BEHAVIORAL SYSTEMS THEORY: A SURVEY

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We survey the so-called behavioral approach to systems and control theory, which was founded by J. C. Willems and his school. The central idea of behavioral systems theory is to put the focus on the set of trajectories of a dynamical system rather than on a specific set of equations modelling the underlying phenomenon. Moreover, all signal components are treated on an equal footing at first, and their partition into inputs and outputs is derived from the system law, in a way that admits several valid cause-effect interpretations, in general.

Keywords: linear systems, behavioral approach.

1. Introduction

In his series of papers (Willems, 1986/87), J. C. Willems introduced a novel approach to systems and control theory, called the behavioral approach. Now, two decades later, many researchers, both applied mathematicians and theoretically inclined engineers, have taken up his ideas and have formed a lively community devoted to advancing and extending behavioral systems theory. The survey papers (Willems, 1991) and (Fuhrmann, 2002), as well as the textbook (Polderman and Willems, 1998), are particularly prominent outcomes of this endeavor.

The present paper offers a brief guided tour through behavioral systems theory. Needless to say, the selection of topics is heavily influenced by the author's personal interests, preferences, and a subjective point of view. No claim of completeness is made concerning both the presented results and the references. Indeed, there is no doubt that many important contributions will remain unmentioned.

2. Starting point

To get started, the reader should be willing to abandon, for the time being, the tendency to view a system as a signal processor: inputs are fed into some "black box", something mysterious (or not so mysterious) happens inside the box, and outputs to be measured and regulated leave the system. This is a very practical and useful point of view, without doubt, for countless engineering applications. Nevertheless, thinking a bit more generally of systems arising in physics, biology, or economics, and of the mathematical models for them that have been studied over the centuries (often without having a concrete engineering application in mind), it seems more natural to see a dynamical system as an interrelation of certain quantities of which it may be hard to tell which is the cause and which the effect (both interpretations may be equally justified).

This leads us to the following very general and comprehensive definition of a dynamical system. Dynamical systems evolve in time, and they interact with their environment through time-dependent functions, called signals or trajectories. Reflecting this general and broad framework, Willems' definition of a dynamical systems involves three ingredients: first, a set $\mathbb{T}$ that is interpreted as a mathematical model of time. Second, a set $\mathbb{W}$ in which the signals take their values. Thus, a trajectory is a function

$$w : \mathbb{T} \to \mathbb{W}, \quad t \mapsto w(t).$$

By $\mathbb{W}^{\mathbb{T}}$, we denote the set of all functions that are defined on $\mathbb{T}$ and that take their values in $\mathbb{W}$. The third and most important part of the definition is the set $\mathcal{B} \subseteq \mathbb{W}^{\mathbb{T}}$ formalizing the set of signals that can occur in the system, i.e., that obey the laws that govern the system (usually, difference or differential equations). Such signals will also be called admissible. Thus we arrive at Willems' famous definition of a system Σ as a triple

$$\Sigma = (\mathbb{T}, \mathbb{W}, \mathcal{B}).$$

In most cases of relevance, $\mathbb{T}$ is a subset of $\mathbb{R}$, and $\mathbb{W}$ is a finite-dimensional vector space, say, $\mathbb{W} = \mathbb{F}^q$, where $\mathbb{F}$ is a field (often $\mathbb{R}$ or $\mathbb{C}$).

A slight modification of this definition was put forward later on (Oberst, 1990): Let $\mathcal{A}$ denote the set of scalar-valued signals, let q denote the number of (scalar) signals occurring in the system and let $\mathcal{B} \subseteq \mathcal{A}^q$ denote the set of q-tuples of such signals that satisfy the system law. Then a system Σ is determined by these three data, $\Sigma = (\mathcal{A}, q, \mathcal{B})$. Of course, for $\mathbb{W} = \mathbb{F}^q$, Willems' definition can be embedded into the modified version by putting $\mathcal{A} = \mathbb{F}^{\mathbb{T}}$. However, not all signals of interest are really functions in the classical sense of the word, for instance, one may want to study distributional solutions to differential equations in order to incorporate phenomena such as "impulses", i.e., Dirac delta distributions, and other generalized functions. For the purposes of this paper, we will use both definitions interchangeably.

3. First steps

A system $\Sigma = (\mathbb{T}, \mathbb{W}, \mathcal{B})$ is called *linear* if $\mathbb{W}$ is an $\mathbb{F}$-vector space ($\mathbb{F}$ being a field) and $\mathcal{B}$ is an $\mathbb{F}$-subspace of $\mathbb{W}^{\mathbb{T}}$. Clearly, this amounts to the formal requirement

$$w_1, w_2 \in \mathbb{W}^{\mathbb{T}}, \lambda_1, \lambda_2 \in \mathbb{F} \quad \Rightarrow \quad \lambda_1 w_1 + \lambda_2 w_2 \in \mathbb{W}^{\mathbb{T}}$$

saying that each linear combination of signals is again a signal and, more importantly,

$$w_1, w_2 \in \mathcal{B}, \lambda_1, \lambda_2 \in \mathbb{F} \quad \Rightarrow \quad \lambda_1 w_1 + \lambda_2 w_2 \in \mathcal{B},$$

that is, every linear combination of admissible signals is again admissible (this is known as the "superposition principle"). Let us assume that $\mathbb{W} = \mathbb{F}^q$ in the following, where $\mathbb{F}$ is a field.

For the definition of time-invariance, we need to restrict to time sets $\mathbb{T} \subseteq \mathbb{R}$ that are closed with respect to addition, that is,

$$t_1, t_2 \in \mathbb{T} \quad \Rightarrow \quad t_1 + t_2 \in \mathbb{T}.$$

Of course, this is satisfied by all of the predominant time sets such as $\mathbb{T} = \mathbb{R}$ or $\mathbb{T} = \mathbb{Z}$. We will restrict to these two cases in the following, and we shall refer to $\mathbb{T} = \mathbb{R}$ as the continuous time case and to $\mathbb{T} = \mathbb{Z}$ as the discrete time case. Then we can define, for any $t \in \mathbb{T}$, the shift operator

$$\sigma^t : \mathbb{W}^{\mathbb{T}} \to \mathbb{W}^{\mathbb{T}}, \quad w \mapsto \sigma^t w,$$

where

$$(\sigma^t w)(\tau) = w(t + \tau) \quad \text{for all } \tau \in \mathbb{T}.$$

A system $\Sigma = (\mathbb{T}, \mathbb{W}, \mathcal{B})$ is called *time-invariant* (or shift-invariant) if

$$w \in \mathcal{B}, t \in \mathbb{T} \quad \Rightarrow \quad \sigma^t w \in \mathcal{B}.$$

An important question in the foundations of behavioral systems theory is the existence of *kernel representations*. Given a linear, shift-invariant subset $\mathcal{B} \subseteq \mathbb{W}^{\mathbb{T}}$, when does there exist a linear system of difference or differential equations with coefficients in the base field $\mathbb{F}$ whose solution set is precisely equal to $\mathcal{B}$?

In discrete time, the answer has been known for quite some time now, see, e.g., (Willems, 1991): The missing property (besides linearity and shift-invariance) is called completeness, and these three properties together are necessary and sufficient for a discrete system to admit a representation

$$\mathcal{B} = \{w \in (\mathbb{F}^q)^{\mathbb{Z}} \mid Rw = 0\}$$

for some Laurent polynomial matrix $R \in \mathbb{F}[s, s^{-1}]^{g \times q}$, which acts on $w \in (\mathbb{F}^q)^{\mathbb{Z}}$ by the unit shift operators $sw := \sigma w := \sigma^1 w$ and $s^{-1} w := \sigma^{-1} w$.

In continuous time, however, the analogous question is still open to some extent. When can a time-invariant linear subspace $\mathcal{B} \subseteq \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^q)$ be written in the form

$$\mathcal{B} = \{w \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^q) \mid Rw = 0\}$$

for some polynomial matrix $R \in \mathbb{R}[s]^{g \times q}$, which acts on $w \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^q)$ by differentiation, that is, $sw := \frac{\mathrm{d}w}{\mathrm{d}t}$? Note that we need to restrict to smooth functions for Rw to make sense for any polynomial matrix of arbitrary degree. The interesting paper (Lomadze, 2007) sheds some light on this problem.

The continuous and discrete time cases can be treated in parallel by setting $\mathcal{A} = \mathcal{C}^\infty(\mathbb{R}, \mathbb{R})$ and $\mathcal{D} = \mathbb{R}[s]$ in continuous time and $\mathcal{A} = \mathbb{F}^{\mathbb{Z}}$ and $\mathcal{D} = \mathbb{F}[s, s^{-1}]$ in discrete time, where the action of $d \in \mathcal{D}$ on $a \in \mathcal{A}$ is defined as above. Then

$$\mathcal{B} = \{w \in \mathcal{A}^q \mid Rw = 0\},$$

where $R \in \mathcal{D}^{g \times q}$ is called a behavior with kernel representation, and R is called a representation matrix for $\mathcal{B}$. We deal exclusively with such systems in the following sections.

4. Milestones

Let $\mathcal{A}$ be a signal set that carries a $\mathcal{D}$-module structure, where $\mathcal{D}$ is a commutative ring. The most prominent example is $\mathcal{D} = \mathbb{R}[s]$ and $\mathcal{A} = \mathcal{C}^\infty(\mathbb{R}, \mathbb{R})$, with $sa := \frac{\mathrm{d}a}{\mathrm{d}t}$ as outlined above. (The $\mathcal{D}$-module structure reflects the fact that applying any linear real-coefficient differential operator $d \in \mathcal{D}$ to any smooth function $a \in \mathcal{A}$ will produce da, which is another element of $\mathcal{A}$. Also, we have the common rules of calculating with operators and signals, similarly as in a vector space, except for the fact that, of course, we cannot simply "divide" by a non-zero operator.) Similarly, $\mathcal{D} = \mathbb{F}[s, s^{-1}]$ and $\mathcal{A} = \mathbb{F}^{\mathbb{Z}}$ provides another example.

In the seminal contribution to systems theory (Oberst, 1990), U. Oberst realized that algebraic properties of $\mathcal{A}$ as a module over $\mathcal{D}$ have a great systems theoretic relevance, and that many of the pairs $\mathcal{D}, \mathcal{A}$ that turn up in systems and control theory satisfy a particularly strong kind of duality enabling the translation of algebraic properties to systems properties and vice versa. The property in question is that $\mathcal{A}$, as a $\mathcal{D}$-module, is the so-called injective cogenerator, see (Wood, 2000) for an overview. In particular, this holds for the two pairs $\mathcal{D}, \mathcal{A}$ from above. We will stick to these two cases for the remainder of this section, and we will present two theorems of fundamental importance for systems theory as prototypes of the cornerstones of behavioral systems theory.

Consider $\mathcal{B} = \{w \in \mathcal{A}^q \mid Rw = 0\}$, where $R \in \mathcal{D}^{g\times q}$. The i-th component w_i of w is called a *free variable* of $\mathcal{B}$ if for any choice of $a \in \mathcal{A}$ there exists $w \in \mathcal{B}$ with $w_i = a$. A behavior $\mathcal{B}$ is called *autonomous* if it has no free variables.

Theorem 1. *The following are equivalent:*

1. *$\mathcal{B}$ is autonomous.*
2. *Any representation matrix of $\mathcal{B}$ has full column rank.*
3. *If $w \in \mathcal{B}$ has bounded support, then w must be identically zero.*
4. *If $w \in \mathcal{B}$ satisfies $w(t) = 0$ for all $t < 0$, then w must be identically zero.*
5. *$\mathcal{B}$ is a finite-dimensional $\mathbb{F}$-vector space.*

Notice that a true behaviorist would choose the trajectory-oriented conditions 3 or 4 as the definition of autonomy and the absence of free variables as a derived characterization. From the algebraic point of view, however, the definition given above is easier to access.

In every non-autonomous system $\mathcal{B}$, the signal vector $w \in \mathcal{A}^q$ can be partitioned (possibly after a permutation of the components of w) into two subvectors $w = [u^T, y^T]^T$, where $u \in \mathcal{A}^m$ and $y \in \mathcal{A}^p$ with $m + p = q$, such that we have

$$\forall u \in \mathcal{A}^m \; \exists y \in \mathcal{A}^p : \begin{bmatrix} u \\ y \end{bmatrix} \in \mathcal{B}$$

and, moreover, the system $\{y \in \mathcal{A}^p \mid [0, y^T]^T \in \mathcal{B}\}$ obtained from setting $u \equiv 0$ is autonomous. Clearly, u is a vector of free variables in $\mathcal{B}$, whose number of components is as large as possible, and it is therefore called an *input*. Similarly, y can be interpreted as the system's *output*. The numbers m and $p = q - m$ are invariants of $\mathcal{B}$; indeed, it can be shown that $p = \operatorname{rank}(R)$ and that any two representations of the same $\mathcal{B}$ must have the same rank. In fact, since $\mathcal{D}$ is a principal ideal domain, we may restrict to representation matrices with full row rank (this is a consequence of the Smith form). Thus, a system is autonomous if and only if it possesses a square non-singular representation matrix. Partitioning the full-row-rank representation matrix $R = [-Q, P]$ according to the partition of $w = [u^T, y^T]^T$, we get an *input-output representation*

$$\begin{bmatrix} u \\ y \end{bmatrix} \in \mathcal{B} \quad \Leftrightarrow \quad Py = Qu,$$

which has a solution $y \in \mathcal{A}^p$ for every choice of $u \in \mathcal{A}^m$. Here, P is square and non-singular. Thus, we recover the concept of input-output relations from classical systems theory, but with the important difference that the partition of the components of w into these two classes is not artificially prescribed but derived from the system law. It is important to note that several distinct input-output representations are possible for one and the same behavior. The rational matrix

$$H := P^{-1}Q \in \mathbb{F}(s)^{p\times m}$$

(where $\mathbb{F} = \mathbb{R}$ in continuous time) is called the *transfer matrix* of $\mathcal{B}$ with respect to the chosen input-output decomposition. There always exists at least one choice of inputs and outputs that guarantees that the associated transfer matrix is proper.

One of the central concepts of control theory is controllability. This notion seems to be very strongly linked to specific properties of state space systems. However, this is not the case. Instead of steering the system from one state to another, the appropriate question in the behavioral framework is to force the system to go from one trajectory to another (without violating the system law, of course). A nice overview over the various controllability notions and their relation with the concept of potentials in physics is given in (Shankar, 2002).

We will call $\mathcal{B}$ *controllable* if it has an image representation, that is,

$$\mathcal{B} = \{w \in \mathcal{A}^q \mid \exists \ell \in \mathcal{A}^n : w = M\ell\}$$

for some matrix $M \in \mathcal{D}^{q\times n}$.

Theorem 2. *The following are equivalent:*

1. *$\mathcal{B}$ is controllable.*
2. *Any representation matrix R of $\mathcal{B}$ is a left syzygy matrix, that is, its rows generate the left kernel $\{z \in \mathcal{D}^{1\times q} \mid zM = 0\}$ of some $M \in \mathcal{D}^{q\times n}$.*
3. *For any $w_1, w_2 \in \mathcal{B}$, there exists $0 < \tau \in \mathbb{T}$ and $w \in \mathcal{B}$ such that*

$$w(t) = \begin{cases} w_1(t) & \text{if } t < 0, \\ w_2(t) & \text{if } t > \tau. \end{cases}$$

4. *Any full-row-rank representation matrix* $R \in \mathcal{D}^{p\times q}$ *of* $\mathcal{B}$ *is right invertible, i.e., there exists* $Y \in \mathcal{D}^{q\times p}$ *such that* $RY = I$.

5. *Any full-row-rank representation matrix* $R \in \mathcal{D}^{p\times q}$ *of* $\mathcal{B}$ *satisfies*

$$\operatorname{rank}(R(\lambda)) = p \quad \textit{for all } \lambda \in \overline{\mathbb{F}},$$

where $\overline{\mathbb{F}}$ *is the algebraic closure of* $\mathbb{F}$*. (This is for continuous time. For the discrete case, we have the same requirement for* $\lambda \in \overline{\mathbb{F}} \setminus \{0\}$*, which is due to the fact that we work over* $\mathbb{Z}$ *here and not over* $\mathbb{N}$ *as in the classical state space setting, and thus we are dealing with Laurent polynomial matrices.)*

Again, only the assertion 3 is formulated entirely in terms of the system trajectories and should be seen as the truly behavioral controllability condition. If it is satisfied, one says that the trajectories w_1, w_2 can be concatenated, and one calls w a connecting trajectory. Similarly as with state space systems, τ can in fact be chosen independently of the specific selection of w_1, w_2 for the system class under consideration and, moreover, τ can be made arbitrarily small in continuous time, but not in discrete time.

For state space systems $\dot{x} = Ax + Bu$, where $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$, we set $w = [x^T, u^T]^T$ and $R = [sI - A, -B] \in \mathbb{R}[s]^{n\times(n+m)}$. In view of the condition 5, recalling that $\overline{\mathbb{R}} = \mathbb{C}$, the behavioral controllability concept coincides with the classical state space notion (via the so-called Hautus test).

5. Further paths

The basic theory can be extended into countless directions. The following material highlights only three of them.

5.1. Multidimensional systems. In her thesis (Rocha, 1990), P. Rocha started the study of two-dimensional systems from the behavioral point of view. That is, we have two independent variables rather than only one, and thus, for instance, in discrete time we have $\mathbb{T} = \mathbb{Z}^2$.

The paper (Oberst, 1990) introduced a rigorous approach to discrete and continuous n-dimensional behaviors with kernel representations, that is,

$$\mathcal{B} = \{w \in \mathcal{A}^q \mid Rw = 0\}$$

with $\mathcal{A} = \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})$ and $\mathcal{D} = \mathbb{R}[s_1, \ldots, s_n]$ for continuous time (where $s_i a = \frac{\partial a}{\partial t_i}$) and $\mathcal{A} = \mathbb{F}^{\mathbb{Z}^n}$ and $\mathcal{D} = \mathbb{F}[s_1, \ldots, s_n, s_1^{-1}, \ldots, s_n^{-1}]$ in discrete time (where $s_i a = \sigma_i a$ is the unit shift with respect to the i-th variable).

The main difficulty from the algebraic point of view is that $\mathcal{D}$ is not a principal ideal domain here, and hence results using the Smith form of univariate polynomial matrices do not carry over to this setting. Nevertheless, autonomy and controllability have been fully characterized also in this situation, by various authors, see, e.g., (Pommaret and Quadrat, 1999; Wood *et al.*, 1999; Zerz, 2000) and the references therein. It turns out that these notions have to be refined into several weaker and stronger forms: For instance, according to Theorem 1, one-dimensional systems are autonomous if and only if they are finite-dimensional as vector spaces over the underlying field. For systems in dimension $n \geq 2$, however, finite-dimensionality over $\mathbb{F}$ implies the absence of free variables but not conversely, and thus two non-equivalent autonomy-related properties have to be distinguished here. However, the equivalence of the first three assertions of Theorem 1 holds also in the multidimensional setting.

Similar things happen for controllability, see, e.g., (Rocha and Wood, 1997). In (Pillai and Shankar, 1999), we find a neat analytic interpretation of controllability in the multidimensional continuous case: $\mathcal{B}$ is controllable (i.e., it has an image representation) if and only if for all $w_1, w_2 \in \mathcal{B}$ and all open sets $U_1, U_2 \subset \mathbb{R}^n$ whose closures are disjoint there exists $w \in \mathcal{B}$ such that

$$w(t) = \begin{cases} w_1(t) & \text{if } t \in U_1, \\ w_2(t) & \text{if } t \in U_2, \end{cases}$$

thus giving a nice multidimensional generalization of the one-dimensional behavioral controllability paradigm (i.e., concatenability of trajectories). In this sense, the equivalence of the assertions 1–3 of Theorem 2 is preserved in the multidimensional setting (and likewise in the discrete case, with the difference that the sets U_i are required to be sufficiently far apart). However, even in a controllable system that can be represented by a full-row-rank matrix R this R is not necessarily right invertible. In fact, the multidimensional generalizations of the conditions 4 and 5 of Theorem 2 characterize a property that is stronger than controllability.

5.2. Continuous time-varying systems. Dropping the assumption of time-invariance, the simplest class of time-varying systems arises from admitting coefficients from a differential field $\mathbb{K}$, say $\mathbb{K} = \mathbb{R}(t)$, the field of rational functions. Then we have the ring of linear differential operators

$$\mathcal{D} = \mathbb{K}[s] = \mathbb{R}(t)[s],$$

where s represents once more the differentiation operator. A suitable signal set $\mathcal{A}$ is the set of functions $a : \mathbb{R} \to \mathbb{R}$ that are smooth except for a finite set of exception points $E(a)$, that is, $a \in \mathcal{C}^\infty(\mathbb{R} \setminus E(a), \mathbb{R})$. Formally, $\mathcal{D}$ looks like an ordinary polynomial ring, but the important difference is that a coefficient $k \in \mathbb{K}$ does not commute with the operator $\frac{\mathrm{d}}{\mathrm{d}t}$ unless k happens to be constant. This is due to the product rule of differentiation $\frac{\mathrm{d}}{\mathrm{d}t}(ka) = k'a + k\frac{\mathrm{d}a}{\mathrm{d}t}$ for $k \in \mathbb{K}$ and $a \in \mathcal{A}$, which implies that

$$\frac{\mathrm{d}}{\mathrm{d}t}k - k\frac{\mathrm{d}}{\mathrm{d}t} = k'.$$

Thus the main difficulty is that the ring $\mathcal{D}$ is not commutative here. Several authors have studied systems of this type (Bourlès, 2005; Ilchmann and Mehrmann, 2005; Pommaret and Quadrat, 1998; Zerz, 2006). The main tool is a non-commutative analogue of the Smith form which is known as the Jacobson form. This makes it possible to translate most results from classical one-dimensional systems to this setting, at least apart from the singularities of the coefficients and the trajectories, i.e., in a generic sense. The algebraic properties of multidimensional linear systems with time-varying coefficients were studied in (Chyzak *et al.*, 2005; Pommaret and Quadrat, 1999).

5.3. Discrete systems over finite rings. For applications in coding theory (Kuijper and Polderman, 2004; Lu *et al.*, 2004), it is of interest to replace the coefficient field $\mathbb{F}$ of a discrete system by a finite ring of the form $\mathbb{Z}_m = \mathbb{Z}/m\mathbb{Z}$, where $m > 1$ is an integer. Thus, we set $\mathcal{D} = \mathbb{Z}_m[s, s^{-1}]$, and s acts on $a \in \mathcal{A} = (\mathbb{Z}_m)^{\mathbb{Z}}$ by the unit shift operator $sa = \sigma a$. The main difficulty from the algebraic point of view is that $\mathcal{D}$ is not a domain, i.e., it contains zero-divisors. Similarly as with multidimensional systems, we cannot use the Smith form and thus, in particular, we cannot restrict to representation matrices with full row rank. In fact, the sheer notion of rank becomes ambiguous, since there are several distinct rank concepts for matrices over rings with zero-divisors. Also, we have two non-equivalent autonomy notions, the stronger one corresponding to the absence of bounded support trajectories besides zero, and the weaker one amounting to the absence of free variables (Kuijper *et al.*, 2006; Zerz, 2007). In terms of a representation matrix $R \in \mathcal{D}^{g\times q}$, the stronger notion is characterized by the existence of $X \in \mathcal{D}^{q\times g}$ and a non-zero-divisor $d \in \mathcal{D}$ such that $XR = dI$ while the weaker by the existence of $X \in \mathcal{D}^{q\times g}$ and non-zero elements $d_1, \ldots, d_q \in \mathcal{D}$ with $XR = \mathrm{diag}(d_1, \ldots, d_q)$. Clearly, both conditions amount to saying that R has full column rank if $\mathcal{D}$ is a domain (for $\mathcal{D} = \mathbb{Z}_m[s, s^{-1}]$; however, this is true if and only if m is prime, and then $\mathbb{Z}_m$ is already a field, i.e., we are back to the classical case). First results on the controllability of such systems will be reported elsewhere.

6. Where to go from here

In this short outlook section, we mention a few recent developments of behavioral systems theory and several important topics that could not be treated in this survey for brevity (the references are pars pro toto). The wide areas of stability and stabilization (Oberst, 2006) and of controller synthesis (Praagman *et al.*, 2007) in the behavioral context were not even touched. The same holds for the equally significant topics of observability and observer design (Valcher and Willems, 1999). The notion of state is of paramount importance in systems theory and admits a nice behavioral interpretation (Rapisarda and Willems, 1997; Fuhrmann *et al.*, 2007). Recently, periodic systems (Aleixo *et al.*, 2007) and rational representations (Willems and Yamamoto, 2007) have been studied within the behavioral framework, providing additional impetus for future research.

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