensions of the matrices $B_{2k},\ k = 0, 1, \dots, r$ and C_2 are determined by the dimension $m \times p$ of the transfer matrix $T(s)$. A dynamical-static realization of $T(s)$ is minimal if and only if the realization $A_1, B_1, \bar{C}_1$ of $T_{sp}(s)$ is minimal.

From the above discussion we have the following result:

Theorem 1. *For a given improper transfer matrix $T(s) \in \mathbb{R}^{p\times m}(s)$ there always exists a minimal dynamical-static realization A_1, A_{21}, B_1, B_{2k}, $k = 0, 1, \dots, r$, C_1 and C_2. This realization can be computed using Procedure 1.*

Example 1. Find a minimal dynamical-static realization of the transfer matrix

$$T(s) = \begin{bmatrix} \dfrac{s^3+s^2+1}{s} & \dfrac{s^2+2s+3}{s+1} \\ \dfrac{2s^2+4s+2}{s+2} & \dfrac{s^3+2s^2+s+3}{s+2} \end{bmatrix}. \quad (15)$$

Using Procedure 1, we obtain the following: *Step 1.* The transfer matrix (15) can be decomposed into the polynomial part

$$\begin{aligned} P(s) &= \begin{bmatrix} s^2+s & s+1 \\ 2s & s^2+1 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix} s + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} s^2 \\ &= P_0 + P_1 s + P_2 s^2 \end{aligned} \quad (16)$$

and the strictly proper part

$$T_{sp}(s) = \begin{bmatrix} \dfrac{1}{s} & \dfrac{2}{s+1} \\ \dfrac{2}{s+2} & \dfrac{1}{s+2} \end{bmatrix}. \quad (17)$$

Step 2. A minimal realization of (17) has the form

$$A_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -2 & 0 \\ 0 & 0 & -1 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 1 & 2 \\ 2 & 1 \\ 0 & -2 \end{bmatrix}, \quad \bar{C}_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad (18)$$

Step 3. In this case we choose, e.g.,

$$C_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad A_{21} = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & -1 \end{bmatrix}. \quad (19)$$

Then from (14) we obtain

$$C_1 = \bar{C}_1 - C_2 A_{21} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix},$$

$$B_{20} = P_0 = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \quad B_{21} = P_1 = \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix},$$

$$B_{22} = P_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \quad (20)$$

The desired minimal dynamical-static realization of the transfer matrix (15) is given by (18)–(20).

4. Concluding Remarks

The problem of computing a minimal realization of a singular system decomposed into the standard dynamical system (3) and the static system (4) of a given improper transfer matrix was formulated and solved. A new notion of the minimal dynamical-static realization of a given transfer matrix was introduced. It was shown that there always exist a minimal dynamical-static realization of a given improper transfer matrix. A procedure for computing a minimal dynamical-static realization of a given improper transfer matrix was proposed and illustrated by a numerical example. With slight modifications (by substitution of s by z and of the derivative by the shifting operator) the proposed method can be extended to discrete-time linear systems.

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References

Christodoulou M.A. and Mertzios B.G. (1985): *Realization of singular systems via Markov parameters.* — Int. J. Contr., Vol. 42, No. 6, pp. 1433–1441.

Kaczorek T. (1992): *Linear Control Systems, Vol. 1.* — New York: Wiley.

Kailath T. (1980): *Linear Systems.* — Englewood Cliffs: Prentice-Hall

Roman J.R. and Bullock T.E. (1975): *Minimal partial realization in canonical form.* — IEEE Trans. Automat. Contr., Vol. AC–20, No. 4, pp. 529–533.

Sinha Naresk K. (1975): *Minimal realization of transfer function matrices: A comparative study of different methods.* — Int. J. Contr., Vol. 22, No. 5, pp. 627–639.

Wolovich W.A. and Guidorsi R. (1977): *A general algorithm for determining state-space representations.* — Automatica, Vol. 13, pp. 295–199.

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