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ULTRA REGULAR COVERING SPACE AND ITS AUTOMORPHISM GROUP

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In order to classify digital spaces in terms of digital-homotopic theoretical tools, a recent paper by Han (2006b) (see also the works of Boxer and Karaca (2008) as well as Han (2007b)) established the notion of regular covering space from the viewpoint of digital covering theory and studied an automorphism group (or Deck's discrete transformation group) of a digital covering. By using these tools, we can calculate digital fundamental groups of some digital spaces and classify digital covering spaces satisfying a radius 2 local isomorphism (Boxer and Karaca, 2008; Han, 2006b; 2008b; 2008d; 2009b). However, for a digital covering which does not satisfy a radius 2 local isomorphism, the study of a digital fundamental group of a digital space and its automorphism group remains open. In order to examine this problem, the present paper establishes the notion of an ultra regular covering space, studies its various properties and calculates an automorphism group of the ultra regular covering space. In particular, the paper develops the notion of compatible adjacency of a digital wedge. By comparing an ultra regular covering space with a regular covering space, we can propose strong merits of the former.

Keywords: digital image, digital isomorphism, (ultra) regular covering space, digital covering space, simply k-connected, Deck's discrete transformation group, compatible adjacency, digital wedge, automorphism group.

1. Introduction

Let $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{R}$ denote the sets of natural numbers, integers and real numbers, respectively. Let $\mathbb{Z}^n$ denote the set of points in the n-dimensional Euclidean space with integer coordinates. Useful tools from algebraic topology and geometric topology for studying digital topological properties of a (binary) digital space include a digital covering space, a (digital) k-fundamental group, a digital k-surface and so forth. These have been studied in numerous papers (Boxer, 1999; Boxer and Karaca, 2008; Han, 2005b; 2005c; 2005d; 2006a; 2006b; 2006c; 2006d; 2007a; 2007b; 2008a; 2008b; 2008c; 2008d; 2009a; 2009b; 2009c; 2010a; 2010b; 2010c; Malgouyres and Lenoir, 2000; Khalimsky, 1987; Rosenfeld and Klette, 2003).

Motivated by a regular covering space in algebraic topology (Spanier, 1966), its digital version was established in digital covering theory (Han, 2006b) (see also Han, 2007b), which plays an important role in classifying digital covering spaces (Boxer and Karaca, 2008; Han, 2010a). In algebraic topology, for a circle S^1 the existence problem of its regular covering space has substantially contributed to the study of a covering space

in topology (Massey, 1977; Spanier, 1966). It is well known that for a covering $(\tilde{X}, p, X)$, if the classical fundamental group $\pi_1(X)$ is an abelian group, then the total space $\tilde{X}$ is regular (Massey, 1977; Spanier, 1966). Unlike these properties, their digital versions have some intrinsic features (Han, 2006b; 2007b; 2009a; 2009b; 2009c; 2010a; 2010b; 2010c; In-Soo Kim and Han, 2008). Boxer and Karaca (2008) as well as Han (2006b; 2007b; 2008a; 2008b; 2009b) studied an automorphism group of a radius 2 (digital) covering (E, p, B). In addition, Boxer and Karaca (2008) studied a classification of digital spaces by using the conjugacy class corresponding to a digital covering. Furthermore, Han (2009c) developed the generalized universal (briefly, GU-)covering property which improves the universal $(2, k)$-covering property of Boxer (2006).

Main applications of digital covering theory include the calculations of both a digital fundamental group of a digital space and an automorphism group of a digital covering. For many digital coverings (E, p, B) except that satisfying a radius 2 local isomorphism the study of both their automorphism groups and digital fundamental groups of E and B remains open. In order to answer this query, the paper establishes the notion of an ultra regular

digital covering space and studies an automorphism group of a digital covering (E, p, B) which does not satisfy a radius 2 local isomorphism. This can play an important role in classifying digital covering spaces.

The paper is organized as follows. Section 2 provides some basic notions. Section 3 reviews some results related to the study of a radius 2 (digital) covering and investigates some properties of a regular covering space from the viewpoint of digital covering theory. Section 4 develops the notion of compatible k-adjacency of a digital wedge which can be used for studying an automorphism group of an ultra regular covering space in Section 5. In addition, we discuss a limitation of the digital version of a regular covering in algebraic topology. Section 5 develops the notion of an ultra regular covering space and studies automorphism groups of several kinds of digital coverings. Section 6 compares an ultra regular covering space and a regular covering space, and refers to strong merits of the former. Finally, Section 7 concludes the paper with a summary.

2. Preliminaries

To study a multidimensional digital space $X \subset \mathbb{Z}^n$, let us now recall the k-adjacency relations of $\mathbb{Z}^n$ as well as some essential terminology such as a digital isomorphism, a digital homotopy, a strong k-deformation retract and so forth. Motivated by the k-adjacency relations of $2D$ and $3D$ digital spaces (Kong and Rosenfeld, 1996; Rosenfeld, 1979), the k-adjacency relations of $\mathbb{Z}^n$ were established (Han, 2003) (see also Han, 2005c; 2008d; 2010c). For a natural number m with $1 \leq m \leq n$, two distinct points

$$p = (p_1, p_2, \cdots, p_n), q = (q_1, q_2, \cdots, q_n) \in \mathbb{Z}^n$$

are called $k(m,n)$- (briefly, k-) adjacent if

- there are at most m indices i such that $|p_i - q_i| = 1$ and
- for all other indices i such that $|p_i - q_i| \neq 1, p_i = q_i$.

Concretely, we can observe that the $k(m,n)$ (or k)-adjacency relations of $\mathbb{Z}^n$ are determined according to the two numbers $m, n \in \mathbb{N}$ (Han, 2003) (also Han, 2005c; 2008d; 2010c), as follows.

Proposition 1. (Han, 2008d) *By using the above operator, we can obtain the k-adjacency of $\mathbb{Z}^n$ as follows:*

$$k := k(m,n) = \sum_{i=n-m}^{n-1} 2^{n-i} C_i^n,$$

where

$$C_i^n = \frac{n!}{(n-i)!\, i!}.$$

In general, a pair (X, k) is assumed to be a (binary) digital space (or digital image) with k-adjacency in a quadruple $(\mathbb{Z}^n, k, \bar{k}, X)$, where $(k, \bar{k}) \in \{(k, 2n), (2n, 3^n - 1)\}$ with $k \neq \bar{k}$, k represents an adjacency relation for X, and $\bar{k}$ represents an adjacency relation for $\mathbb{Z}^n - X$ (Kong and Rosenfeld, 1996). More precisely, owing to the digital k-connectivity paradox found in the work of Kong and Rosenfeld (1996), we remind $k \neq \bar{k}$ except the case $(\mathbb{Z}, 2, 2, X)$. For $\{a, b\} \subset \mathbb{Z}$ with $a \lneq b$, $[a, b]_{\mathbb{Z}} = \{a \leq n \leq b | n \in \mathbb{Z}\}$ is considered in $(\mathbb{Z}, 2, 2, [a, b]_{\mathbb{Z}})$ (Boxer, 1999). But in this paper we are not concerned with $\bar{k}$-adjacency between two points in $\mathbb{Z}^n - X$.

We say that two subsets (A, k) and (B, k) of (X, k) are k-adjacent to each other if $A \cap B = \emptyset$ and there are points $a \in A$ and $b \in B$ such that a and b are k-adjacent to each other (Kong and Rosenfeld, 1996). We say that a set $X \subset \mathbb{Z}^n$ is k-connected if it is not a union of two disjoint non-empty sets that are not k-adjacent to each other (Kong and Rosenfeld, 1996). For an adjacency relation k of $\mathbb{Z}^n$, a simple k-path with $l + 1$ elements in $\mathbb{Z}^n$ is assumed to be an injective sequence $(x_i)_{i \in [0,l]_{\mathbb{Z}}} \subset \mathbb{Z}^n$ such that x_i and x_j are k-adjacent if and only if either $j = i+1$ or $i = j + 1$ (Kong and Rosenfeld, 1996). If $x_0 = x$ and $x_l = y$, then we say that the length of the simple k-path, denoted by $l_k(x, y)$, is the number l. A simple closed k-curve with l elements in $\mathbb{Z}^n$, denoted by $SC_k^{n,l}$ (Han, 2006b), is the simple k-path $(x_i)_{i \in [0,l-1]_{\mathbb{Z}}}$, where x_i and x_j are k-adjacent if and only if $j = i + 1 (\mathrm{mod}\ l)$ or $i = j + 1 (\mathrm{mod}\ l)$ (Kong and Rosenfeld, 1996).

In the study of digital continuity and various properties of a digital space (Han, 2006a; 2006d), we have often used the following digital k-neighborhood of a point $x \in X$ with radius $\varepsilon \in \mathbb{N}$ (Han, 2003) (see also Han, 2005c): For a digital space (X, k) in $\mathbb{Z}^n$, the digital k-neighborhood of $x_0 \in X$ with radius ε is defined in X to be the following subset of X: $N_k(x_0, \varepsilon) = \{x \in X \mid l_k(x_0, x) \leq \varepsilon\} \cup \{x_0\}$, where $l_k(x_0, x)$ is the length of a shortest simple k-path from x_0 to x and $\varepsilon \in \mathbb{N}$.

Motivated by both the digital continuity of Rosenfeld (1979) and the (k_0, k_1)-continuity of Boxer (1999), we can present digital continuity which can be substantially used for studying digital spaces in $\mathbb{Z}^n, n \in \mathbb{N}$, as follows.

Proposition 2. (Han, 2008d) *Let (X, k_0) and (Y, k_1) be digital spaces in $\mathbb{Z}^{n_0}$ and $\mathbb{Z}^{n_1}$, respectively. A function $f : X \to Y$ is (k_0, k_1)-continuous if and only if for every $x \in X$, $f(N_{k_0}(x, 1)) \subset N_{k_1}(f(x), 1)$.*

Since a digital space (X, k) can be considered to be a digital k-graph, we may use the term a (k_0, k_1)-isomorphism as in the work of Han (2005d) (see also Boxer, 2006) rather than a (k_0, k_1)-homeomorphism as used by Boxer (1999), as follows.

Definition 1. (Han, 2005d, see also Boxer, 2006) For two digital spaces (X, k_0) in $\mathbb{Z}^{n_0}$ and (Y, k_1) in $\mathbb{Z}^{n_1}$, a

map $h : X \to Y$ is called a (k_0, k_1)-*isomorphism* if h is a (k_0, k_1)-continuous bijection and, further, $h^{-1} : Y \to X$ is (k_1, k_0)-continuous. Then we use the notation $X \approx_{(k_0,k_1)} Y$. If $n_0 = n_1$ and $k_0 = k_1$, then we speak about a k_0-isomorphism and use the notation $X \approx_{k_0} Y$.

For a digital space (X, k) and $A \subset X$, (X, A) is called a *digital space pair with k-adjacency* (Han, 2006a). Furthermore, if A is a singleton set $\{x_0\}$, then (X, x_0) is called a *pointed digital space* (Kong and Rosenfeld, 1996). Based on the pointed digital homotopy of Boxer (1999), the following notion of k-homotopy relative to a subset $A \subset X$ has been often used in studying a k-homotopic thinning and a strong k-deformation retract of a digital space (X, k) in $\mathbb{Z}^n$ (Han, 2008d).

Definition 2. (Han, 2006a, see also Han, 2006b; 2007a) Let $((X, A), k_0)$ and (Y, k_1) be a digital space pair and a digital space, respectively. Let $f, g : X \to Y$ be (k_0, k_1)-continuous functions. Suppose there exist $m \in \mathbb{N}$ and a function $F : X \times [0, m]_{\mathbb{Z}} \to Y$ such that

- For all $x \in X$, $F(x, 0) = f(x)$ and $F(x, m) = g(x)$.

- For all $x \in X$, the induced function $F_x : [0, m]_{\mathbb{Z}} \to Y$ given by $F_x(t) = F(x, t)$ for all $t \in [0, m]_{\mathbb{Z}}$ is $(2, k_1)$-continuous.

- For all $t \in [0, m]_{\mathbb{Z}}$, the induced function $F_t : X \to Y$ given by $F_t(x) = F(x, t)$ for all $x \in X$ is (k_0, k_1)-continuous. Then we say that F is a (k_0, k_1)-homotopy between f and g (Boxer, 1999).

- Furthermore, for all $t \in [0, m]_{\mathbb{Z}}$ then suppose the induced map F_t on A is a constant which is a prescribed function from A to Y. In other words, $F_t(x) = f(x) = g(x)$ for all $x \in A$ and for all $t \in [0, m]_{\mathbb{Z}}$. Then we call F a (k_0, k_1)-homotopy relative to A between f and g, and we say f and g are (k_0, k_1)-homotopic relative to A in Y, $f \simeq_{(k_0,k_1)\mathrm{rel}A} g$ in symbols.

In Definition 2, if $A = \{x_0\} \subset X$, then we say that F is a pointed (k_0, k_1)-homotopy at $\{x_0\}$ (Boxer, 1999). When f and g are pointed (k_0, k_1)-homotopic in Y, we denote by $f \simeq_{(k_0,k_1)} g$. In addition, if $k_0 = k_1$ and $n_0 = n_1$, then we say that f and g are pointed k_0-homotopic in Y and use the notation $f \simeq_{k_0} g$ and $f \in [g]$ which means the k_0-homotopy class of g. If, for some $x_0 \in X$, 1_X is k-homotopic to the constant map with the space x_0 relative to $\{x_0\}$, then we say that (X, x_0) is pointed k-contractible (Boxer, 1999). Indeed, the notion of k-contractibility is slightly different from the contractibility in Euclidean topology (Boxer, 1999) (see also Han, 2005c).

Definition 3. (Han, 2006b, see also Han, 2007a) For a digital space pair $((X, A), k)$, we say that A is a *strong k-deformation retract of X* if there is a digital k-continuous map r from X onto A such that $F : i \circ r \simeq_{k \cdot rel.A} 1_X$ and $r \circ i = 1_A$. Then a point $x \in X - A$ is called strong k-deformation retractable.

By using the trivial extension presented by Boxer (1999) and the *Khalimsky operation* presented by Khalimsky (1987), Boxer (1999) establishes the k-fundamental group: For a digital space (X, k), consider a k-loop f with a base point x_0; we denote by $[f]_X$ (briefly, $[f]$) the k-homotopy class of f in X. Then for a k-loop f_1 with the same base point $x_0 \in X$, $f_0 \in [f]$ means that the two k-loops f and f_0 have trivial extensions that can be joined by a k-homotopy keeping the end point fixed (Han, 2005c) (see also Boxer, 2006). Furthermore, if $f_1, f_2, g_1, g_2 \in F^k(X, x_0)$, $f_1 \in [f_2]$, and $g_1 \in [g_2]$, then $f_1 * g_1 \in [f_2 * g_2]$, i.e., $[f_1 * g_1] = [f_2 * g_2]$ (Boxer, 1999; Khalimsky, 1987). Then we use the notation $\pi^k(X, x_0) = \{[f] | f \in F^k(X, x_0)\}$ which is a group (Boxer, 1999) with the operation $[f] \cdot [g] = [f * g]$ called the (digital) k-fundamental group of (X, x_0) (Boxer, 1999), where the base point is assumed to be a point which cannot be deleted by a strong k-deformation retract (Han, 2008a). If X is pointed k-contractible, then $\pi^k(X, x_0)$ is trivial (Boxer, 1999).

Let $((X, A), k)$ be a digital space pair with k-adjacency. A map $f : ((X, A), k_0) \to ((Y, B), k_1)$ is called (k_0, k_1)-continuous if f is (k_0, k_1)-continuous and $f(A) \subset B$ (Han, 2006a). If $A = \{a\}$, $B = \{b\}$, we write $(X, A) = (X, a)$, $(Y, B) = (Y, b)$, and we say that f is a pointed (k_0, k_1)-continuous map (Kong and Rosenfeld, 1996). A (k_0, k_1)-continuous map $f : ((X, x_0), k_0) \to ((Y, y_0), k_1)$ induces a group homomorphism $f_* : \pi^{k_0}(X, x_0) \to \pi^{k_1}(Y, y_0)$ given by $f_*([\alpha]) = [f \circ \alpha]$, where $[\alpha] \in \pi^{k_0}(X, x_0)$ (Boxer, 1999). In addition, a (k_0, k_1)-isomorphism $\phi : ((X, x_0), k_0) \to ((Y, y_0), k_1)$ induces a group isomorphism $\phi_* : \pi^{k_0}(X, x_0) \to \pi^{k_1}(Y, y_0)$ (Boxer, 1999).

The following notion of "simply k-connected" found in the work of Han (2005c) has been often used in digital k-homotopy theory and digital covering theory: A pointed k-connected digital space (X, x_0) is called simply k-connected if $\pi^k(X, x_0)$ is a trivial group.

Han (2006b) (see also Han, 2007a) proved that if (A, x_0) is a strong k-deformation retract of (X, x_0), then $\pi^k(X, x_0)$ is isomorphic to $\pi^k(A, x_0)$. Since k-contractibility requires a digital space (X, k) to shrink (k, k)-continuously to a point over a finite time interval, we cannot say that $\mathbb{Z}^n$ is $2n$-contractible, $n \in \mathbb{N}$. However, motivated by simple 2-connectedness of $\mathbb{Z}$ (Han, 2005c), we can obtain that $(\mathbb{Z}^n, 0_n)$ is simply k-connected, where k-adjacency is assumed to be anyone of k-adjacency relations of $\mathbb{Z}^n$.

Motivated by both 8-contractibility of $SC_8^{2,4}$ (Boxer, 1999) and non-8-contractibility of $SC_8^{2,6}$ (Han, 2005c), the paper by Han (2005c) (see also Han, 2006b; 2007a)

proved that $\pi^k(SC_k^{n,l})$ is an infinite cyclic group, where $SC_k^{n,l}$ is not k-contractible. Precisely, $\pi^k(SC_k^{n,l}) \simeq (l\mathbb{Z}, +)$, where $SC_k^{n,l}$ is not k-contractible and "$\simeq$" means a group isomorphism. In addition, $\pi^k(SC_k^{n,4})$ is trivial if $k = 3^n - 1, n \in \mathbb{N} - \{1\}$.

3. Some properties of a regular covering space

Let (X, k) be a digital space in $\mathbb{Z}^n$. In relation to the calculation of $\pi^k(X, x_0)$ and the classification of digital spaces in terms of a digital k-homotopy, we have often used some properties of a digital covering (Boxer, 2006; Boxer and Karaca, 2008; Han, 2005b; 2005c; 2008d; 2009a). In digital covering theory, since each digital space (X, k) is assumed to be k-connected, hereafter, every (X, k) is considered to be k-connected unless stated otherwise. In this section we study some properties of a regular covering space. Let us now recall the typical axiom of a digital covering space, as follows.

Definition 4. (Han, 2005c, see also Han, 2008b) Let (E, k_0) and (B, k_1) be digital spaces in $\mathbb{Z}^{n_0}$ and $\mathbb{Z}^{n_1}$, respectively. Let $p : E \to B$ be a (k_0, k_1)-continuous surjection. Suppose that for any $b \in B$ there exists $\varepsilon \in \mathbb{N}$ such that
(1) for some index set M, $p^{-1}(N_{k_1}(b, \varepsilon)) = \cup_{i \in M} N_{k_0}(e_i, \varepsilon)$ with $e_i \in p^{-1}(b)$;
(2) if $i, j \in M$ and $i \neq j$, then $N_{k_0}(e_i, \varepsilon) \cap N_{k_0}(e_j, \varepsilon)$ is an empty set; and
(3) the restriction map p on $N_{k_0}(e_i, \varepsilon)$ is a (k_0, k_1)-isomorphism for all $i \in M$.
Then the map p is called a (k_0, k_1)-*covering map* and (E, p, B) is said to be a (k_0, k_1)-*covering*.

The k_1-neighborhood $N_{k_1}(b, \varepsilon)$ in Definition 4 is called an elementary k_1-neighborhood of b with some radius ε and E is called a (k_0, k_1)-covering space of p.

Definition 5. (Han, 2006b, see also Han, 2008b) We say that a (k_0, k_1)-covering map $p : (E, e_0) \to (B, b_0)$ is an m-*fold* (k_0, k_1)-*covering map* if the cardinality of the index set M is m.

Definition 5 can be restated as follows: For a (k_0, k_1)-covering map $p : (E, e_0) \to (B, b_0)$, if the set $p^{-1}(b_0)$ has n elements (or the number n can also be called the sheets of the digital covering (Massey, 1977), then the map p is called an m-fold (k_0, k_1)-covering map because any points $b_1, b_2 \in B$ satisfy the following identity in terms of the digital version of the corresponding properties of a covering found in the work of Massey (1977): $\sharp\{p^{-1}(b_1)\} = \sharp\{p^{-1}(b_2)\} = m$, where "$\sharp$" means the cardinality of the given set. For instance, for any $SC_k^{n,l} := (c_t)_{t \in [0, l-1]_{\mathbf{Z}}}$ we observe that $(SC_k^{n,ml} := (a_t)_{t \in [0, ml-1]_{\mathbb{Z}}}, p, SC_k^{n,l})$, given by $p(a_i) = c_{i(mod\, l)}, m \in \mathbb{N}$, is an m-fold (k, k)-covering.

For pointed digital spaces $((E, e_0), k_0)$ and $((B, b_0), k_1)$, if $p : (E, e_0) \to (B, b_0)$ is a (k_0, k_1)-covering map such that $p(e_0) = b_0$, then the map p is a pointed (k_0, k_1)-covering map (Han, 2005c). Hereafter, we assume that each digital covering map is a pointed one unless stated otherwise.

Definition 6. (Han, 2005a, see also Han, 2005b; 2008c) For two digital spaces (X, k_0) in $\mathbb{Z}^{n_0}$ and (Y, k_1) in $\mathbb{Z}^{n_1}$, a (k_0, k_1)-continuous map $h : X \to Y$ is called a local (k_0, k_1)-isomorphism if for any $x \in X$, h maps $N_{k_0}(x, 1)$ (k_0, k_1)-isomorphically onto $N_{k_1}(h(x), 1) \subset Y$. If $n_0 = n_1$ and $k_0 = k_1$, then the map h is called a local k_0-isomorphism.

This local (k_0, k_1)-isomorphism has often been used in studying the preservation of local k_0-properties of a digital space (X, k_0) into its corresponding k_1-ones in digital geometry (Han, 2008c).

Since a (k_0, k_1)-isomorphism is equivalent to a locally (k_0, k_1)-isomorphic bijection (Han, 2005a) (see also Han, 2006b) and a restriction map of a (k_0, k_1)-isomorphism is also a (k_0, k_1)-isomorphism (Han, 2006d), we obtain the following property: If $h : (X, k_0) \to (Y, k_1)$ is a (k_0, k_1)-isomorphism, then the restriction map on $N_{k_0}(x, 1)$, $h|_{N_{k_0}(x,1)} : N_{k_0}(x, 1) \to N_{k_1}(h(x), 1)$ is a (k_0, k_1)-isomorphism (Han, 2005a) (see also Han, 2006d). Thus we obtain the following.

Remark 1.
(1) As discussed by Han (2006b), we may take $\varepsilon = 1$ for the (k_0, k_1)-covering of Definition 4.
(2) As discussed by Han (2009c) (for more details, see another work of Han (2010d)), for the (k_0, k_1)-covering of Definition 4 we can replace "(k_0, k_1)-continuous surjection" with "surjection".

Definition 7. (Han, 2005b) For $n \in \mathbb{N}$, a (k_0, k_1)-covering (E, p, B) is a radius n local isomorphism if the restriction map $p|_{N_{k_0}(e_i, n)} : N_{k_0}(e_i, n) \to N_{k_1}(b, n)$ is a (k_0, k_1)-isomorphism for all $i \in M$, where $e_i \in p^{-1}(b)$.

By Definition 7, we can say that a (k_0, k_1)-covering (E, p, B) is a radius n-(k_0, k_1)-covering if $\varepsilon \geq n$, where the number ε is the same as ε in Definition 4 (Han, 2005b) (see also Han, 2008b).

In view of Definitions 4 and 7, we observe that a (k_0, k_1)-covering satisfying a radius n local isomorphism is equivalent to a radius n-(k_0, k_1)-covering (Han, 2006b).

Since both the unique digital lifting theorem and the digital homotopy lifting theorem will often be used for studying a digital covering space, let us now review them along with related results, as follows. For three digital spaces (E, k_0) in $\mathbb{Z}^{n_0}$, (B, k_1) in $\mathbb{Z}^{n_1}$, and (X, k_2) in $\mathbb{Z}^{n_2}$, let $p : E \to B$ be a (k_0, k_1)-continuous map. For a (k_2, k_1)-continuous map $f : (X, k_2) \to (B, k_1)$, as the digital analogue of the lifting found in the work

of Massey (1977), we say that a digital lifting of f is a (k_2, k_0)-continuous map $\tilde{f} : X \to E$ such that $p \circ \tilde{f} = f$ (Han, 2005c). We now recall the unique digital lifting theorem of Han (2005c), as follows.

Lemma 1. (Han, 2005c) *For pointed digital spaces $((E, e_0), k_0)$ in $\mathbb{Z}^{n_0}$ and $((B, b_0), k_1)$ in $\mathbb{Z}^{n_1}$, let $p : (E, e_0) \to (B, b_0)$ be a pointed (k_0, k_1)-covering map. Any k_1-path $f : [0, m]_{\mathbb{Z}} \to B$ beginning at b_0 has a unique digital lifting to a k_0-path $\tilde{f}$ in E beginning at e_0.*

Moreover, the following digital homotopy lifting theorem was introduced by Han (2005b), which plays an important role in studying digital covering theory.

Lemma 2. (Han, 2006b, see also Han, 2007b) *Let $((E, e_0), k_0)$ and $((B, b_0), k_1)$ be pointed digital spaces. Let $p : (E, e_0) \to (B, b_0)$ be a radius 2-(k_0, k_1)-covering map. For k_0-paths g_0, g_1 in (E, e_0) that start at e_0, if there is a k_1-homotopy in B from $p \circ g_0$ to $p \circ g_1$ that holds the endpoints fixed, then g_0 and g_1 have the same terminal point, and there is a k_0-homotopy in E from g_0 to g_1 that holds the endpoints fixed.*

The following notion has often been used for calculating the k-fundamental group of a digital space (X, k) and classifying digital spaces (Han, 2007a; 2007b; 2008d).

Definition 8. A (k_0, k_1)-covering $((E, e_0), p, (B, b_0))$ is called *regular* if $p_* \pi^{k_0}(E, e_0)$ is a normal subgroup of $\pi^{k_1}(B, b_0)$.

As discussed by Han (2007b) (see also Han, 2008a), by using Massey's program (Massey, 1977), we obtain the following: Let $((E, e_0), p, (B, b_0))$ be a radius 2-(k_0, k_1)-covering such that E is k_0-connected. For any $\tilde{e} \in p^{-1}(b_0)$ and any $\alpha \in \pi^{k_1}(B, b_0)$, defining $\tilde{e} \cdot \alpha \in p^{-1}(b_0)$, we observe that the set $p^{-1}(b_0)$ is a homogeneous right $\pi^{k_1}(B, b_0)$-space (Han, 2007b) (see also Han, 2009b) because the group $\pi^{k_1}(B, b_0)$ operates transitively on the set $p^{-1}(b_0)$ (Han, 2007b) (see also Han, 2008a; 2010c). Precisely, consider $e \in p^{-1}(b_0)$ and $\alpha \in \pi^{k_1}(B, b_0)$. Take a k_1-path $f : [0, m_f]_{\mathbb{Z}} \to (B, b_0)$ such that $[f] = \alpha$, $f(0) = b_0 = f(m_f)$ Then, by Lemma 2, the map f is well-defined. By Lemma 1, there is the unique k_0-path $\tilde{f} : [0, m_f]_{\mathbb{Z}} \to (E, e_0)$ such that $p_*([\tilde{f}]) = [f]$ and $\tilde{f}(0) = e$ with the following.
Define

$$p^{-1}(b_0) \times \pi^{k_1}(B, b_0) \to p^{-1}(b_0) \qquad (1)$$

by

$$(e, \alpha) \to e \cdot \alpha = \tilde{f}(m_f),$$

and by Lemma 2 this action is well defined because this process does not depend on the choice of the map f, where $[f] = \alpha$. Then for any $e \in p^{-1}(b_0)$, we clearly observe the following (Han, 2007b):

$$(e, 1) = e \quad \text{and} \quad ((e, \alpha), \beta) = (e, \alpha \cdot \beta), \qquad (2)$$

where $\alpha, \beta \in \pi^{k_1}(B, b_0)$ and 1 is the identity element. This implies $p^{-1}(b_0)$ admits $\pi^{k_1}(B, b_0)$ as a group of operator (or permutation).

For a (k_0, k_1)-covering map $p : ((E, e_0), k_0) \to ((B, b_0), k_1)$ in order to study a relation between $p^{-1}(b_0)$ and a coset $\pi^{k_1}(B, b_0)/p_* \pi^{k_0}(E, e_0)$, we often use the following property.

Lemma 3. (Massey, 1977) *Let E be a set and G a group. If $E \times G \to E$ is a transitive action, then E is isomorphic to the factor group G/G_{x_0}, where $G_{x_0} := \{g \in G | g(x_0) = x_0\}$ called the isotropy subgroup of G.*

By using Massey's program (Massey, 1977), and Lemmas 1 and 2, we obtain the following.

Theorem 1. (Han, 2009b) *Let $((E, e_0), p, (B, b_0))$ be a pointed radius 2-(k_0, k_1)-covering and (E, k_0) k_0-connected. Then we obtain that*

(1) $\pi^{k_1}(B, b_0)$ operates transitively on $p^{-1}(b_0)$ on the right (Han, 2007b).

(2) The right $\pi^{k_1}(B, b_0)$-space $p^{-1}(b_0)$ is bijective to $\pi^{k_1}(B, b_0)/p_* \pi^{k_0}(E, e_0)$.

Remark 2. (Han, 2009b) Let us recall that, by Theorem 5(2), the group $p^{-1}(b_0)$ found in the works of Han (2007b; 2008a; 2008d), related to the assertion of Theorem 1(2), is clearly isomorphic to the factor group $\pi^{k_1}(B, b_0)/p_* \pi^{k_0}(E, e_0)$ as a right $\pi^{k_1}(B, b_0)$-space, and the papers by Han (2007b; 2008a; 2008d) tell more about the case when $p^{-1}(b_0)$ has the group structure $(p^{-1}(b_0), +)$ derived from the given digital coverings such as $(\mathbb{Z}, p, SC_k^{n,l})$ and $(\mathbb{Z} \times \mathbb{Z}, p_1 \times p_2, SC_{k_1}^{n_1,l_1} \times SC_{k_2}^{n_2,l_2})$, where $(SC_{k_1}^{n_1,l_1} \times SC_{k_2}^{n_2,l_2}, k)$ has the L_S-or L_C-property found in the research by Han (2009a; 2010b; 2006b; 2007a; 2008d). Of course, by Theorem 1, in this case we clearly observe that the given digital covering map p should be regular so that $\pi^{k_1}(B, b_0)/p_* \pi^{k_0}(E, e_0)$ is clearly isomorphic to $N(p_* \pi^{k_0}(E, e_0))/p_* \pi^{k_0}(E, e_0)$.

In algebraic topology, both Deck's transformation group of a covering map and a universal covering space have strongly contributed to the study of the classification of topological spaces. Motivated by the covering homomorphism of Spanier (1966), the notion of (k_1, k_2)-homomorphism from a (k_1, k)-covering (E_1, p_1, B) into a (k_2, k)-covering (E_2, p_2, B) was introduced by Han (2007a) (see also Boxer and Karaca, 2008) and has contributed to the establishment of an automorphism group of a digital covering map in the work of Han (2008b) as well as In-Soo Kim and Han (2008), which is so different from that of a covering space in algebraic topology.

An automorphism group of a digital (k_0, k_1)-covering has also substantial advantages which make it convenient and efficient for calculating the digital fundamental groups of a digital spaces and classifying digital

spaces (Han, 2006b; 2008a; 2008b, 2010b). For three digital spaces (B, k), (E_1, k_1) and (E_2, k_2), let (E_1, p_1, B) and (E_2, p_2, B) be (k_1, k)- and (k_2, k)-coverings, respectively. Then we say that a (k_1, k_2)-continuous map $\phi : E_1 \to E_2$ such that $p_2 \circ \phi = p_1$ is a (k_1, k_2)-covering homomorphism from (E_1, p_1, B) into (E_2, p_2, B) (Han, 2007a) (see also Boxer and Karaca, 2008). As a special case of this (k_1, k_2)-covering homomorphism, we obtain the digital version of Deck's transformation group of a covering map in algebraic topology (Spanier, 1966). Furthermore, by using the generalized universal covering property of Han (2009c), we can classify digital covering spaces.

Definition 9. (Han, 2008b, see also Boxer and Karaca, 2008; Kim and Han, 2008) Consider a (k_0, k_1)-covering map $p : ((E, e_0) \to ((B, b_0)$. A self k_0-isomorphism of the (k_0, k_1)-covering map p, denoted by $h : (E, k_0) \to (E, k_0)$, is called a k_0-covering transformation or an automorphism of a digital covering map p if $p = p \circ h$, where $\circ$ means the composition. The set of the automorphisms of a digital covering map with composition operation is obviously a group which is denoted by $Aut(E|B)$ (or $Aut(E, p, B)$).

In the study of an automorphism group of a digital (k_0, k_1)-covering, by using (1) and (2), motivated by various properties of covering space found in the work of Massey (1977), we can obtain the following.

Theorem 2. (Han, 2009b)
(1) Let $((E, e_0), p, (B, b_0))$ be a pointed radius 2-(k_0, k_1)-covering and (E, k_0) k_0-connected. Then $Aut(E|B)$ is isomorphic to $Aut(p^{-1}(b_0))$ induced from the map $\phi \in Aut(E|B)$, where $Aut(p^{-1}(b_0))$ is considered as a right $\pi^{k_1}(B, b_0)$-space.
(2) Let $((E, e_0), p, (B, b_0))$ be a pointed radius 2-(k_0, k_1)-covering and (E, k_0) k_0-connected. Then we obtain $Aut(E|B) \simeq N(p_*\pi^{k_0}(E, e_0))/p_*\pi^{k_0}(E, e_0)$, where $N(p_*\pi^{k_0}(E, e_0))$ is the normalizer of $p_*\pi^{k_0}(E, e_0)$ in $\pi^{k_1}(B, b_0)$.
(3) Let $((E, e_0), p, (B, b_0))$ be a pointed radius 2-(k_0, k_1)-covering which is regular and (E, k_0) k_0-connected. Then we obtain

$$Aut(E|B) \simeq Aut(p^{-1}(b_0))$$
$$\simeq \pi^{k_1}(B, b_0)/p_*\pi^{k_0}(E, e_0).$$

(4) Let $((E, e_0), p, (B, b_0))$ be a pointed radius 2-(k_0, k_1)-covering and (E, k_0) k_0-connected. Then we obtain the following: $((E, e_0), p, (B, b_0))$ is regular if and only if $p_*\pi^{k_0}(E, e_0) = p_*\pi^{k_0}(E, e_1)$, where $p(e_0) = p(e_1) = b_0$.

By using Theorem 2, we can study automorphism groups of many digital coverings including a (k_0, k_1)-covering which does not satisfy a radius 2 local isomorphism in Section 5.

4. Compatible adjacency of a digital wedge

Since a digital wedge can play an important role in studying an automorphism group of a digital covering, let us now recall a digital wedge discussed by Han (2005c) (see also Boxer, 2006; Han, 2009c). For digital spaces (X_i, k_i) in $\mathbb{Z}^{n_i}$, $i \in \{0, 1\}$, the notion of digital wedge of (X_i, k_i) was introduced by Han (2005c). In relation to the study of automorphism groups of both an ultra regular and a regular covering space in Sections 5 and 6, motivated by the former version of Han (2009c), we need to develop a notion of compatible k-adjacency of a digital wedge as follows.

Definition 10. For pointed digital spaces $((X, x_0), k_0)$ in $\mathbb{Z}^{n_0}$ and $((Y, y_0), k_1)$ in $\mathbb{Z}^{n_1}$, the wedge of (X, k_0) and (Y, k_1), written $(X \vee Y, (x_0, y_0))$, is the digital space in $\mathbb{Z}^n$,

$$\{(x, y) \in X \times Y \mid x = x_0 \text{ or } y = y_0\}, \qquad (3)$$

with compatible $k(m, n)$(or k)-adjacency relative to both (X, k_0) and (Y, k_1), and the only one point (x_0, y_0) in common with the following property:
(W1) The $k(m, n)$ (or k)-adjacency is determined by the numbers m and n with $n = \max\{n_0, n_1\}$, $m = \max\{m_0, m_1\}$ satisfying $(W1{-}1)$ below, where the numbers m_i are taken from the k_i(or $k(m_i, n_i)$)-adjacency relations of the given digital spaces $((X, x_0), k_0)$ and $((Y, y_0), k_1)$, $i \in \{0, 1\}$.

(W 1-1) In view of (3), induced from the projection maps, we can consider the natural projection maps,

$$W_X : (X \vee Y, (x_0, y_0)) \to (X, x_0)$$

and

$$W_Y : (X \vee Y, (x_0, y_0)) \to (Y, y_0).$$

In relation to the establishment of a compatible k-adjacency of the digital wedge $(X \vee Y, (x_0, y_0))$, the restriction maps of W_X and W_Y on $(X \times \{y_0\}, (x_0, y_0)) \subset (X \vee Y, (x_0, y_0))$ and $(\{x_0\} \times Y, (x_0, y_0)) \subset (X \vee Y, (x_0, y_0))$ satisfy the following properties, respectively,

(1) $W_X|_{X \times \{y_0\}} : (X \times \{y_0\}, k) \to (X, k_0)$ is a (k, k_0)-isomorphism, and

(2) $W_Y|_{\{x_0\} \times Y} : (\{x_0\} \times Y, k) \to (Y, k_1)$ is a (k, k_1)-isomorphism.

(W2) Any two distinct elements $x(\neq x_0) \in X \subset X \vee Y$ and $y(\neq y_0) \in Y \subset X \vee Y$ are not $k(m, n)$ (or k)-adjacent to each other.

Example 1. For several types of simple closed k-curves in $\mathbb{Z}^n$, $n \in \{2, 3\}$, (see Fig. 1), we can observe the following compatible k-adjacencies of digital wedges:

(1) $(MSC_{18} \vee SC_8^{2,4}, 18)$.

(2) $(SC_8^{2,6} \vee SC_{26}^{2,4}, 26)$.

(3) There is no compatible k-adjacency of $(MSC_{18} \vee SC_{26}^{2,4}, k)$.

(4) There is no compatible k-adjacency of $(SC_4^{2,8} \vee SC_8^{2,6}, k)$.

Proof. (1) and (2) are clearly proved.
(3) In terms of (W1) of Definition 10, $MSC_{18} \vee SC_{26}^{2,4}$ should be considered in $\mathbb{Z}^3$. Now we only examine if the digital wedge $MSC_{18} \vee SC_{26}^{2,4}$ has a compatible 26-adjacency because 18- and 6-adjacencies cannot be valid contrary to (W1) of Definition 10. Suppose that it has a compatible 26-adjacency. Then it clearly violates the property (W1-1) of Definition 10.
(4) With a method similar to the proof of (3), we can prove that $SC_4^{2,8} \vee SC_8^{2,6} \subset \mathbf{Z}^2$ cannot have any compatible k-adjacency, $k \in \{4, 8\}$. Precisely, owing to the properties (W1) (resp. (W1-1)), $SC_4^{2,8} \vee SC_8^{2,6}$ cannot have the compatible 4-(resp., 8-)adjacency. ∎

Remark 3. The notion of compatible adjacency of a digital wedge can be considered to be the most reasonable one of a digital wedge. In addition, there is no need for a uniqueness of a compatible adjacency of a digital wedge.
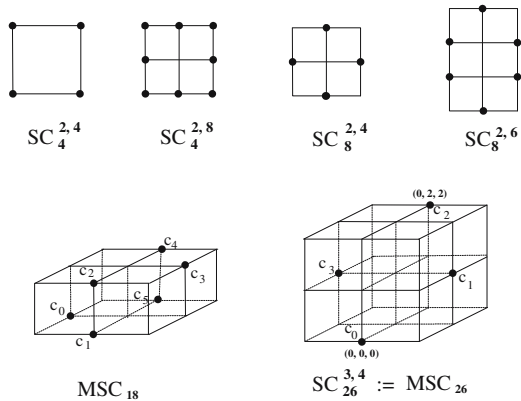

Fig. 1. Several kinds of simple closed k-curves (Han, 2005b; 2006d; 2010b).

In view of Example 1 and Remark 3, we obtain the following:

Remark 4. A compatible k-adjacency of a digital wedge has the following properties:
(1) By comparing with the k-adjacency of a wedge product Han (2009c), we can observe that the current compatible adjacency requires further (W1-1).
(2) Consider $SC_{k_i}^{n_i,l_i}$ with $k_i = 3^{n_i} - 1, i \in \{0, 1\}$. Assume $n_0 \leq n_1$. Then we always have a compatible k_1-adjacency of the digital wedge $SC_{k_0}^{n_0,l_0} \vee SC_{k_1}^{n_1,l_1} \subset \mathbb{Z}^{n_1}$.
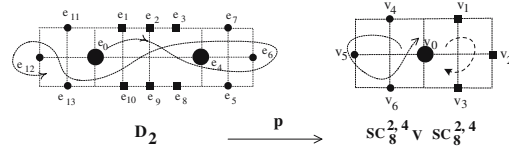

Fig. 2. $(8,8)$-covering $(D_2, p, SC_8^{2,4} \vee SC_8^{2,4})$, which does not satisfy a radius 2 local isomorphism (Han, 2009b).

(3) Consider $SC_{k_i}^{n_i,l_i}$ with $k_i(m_i, n_i)$-adjacency, $i \in \{0, 1\}$. Assume $n_0 \leq n_1$. If $m_0 = m_1$, then we always have a compatible $k(m_1, n_1)$-adjacency of the digital wedge $SC_{k_0}^{n_0,l_0} \vee SC_{k_1}^{n_1,l_1} \subset \mathbb{Z}^{n_1}$ (Han, 2009c).

4.1. Limitation of a regular covering space in digital covering theory.
In spite of the study of various properties of a regular covering space in Section 3, in relation to the study of an automorphism group of a digital covering space, a digital regular covering space has a limitation. More precisely, as discussed in (1), (2), Theorems 1 and 2, the study of an automorphism group of a digital covering requires to satisfy a radius 2 local isomorphism of a given digital covering. Thus, if a (k_0, k_1)-covering does not satisfy a radius 2 local isomorphism, then we have an obstacle to the study of digital homotopic properties of a digital covering as well as its automorphism group (see Boxer, 2006; Boxer and Karaca, 2008; Han, 2005b; 2006b). To be specific, let us now consider the two $(8, 8)$-coverings $(SC_8^{2,12}, p, SC_8^{2,4})$ and $(D_2, p, SC_8^{2,4} \vee SC_8^{2,4})$ in Fig. 2, in which none of them can satisfy a radius 2 local isomorphism. In other words, the digital covering $(SC_8^{2,12}, p, SC_8^{2,4})$ cannot be a radius 2-$(8, 8)$-covering. Namely, in Fig. 2, assume that $p : D_2 \to SC_8^{2,4} \vee SC_8^{2,4}$ maps each of solid squares, small solid circles and big solid circles of D_2 into the corresponding ones in $SC_8^{2,4} \vee SC_8^{2,4}$. Then we can observe that $(D_2, p, SC_8^{2,4} \vee SC_8^{2,4})$ cannot satisfy a radius 2 local isomorphism, either. Consequently, we have to establish another notion instead of a regular (k_0, k_1)-covering (see Section 5).

5. Ultra regular (k_0, k_1)-covering space and its automorphism group

In this section we develop the notion of an ultra regular covering space and investigate its properties related to the study of its automorphism group. As discussed in Section 3, for a radius 2 covering (E, p, B), its automorphism group was studied by Han (2008a) (see also Boxer and Karaca, 2008; Han, 2009b). Meanwhile, if a digital covering (E, p, B) does not satisfy a radius 2 local isomorphism (see Fig. 2), then its automorphism group has intrinsic features, which remains to be studied. In addition, for a digital covering space which satisfies a radius 2 local

isomorphism, its automorphism group can also be studied (see Example 2(2)). Thus, in order to study this problem, this section establishes the notion of an ultra regular digital covering space and studies its automorphism group, which can play an important role in classifying digital covering spaces. Motivated by the transitivity of an automorphism of a covering space found work of in Massey (1977), we can define the following.

Definition 11. For a (k_0, k_1)-covering $((E, e_0), p, (B, b_0))$, we say that $Aut(E|B)$ acts transitively on $p^{-1}(b_0)$ if for any two distinct points e_0 and e_1 in $p^{-1}(b_0)$ there is $\phi \in Aut(E|B)$ such that $\phi(e_0) = e_1$.

In general, for a (k_0, k_1)-covering $((E, e_0), p, (B, b_0))$ $Aut(E|B)$ need not act transitively on $p^{-1}(b_0)$ (see Example 2(1)).

Example 2.
(1) Consider the map $p_1 : E_1 \rightarrow SC_8^{2,6} \vee SC_8^{2,6}$ given by Fig. 3(a). To be specific, assume that p_1 maps each of solid squares, small solid circles, and big solid circles of E_1 into the corresponding ones in $SC_8^{2,6} \vee SC_8^{2,6}$ along with the arrows in E_1. Then, for the $(8, 8)$-covering $(E_1, p_1, SC_8^{2,6} \vee SC_8^{2,6})$, we can observe that $Aut(E_1|SC_8^{2,6} \vee SC_8^{2,6})$ cannot act transitively on $p^{-1}(v_0)$ for the point $v_0 \in SC_8^{2,6} \vee SC_8^{2,6}$ (see Fig. 3(a)). More precisely, for two distinct points e_i and e_j in $p^{-1}(v_0)$ (see the points $(0, 0), (6, 1)$ in $p^{-1}(v_0)$), there is no $\phi \in Aut(E_1|SC_8^{2,6} \vee SC_8^{2,6})$ such that $\phi(e_i) = e_j$.

(2) Consider the map $p_2 : E_2 \rightarrow SC_8^{2,6} \vee SC_8^{2,6}$ given by Fig. 3(b). Then the $(8, 8)$-covering $(E_2, p_2, SC_8^{2,6} \vee SC_8^{2,6})$ is a radius 2-$(8, 8)$-covering. For any two distinct points e_i and e_j (e.g., the points $e_0, e_6 \in p^{-1}(v_0) \subset E_2$), there is always $\phi \in Aut(E_2|SC_8^{2,6} \vee SC_8^{2,6})$ such that $\phi(e_i) = e_j$.

In view of Example 2, we can clearly observe that, for a (k_0, k_1)-covering map $p : (E, e_0) \rightarrow (B, b_0)$ and two distinct points e_0 and e_1 in $p^{-1}(b_0)$, there may not be an element $\phi \in Aut(E|B)$ such that $\phi(e_0) = e_1$. Unlike Example 2, motivated by Lemma 8.1 of Massey (1977), by Theorem 2, we clearly obtain the following.

Lemma 4. *If a radius* 2-(k_0, k_1)-*covering map* $p : (E, e_0) \rightarrow (B, b_0)$ *is regular, then* $Aut(E|B)$ *acts transitively on* $p^{-1}(b_0)$.

For a (k_0, k_1)-covering map $p : (E, e_0) \rightarrow (B, b_0)$ which does not satisfy a radius 2 local isomorphism, the study of $Aut(E|B)$ remains to be approached. In order to deal with this problem, we need to make the version of Han (2006b) advanced into the following notion, which is different from a regular (k_0, k_1)-covering of Han (2006b).

Definition 12. A (k_0, k_1)-covering $((E, e_0), p, (B, b_0))$ is called an *ultra regular* (briefly, UR-) (k_0, k_1)-covering if $Aut(E|B)$ acts transitively on $p^{-1}(b_0)$.

Let us now study an important property of a UR-(k_0, k_1)-covering which proposes a method of determining if a (k_0, k_1)-covering is a UR-(k_0, k_1)-one. The following theorem can characterize a UR-(k_0, k_1)-covering.

Theorem 3. *The following are equivalent:*
(1) A (k', k)-covering $((E, e_0)), p, (B, b_0))$ is ultra regular.
(2) For a (k', k)-covering $((E, e_0)), p, (B, b_0))$ assume a closed k-curve $\alpha : [0, m]_\mathbb{Z} \rightarrow (B, k)$ with $\alpha(0) = b_0 \in B$. Either each of all liftings of α on (E, k') is a k'-closed curve or none of them is a k'-closed one.

Proof.
Case 1. Let us assume that a given covering $((E, e_0)), p, (B, b_0))$ is a radius 2-(k', k)-covering. Then the assertion is clear. Precisely, in this case, since the property (2) of the current theorem implies that for any points $e_0 \in p^{-1}(b_0)$ the subgroups $p_*\pi^{k'}(E, e_0) \subset \pi^k(B, b_0)$ are normal, they are the same. Finally, by Theorem 1(2), we can observe that the given (k', k)-covering (E, p, B) is regular. Consequently, by Lemma 4, the proof is completed.
Case 2. Let us now prove the case that a given covering $((E, e_0)), p, (B, b_0))$ is not a radius 2-(k', k)-covering.
$(1) \Rightarrow (2)$: For a closed k-curve $\alpha : [0, m]_\mathbb{Z} \rightarrow (B, k)$ with $\alpha(0) = b_0 \in B$, assume that there are digital liftings of α on (E, k') of which one of them is a k'-closed curve and another of them is a k'-path that is not a k'-closed curve. Then the given (k', k)-covering (E, p, B) cannot be a UR-(k', k)-covering because $Aut(E|B)$ cannot act transitively on $p^{-1}(b_0)$. For instance, owing to the point $(0, 0) \in p^{-1}(v_0)$, the digital covering $(E_1, p_1, SC_8^{2,6} \vee SC_8^{2,6})$ in Fig. 3(a) cannot be a UR-$(8, 8)$-covering.
$(2) \Rightarrow (1)$: For a (k', k)-covering (E, p, B) satisfying the property (2) we can clearly observe that $Aut(E|B)$ acts transitively on $p^{-1}(b_0)$, which completes the proof. For instance, see the $(8, 8)$-covering $(D_2, p, SC_8^{2,4} \vee SC_8^{2,4})$ in Fig. 2. ∎

Due to Theorem 3, hereafter, regardless of the requirement of a radius 2 local isomorphism of a (k', k)-covering, we can have a very convenient method of determining if a given digital covering is UR-(k', k)-regular. Thus, hereafter, by using Theorem 3, for a (k', k)-covering (E, p, B) we can study $Aut(E|B)$ without using digital homotopic tools.

By Theorem 1, we obtain the following.

Corollary 1.
(1) For any $SC_k^{n,l}$ the $(2, k)$-covering map $p : \mathbb{Z} \rightarrow SC_k^{n,l} := (c_t)_{t \in [0, l-1]_\mathbf{z}}$ given by $p(t) = c_{t(\text{mod } l)}$, $t \in \mathbb{Z}$ is ultra regular.
(2) The map $p : SC_k^{n,ml} := (a_t)_{t \in [0, ml-1]_\mathbf{z}} \rightarrow SC_k^{n,l} := (c_t)_{t \in [0, l-1]_\mathbb{Z}}$ given by given by $p(a_i) = c_{i(\text{mod } l)}, m \in \mathbb{N}$ is ultra regular, where $m \in \mathbb{N}$ and $SC_k^{n,l}$ need not be k-contractible.
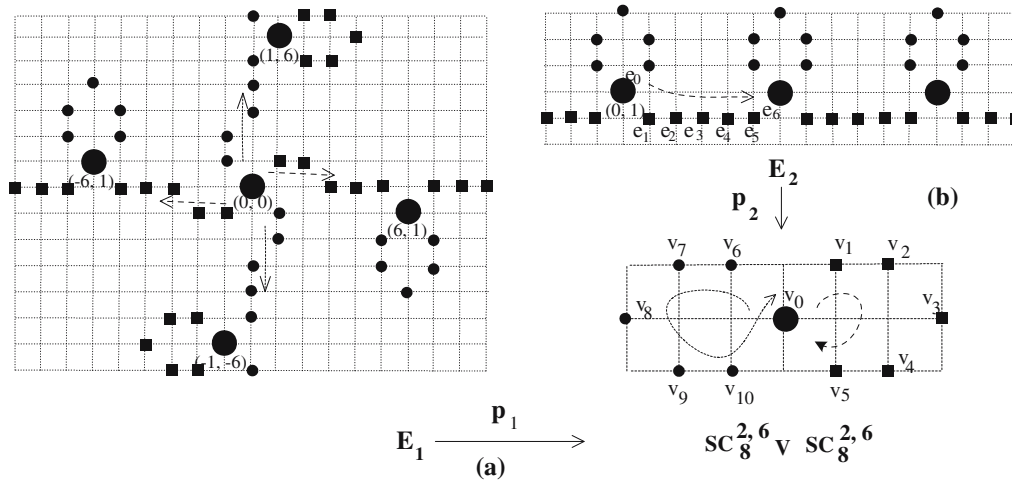
Fig. 3. Infinite fold $(8,8)$-covering $(E_1, p_1, SC_8^{2,6} \vee SC_8^{2,6})$ (Han, 2005c), which does not allow an automorphism with transitive action (a); infinite fold $(8,8)$-covering $(E_2, p_2, SC_8^{2,6} \vee SC_8^{2,6})$ from the work of Boxer (2006), which is both ultra regular and regular (b).

(3) The map $p : D_2 \rightarrow SC_8^{2,4} \vee SC_8^{2,4}$ in Fig. 2 is a UR-$(8,8)$-covering.

Proof.
(1) Take a base point $c_0 \in SC_k^{n,l}$. Then we can obtain $p^{-1}(c_0) := l\mathbb{Z} \subset \mathbb{Z}$. By Theorem 3, we can clearly observe that $Aut(E|B) := l\mathbb{Z}$ acts transitively on $p^{-1}(c_0)$.
(2) By a similar method as the proof of (1), the proof can be clearly completed by using the circulation on $SC_k^{n,ml}$ depending on the points $p^{-1}(c_0)$.
(3) The assertion follows from Theorem 3.   ∎

6. Merits of an ultra regular covering space

For a digital space (X, k_1), how can we describe a difference between a UR-(k_0, k_1)-covering and a regular (k_0, k_1)-covering over (X, k_1)? In the light of Theorem 3, this section discusses some merits of a UR-(k_0, k_1)-covering. In Section 3, for a radius 2-(k_0, k_1)-covering, its regularity has been studied. But a UR-(k_0, k_1)-covering need not require a radius 2 local isomorphism. In view of this difference, we can observe that a UR-(k_0, k_1)-covering has strong merits of classifying digital covering space. By comparing an ultra regular covering space with a regular covering space, we obtain the following.

Theorem 4.
(1) A regular (k_0, k_1)-covering space does not imply a UR-(k_0, k_1)-covering space.
(2) For a digital space (X, k_1), let $R_2(X)$ denote the set of all radius 2-(k_0, k_1)-coverings over (X, k_1). Then we obtain the following: In $R_2(X)$ a UR-(k_0, k_1)-covering is equivalent to a regular (k_0, k_1)-covering.

Proof.
(1) As an example, consider the infinite fold $(8, 8)$-covering map $p : E_5 \rightarrow SC_8^{2,4} \vee SC_8^{2,4}$ in Fig. 4(c). Precisely, assume that p_5 maps each of solid squares, small solid circles, and big solid circles of E_5 into the corresponding ones in $SC_8^{2,4} \vee SC_8^{2,4}$. Since both $\pi^8(E_5)$ and $\pi^8(SC_8^{2,4} \vee SC_8^{2,4})$ are trivial, $(E_5, p_5, SC_8^{2,4} \vee SC_8^{2,4})$ is a regular $(8,8)$-covering. But it cannot be a UR-$(8,8)$-covering because $Aut(E_5, SC_8^{2,4} \vee SC_8^{2,4})$ is trivial. Precisely, the $(8,8)$-covering $(E_5, p, SC_8^{2,4} \vee SC_8^{2,4})$ has the only digital isomorphism such as the identity. To be specific, if there is a digital automorphism $h : E_5 \rightarrow E_5$, then any 8-loop in $SC_8^{2,4} \vee SC_8^{2,4}$ that lifts to an 8-loop in E_5 at $(0,0)$ also lifts to an 8-loop when the lift begins at $h(0,0)$. In this case we clearly observe that h is exactly the identity.
(2) In $R_2(X)$, by Theorem 2 and Lemma 4, it is clear that a UR-(k_0, k_1)-covering (E, p, X) is equivalent to a regular (k_0, k_1)-covering.   ∎

In the light of Theorem 4, if a (k_0, k_1)-covering is not a radius 2-(k_0, k_1)-covering, then a comparison between an ultra regular covering space and a regular covering space depends on the situation. To study a UR-(k_0, k)-covering $((E, k_0), p, SC_{k_1}^{n_1,l_1} \vee SC_{k_2}^{n_2,l_2})$ up to a digital covering isomorphism, let us recall that the k-fundamental group of $(SC_k^{n,l_0} \vee SC_k^{n,l_1}, k)$ is a free group with two generators (Han, 2005c) (see also Han, 2007a), where SC_k^{n,l_i} is not k-contractible, $i \in \{0, 1\}$ and the k-adjacency of $SC_{k_1}^{n_1,l_1} \vee SC_{k_2}^{n_2,l_2}$) is assumed to be compatible according to Definition 10. By using this property and Theorem 3, let us demonstrate Theorem 4 with the following examples.

Example 3. For illustration, consider the $(8,8)$-covering $(E_3, p_3, SC_8^{2,4} \vee SC_8^{2,4})$ in Fig. 4(a), which is not a radius 2-$(8,8)$-covering. Then we can observe that it is not a
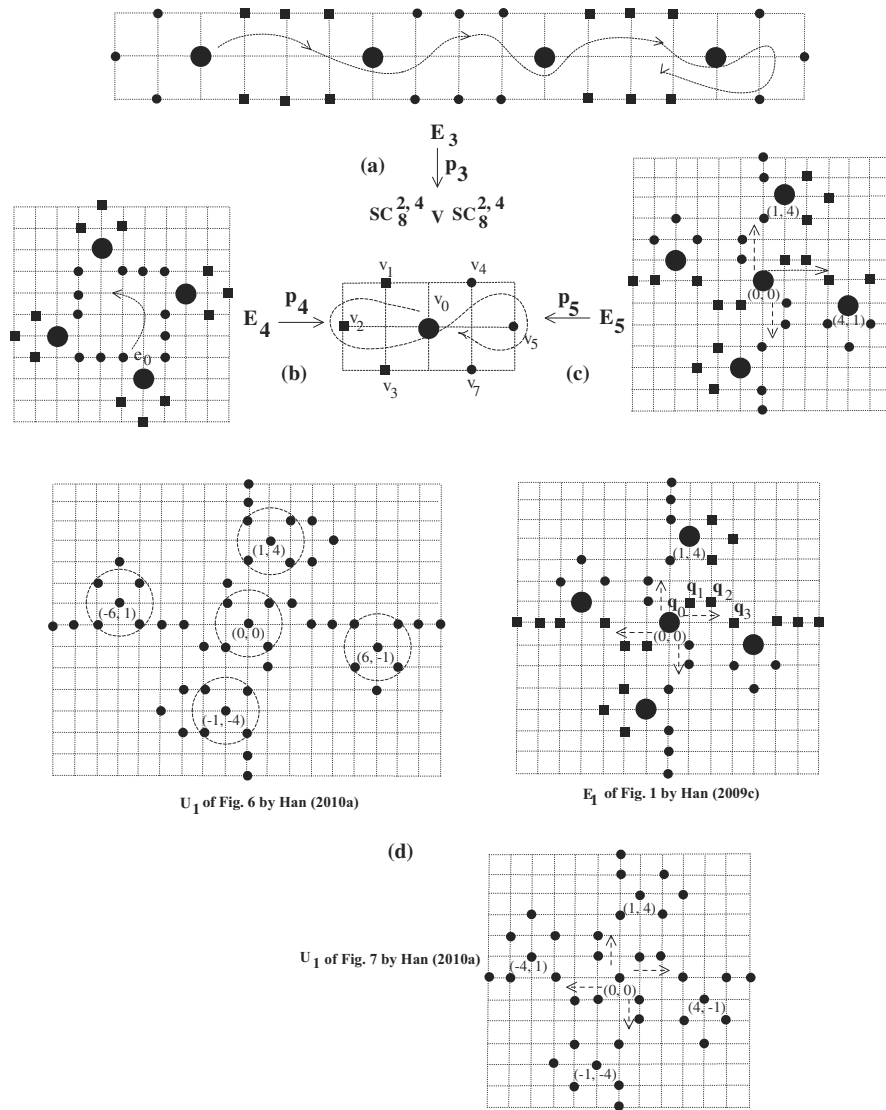
Fig. 4. 4-fold $(8,8)$-covering $(E_3, p_3, SC_8^{2,4} \vee SC_8^{2,4})$, which is not UR-regular (a); 4-fold $(8,8)$-covering $(E_4, p_4, SC_8^{2,4} \vee SC_8^{2,4})$ found in the work of Han (2009c), which is UR-regular (b); infinite fold $(8,8)$-covering $(E_5, p_5, SC_8^{2,4} \vee SC_8^{2,4})$, which is not UR-regular but regular (c); correction of Fig. E_1 and U_1 presented by Han (2009c; 2010a) (d).

UR-$(8,8)$-covering.

By Theorem 4(2), we can observe the following.

Example 4.
(1) Consider the $(8,8)$-covering $(E_4, p_4, SC_8^{2,4} \vee SC_8^{2,4})$ in Fig. 4(b), which is not a radius 2-$(8,8)$-covering. Then we can observe that it is a UR-$(8,8)$-covering.
(2) Consider the $(8,8)$-covering $(E_5, p_5, SC_8^{2,4} \vee SC_8^{2,4})$ in Fig. 4(c), which is not a radius 2-$(8,8)$-covering. Then we can observe that it is not a UR-$(8,8)$-covering but a regular $(8,8)$-covering.

Remark 5. In relation to the study of an automorphism group of a (k_0, k_1)-covering, since Theorem 3 does not require a radius 2 local (k_0, k_1)-isomorphism of the given digital covering as well as digital homotopic properties

such as Theorem 2, the notion of a UR-(k_0, k_1)-covering is so useful (see Theorem 3).

Remark 6. (Correcting) Since the two objects U_1 of Fig. 6 and U_1 of Fig. 7 found in the work of Han (2010a) are misprinted at the point $(0,0) \in \mathbb{Z}^2$, we can now correct them (see Fig. 4(d)). With the same criterion, the objects E_1 of Fig. 1 found in the paper by Han (2009c) should be corrected at the point $(0,0) \in \mathbb{Z}^2$ (motivated by Fig. 4 of Han (2005c)).

7. Concluding remarks and further work

In relation to the study of an automorphism group of a digital covering and the classification of digital spaces, the

paper has studied the following. First, we have established the notion of compatible adjacency of a digital wedge which can be used for studying an automorphism group of a digital covering. Second, we have established the notion of a UR-(k', k)-covering space and investigated some properties of an ultra regular covering space over $SC_k^{n,l}$ or $SC_{k_1}^{n_1,l_1} \vee SC_{k_2}^{n_2,l_2}$. Then, by comparing an ultra regular covering with a regular covering, we have proposed some merits of an ultra regular covering. Also, by using the GU-(k', k)-covering property of $(E, p, SC_k^{n,l})$, where (E, k') is $(k', 2)$-isomorphic to $(\mathbb{Z}, 2)$, we have demonstrated that an automorphism group of a (k', k)-covering map can be strongly used in the classification of digital covering spaces over $SC_k^{n,l}$ or a digital wedge without any limitation of a radius 2 local isomorphism of a digital covering.

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