amcs

ERGODIC THEORY APPROACH TO CHAOS: REMARKS AND COMPUTATIONAL ASPECTS

PAWEŁ J. MITKOWSKI [*], WOJCIECH MITKOWSKI [**]

[*] Faculty of Electrical Engineering, Automatics, Computer Science and Electronics
AGH University of Science and Technology, al. Mickiewicza 30/B-1, 30-059 Cracow, Poland
e-mail: `pawel.mitkowski@gmail.com`

[**] Department of Automatics
AGH University of Science and Technology, al. Mickiewicza 30, 30-059 Cracow, Poland
e-mail: `wojciech.mitkowski@agh.edu.pl`

We discuss basic notions of the ergodic theory approach to chaos. Based on simple examples we show some characteristic features of ergodic and mixing behaviour. Then we investigate an infinite dimensional model (delay differential equation) of erythropoiesis (red blood cell production process) formulated by Lasota. We show its computational analysis on the previously presented theory and examples. Our calculations suggest that the infinite dimensional model considered possesses an attractor of a nonsimple structure, supporting an invariant mixing measure. This observation verifies Lasota's conjecture concerning nontrivial ergodic properties of the model.

Keywords: ergodic theory, chaos, invariant measures, attractors, delay differential equations.

1. Introduction

In the literature concerning dynamical systems we can find many definitions of chaos in various approaches (Rudnicki, 2004; Devaney, 1987; Bronsztejn *et al.*, 2004). Our central issue here will be the ergodic theory approach. Ergodic theory in general has its origin in physical systems of a large number of particles, where microscopic chaos leads to macroscopic (statistical) regularity. As the beginning of ergodic theory, the moment when Boltzmann formulated his famous *ergodic hypothesis*, in 1868 (see, e.g., Nadzieja, 1996; Górnicki, 2001) or in 1871 (Lebowitz and Penrose, 1973), can probably be considered. For more information about the ergodic hypothesis, consult also the works of Birkhoff and Koopman (1932) as well as Dorfman (2001).

2. Ergodic theory and chaos: Basic facts

One of the most fundamental notions in ergodic theory is that of *invariant measure* (see Lasota and Mackey, 1994; Fomin *et al.*, 1987; Bronsztejn *et al.*, 2004; Rudnicki, 2004; Dawidowicz, 2007), which is a consequence of Liouville's theorem (see, e.g., Szlenk, 1982; Landau and Lifszyc, 2007; Arnold, 1989; Nadzieja, 1996; Dorf-

man, 2001). Transformations (or flows) with an invariant measure display three main levels of irregular behaviour, i.e., (ranging from the lowest to the highest) *ergodicity*, *mixing* and *exactness*. Between ergodicity and mixing we can also distinguish *light mixing*, *mild mixing* and *weak mixing* (Lasota and Mackey, 1994; Silva, 2010) and, on the level similar to exactness, the type of *K-flows* (or *K-property*, *K-automorphism*) (cf. Rudnicki 1985a; 1985b; 2004; Lasota and Mackey,1994). In this article we will consider only ergodicity and mixing. First we formalize these notions and show some simple examples of ergodic and mixing transformations. Then in Section 3. we analyze an infinite dimensional system which additionally has interesting medical (hematological) interpretations.

By $\{S_t\}_{t\geq 0}$ we denote a *semidynamical system* or a *semiflow* on the metric space X, i.e.,

(i) $S_0(x) = x$ for all $x \in X$;

(ii) $S_t(S_{t'}(x)) = S_{t+t'}(x)$ for all $x \in X$, and $t, t' \in \mathbb{R}^+$;

(iii) $S: X \times \mathbb{R}^+ \to X$ is a continuous function of (t, x).

By a measure on X we mean any probability measure defined on the σ-algebra $\mathcal{B}(X)$ of Borel subsets of X. A

measure μ is called *invariant* under a semiflow $\{S_t\}_{t\geq 0}$, if $\mu(A) = \mu(S_t^{-1}(A))$ for each $t \geq 0$ and each $A \in \mathcal{B}$.

2.1. Ergodicity. A Borel set A is called invariant with respect to the semiflow $\{S_t\}_{t\geq 0}$ if $S_t^{-1}(A) = A$ for all $t \geq 0$. We now denote by (S, μ) a semiflow $\{S_t\}_{t\geq 0}$ with an invariant measure μ. The semiflow (S, μ) is *ergodic* (we say also that the measure is ergodic) if the measure $\mu(A)$ of any invariant set A equals 0 or 1. Let us now consider two simple examples.

Example 1. Let $S \colon [0, 2\pi) \to [0, 2\pi)$ be a transformation generating rotation through an angle ϕ on a circle with unit radius (see Lasota and Mackey, 1994; Bronsztejn *et al.*, 2004; Devaney, 1987; Dorfman, 2001):

$$S(x) = x + \phi \qquad (\text{mod } 2\pi). \tag{1}$$

If $\phi/2\pi$ is rational, we can find invariant sets which have measure different from 0 or 1, and thus S is not ergodic. However, if $\phi/2\pi$ is irrational, then S is ergodic (for a proof, see the work of Lasota and Mackey (1994, p. 75) or Devaney (1987, p. 21)). If we take, e.g., $\phi = \sqrt{2}$ and pick an arbitrary point on the circle, we can observe that successive iterations of this point under the action of S will densely fill the whole available space (circle) (see Fig. 1). ♦

Example 2. To understand better the typical features of ergodic behaviour, let us consider the following transformation (see Lasota and Mackey, 1994, p. 68):

$$S(x, y) = (\sqrt{2} + x, \sqrt{3} + y) \qquad (\text{mod } 1). \tag{2}$$

This is an extension of the rotational transformation (1) on the space $[0,1] \times [0,1] \to [0,1] \times [0,1]$. In Fig. 2 we can observe the result of the action of S on the ensemble of 10^3 points distributed randomly in the area $[0, 0.1] \times [0, 0.1]$. The transformation (2) shifts the initial area and does not spread the points over the space. When we measure the Euclidean distance during iterations between two arbitrarily chosen close points, we notice that it is constant (Fig. 2(d)). Thus the popular criterion of chaos, i.e., sensitivity to initial conditions, is not a property of ergodic transformations. Their property is the dense trajectory (we formalize this fact in the last paragraph of this section). ♦

One of the most important theorems in ergodic theory is the *Birkhoff individual ergodic theorem* (Birkhoff, 1931a; 1931b; Birkhoff and Koopman, 1932; Lasota and Mackey, 1994, Fomin *et al.*, 1987; Szlenk 1982; Dawidowicz, 2007; Nadzieja, 1996; Gornicki, 2001; Dorfman, 2001). Here we cite a popular extension of this theorem (see Lasota and Mackey, 1994, p. 64; Fomin *et al.*, 1987, p. 46). Recall that by (S, μ) we denote a semiflow $\{S_t\}_{t\geq 0}$ with an invariant measure μ.

Theorem 1. (Extension of the Birkhoff theorem) *Let* (S, μ) *be ergodic. Then, for each μ-integrable function* $f \colon X \to \mathbb{R}$*, the mean of f along the trajectory of S is equal almost everywhere to the mean of f over the space X, that is,*

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T f(S_t(x))\, \mathrm{d}t = \frac{1}{\mu(X)} \int_X f(x)\, \mu(\mathrm{d}x), \tag{3}$$

μ-*almost everywhere.*

If we substitute $f = \mathbf{1}_A$ in Eqn. (3) ($\mathbf{1}_A$ is the characteristic function of A) (see Lasota and Mackey, 1994; Rudnicki, 2004; Dawidowicz, 2007), then the left-hand
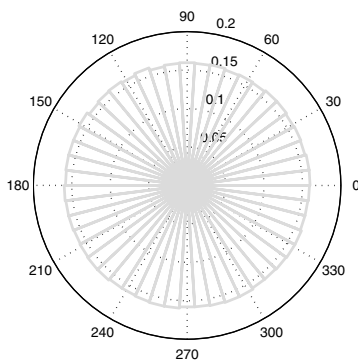

Fig. 1. Normalized (to the probability density function) round histogram (bars inside the circle) showing that a single point under the action of the ergodic transformation (1) with $\phi = \sqrt{2}$ fills densely the whole circle.
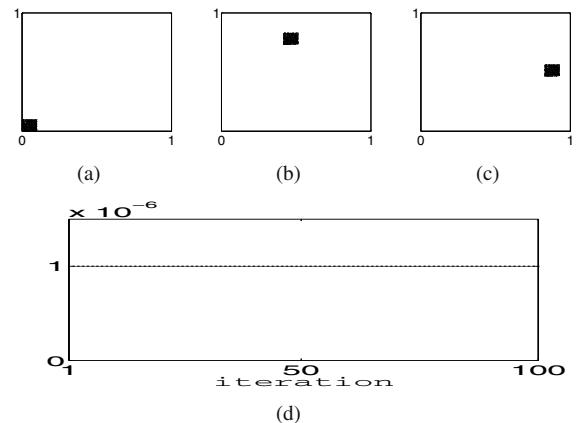

Fig. 2. Iterations of the ergodic transformation (2) acting on an ensemble of 10^3 points randomly distributed in $[0, 0.1] \times [0, 0.1]$: 1st iteration (a), 2nd iteration (b), 3rd iteration (c), Euclidean metric between two arbitrarily chosen close points from the ensemble (d).

side of (3) is the mean time of visiting the set A and the right-hand side is $\mu(A)$, and this corresponds to ergodicity in the sense of Boltzmann, which roughly speaking is the mean time that a particle of a physical system spends in some region and it is proportional to its natural probabilistic measure (Dawidowicz, 2007; Dorfman, 2001; Nadzieja, 1996; Górnicki, 2001; Birkhoff and Koopman, 1932; Lebowitz and Penrose, 1973)

We can see that ergodic behaviour in the "pure" form does not need to be very irregular and unpredictable. In fact, an invariant and ergodic measure should have some additional properties to be interesting from the point of view of dynamics. Briefly speaking, it should be nontrivial—for example, we intuitively understand that to have interesting dynamics the measure should not be concentrated on a single point. According to our knowledge, two approaches to this problem appear in the literature. In the main ideas, both seem to be similar, but in the literature exist separately. One is the theory of Prodi (1960) (and Foias (1973)), which says that stationary turbulence occurs when the flow admits nontrivial invariant ergodic measure. This theory was strongly developed by Lasota (1979; 1981) (see also Lasota and Yorke, 1977; Lasota and Myjak, 2002; Lasota and Szarek, 2004) and further by Rudnicki (1985a; 1988; 2009) (see also Myjak and Rudnicki, 2002) as well as Dawidowicz (1992a; 1992b) (see also Dawidowicz *et al.*, 2007). Another one uses the notion of *SRB* (*Sinai, Ruelle, Bowen*) *measures* (see, e.g., Bronsztejn *et al.*, 2004; Dorfman, 2001; Taylor, 2004; Tucker, 1999). Roughly speaking, both the approaches say that to have interesting dynamics the support of the measure should be possibly a large set.

Let us now assume that X is a separable metric space and μ is a probability Borel measure on X such that $\operatorname{supp}\mu = X$. We can state that (see Rudnicki, 2004, p. 727, Proposition 1), if a semiflow (S, μ) is ergodic, then for μ-almost all x the trajectory $S_t(x)$, $t \geq 0$ is dense.

2.2. Mixing. Now we will consider the notion of *mixing*, which exhibits a higher level of irregular behaviour than ergodicity. The literature says that the concept of a mixing system was introduced by J.W. Gibbs (see, e.g., Dorfman, 2001, p. 18, 65). A semiflow (S, μ) is mixing (see, e.g., Lasota and Mackey, 1994; Rudnicki, 2004; Bronsztejn *et al.*, 2004) if

$$\lim_{t \to \infty} \mu(A \cap S_t^{-1}(B)) = \mu(A)\mu(B) \quad \text{for all } A, B \in \mathcal{B}.$$
(4)

This means that the fraction of points which at $t = 0$ are in A and for large t are in B is given by the product of the measures of A and B in X. Mixing systems are also ergodic.

Example 3. Let us consider the mixing transformation

(see Lasota and Mackey, 1994, p. 57, pp. 65–68)

$$S(x, y) = (x + y, x + 2y) \qquad (\text{mod } 1). \qquad (5)$$

This is an example of the Anosov diffeomorphism (Anosov, 1963) (see also Bronsztejn *et al.*, 2004, p. 903). In Fig. 3 we can see the first the fifth and the tenth iteration of the mixing tranformation (5) acting on the ensemble of 10^3 points distributed randomly in the area $[0, 0.1] \times [0, 0.1]$. The points are being spread over the space and afterwards that transformation is literally mixing these points in the whole space. The Euclidean distance between close points first grows quickly and then fluctuates irregularly (Fig. 3 (d)). The difference between the ergodic transformation (2) (cf. Fig. 2) is noticeable. Typical for mixing is the sensitivity to initial conditions (we will formalize this fact further on). ◆

We can say more about the chaoticity of mixing systems. First let us recall the following definition (Auslander and Yorke, 1980) (see also Rudnicki, 2004).

Definition 1. The flow is *chaotic in the sense of Auslander and Yorke* if

 (i) there exists a dense trajectory, and

 (ii) each trajectory is unstable.

Instability here means that there exists a constant $\eta > 0$ such that for each point $x \in X$ and for each $\epsilon > 0$ there exists a point $y \in B(x, \epsilon)$ and $t > 0$ such that $\rho(S_t(x), S_t(y)) > \eta$, where ρ is the metric in X and $B(x, r)$ is the open ball in X with center x and radius $r > 0$. Instability can be also described here as the sensitivity to initial conditions, which is a "popular" criterion of chaos. Now, with the assumption that X is a separable metric space and μ is a probability Borel measure on X such
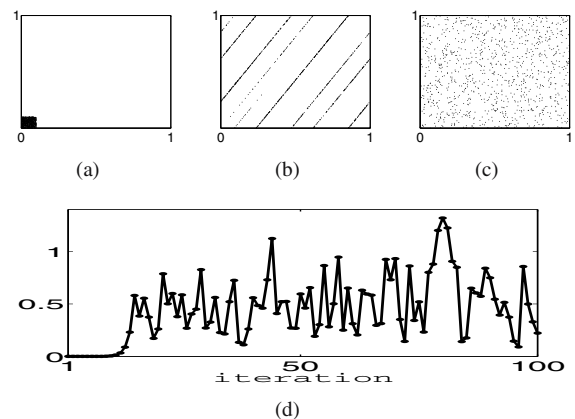

Fig. 3. Iterations of the mixing transformation (5) acting on an ensemble of 10^3 points randomly distributed in $[0, 0.1] \times [0, 0.1]$: 1st iteration (a), 5th iteration (b), 10th iteration (c), Euclidean metric between two arbitrarly chosen close points from the ensemble (d).

that $\operatorname{supp} \mu = X$, we can state that (see Rudnicki, 2004, p. 727, Proposition 1), if a semiflow (S, μ) is mixing, then the semiflow $\{S_t\}_{t \geq 0}$ is chaotic in the sense of Auslander and Yorke.

Example 4. Once again let us consider the mixing transformation (5) from Example 3. Let us consider a correlation coefficient in the form (see de Larminat and Thomas, 1983)

$$\gamma_{xy}(\tau) = \frac{c_{xy}(\tau)}{\sigma_x \sigma y}, \quad \tau = 0, 1, 2, \ldots, \quad (6)$$

where

$$c_{xy}(\tau) = \lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} (x_i - x_0)(y_{i+\tau} - y_0(\tau)), \quad (7)$$

$$x_0 = \lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} x_i, \quad y_0(\tau) = \lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} y_{i+\tau} \quad (8)$$

and

$$\sigma_x = \sqrt{\lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} (x_i - x_0)^2}, \quad (9)$$

$$\sigma_y = \sqrt{\lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} (y_{i+\tau} - y_0(\tau))^2}. \quad (10)$$

Once again the tranformation (5) is acting on the ensemble of points (this time 10^4 for higher accuracy). After a few iterations it reaches the statistical equilibrium on the ensemble and with further iterations it is "mixing" the ensemble in the space. We take a sequence x_i of the euclidean norms for the ensemble in the equilibrium, so we have a sequence of 10^4 values. $y_{i+\tau}$ for $\tau = 0$ is the same as x_i and for $\tau = 1, 2, \ldots$ it forms a sequence for further iterations. So using the formula (6) we obtain a correlation function where for $\tau = 0$ we have correlation x_i with x_i (Fig. 4(c)) and for $\tau = 1, 2, \ldots$ we have correlation between x_i and $y_{i+\tau}$ which is moving away in time. The result is visible in Fig. 4(a).

We can see that the correlation function (6) for the ensemble decreases to a value near 0 very quickly (already in the 2nd iteration). When we draw the spread of the ensembles on the space for $\tau > 0$, e.g., $\tau = 5$, we can see that points are correlated neither linearly nor in any other way (Fig. 4(d)). Since the mixing transformation is also ergodic, we can change averages over the ensemble to averages along a single trajectory. So instead of calculating a correlation function for the whole ensembles, we can calculate it for a single trajectory and its time shifts. The result is presented in Fig. 4(b); we can see that the correlation functions in both cases (ensemble and single trajectory) are almost the same. Such a rapid decrease in correlation is typical for mixing systems (see Bronsztejn *et al.*, 2004; Rudnicki, 2004; 1988).   ♦
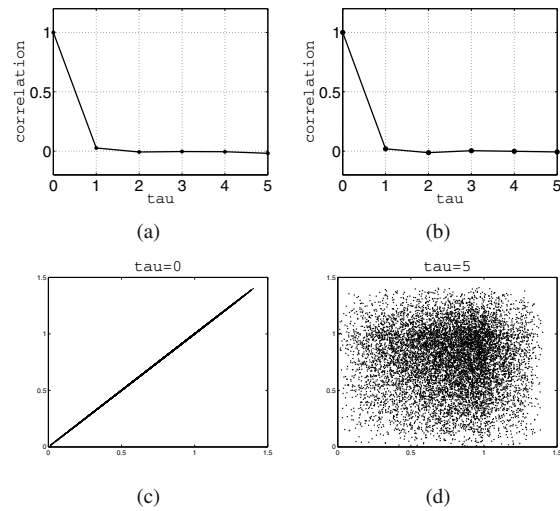

Fig. 4. Rapid decrease in the correlation for the mixing transformation (5) for an ensemble of 10^4 points (a), correlation for a single trajectory and its time shift (b), spread of points of the ensemble for $\tau = 0$, i.e., correlation of the "initial" ensemble with itself (c), spread of points of the ensemble for $\tau = 5$ (d).
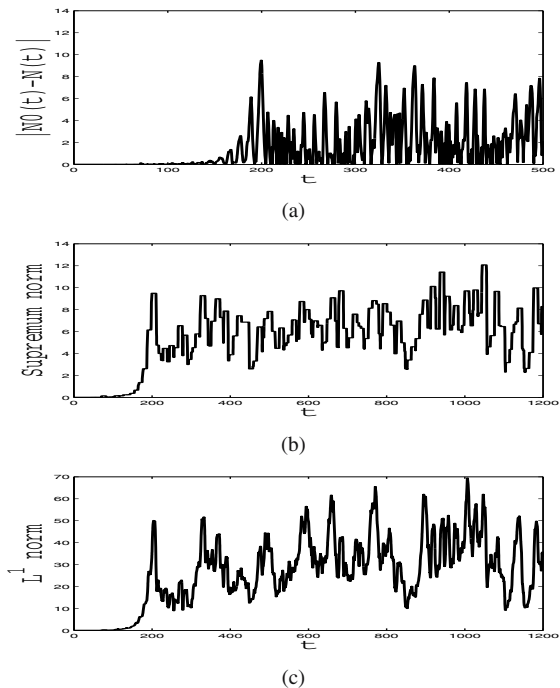
3. Infinite dimensional case

Fig. 5. Two trajectories of Eqn. (11) for constant initial functions different by 0.0001 of the absolute value of the distance between the values $N(t)$ (a), distance in the supremum norm (b), distance in the L^1 norm (c).

Let us now consider the delay blood cell production model formulated by Lasota (1977):

$$\frac{\mathrm{d}N(t)}{\mathrm{d}t} = -\sigma \cdot N(t) + (\rho \cdot N(t-h))^s \cdot e^{-\gamma \cdot N(t-h)}. \quad (11)$$

Biological interpretations of this equation have their origin in the famous research of Ważewska-Czyżewska and Lasota (1976) into mathematical modelling of the dynamics of erythropoiesis, which is a process of red blood cells (erythrocytes) formation in the bone marrow. For further insight into this research, consult the works of Ważewska-Czyżewska (1983) and Lasota *et al.* (1981). $N(t) \in \mathbb{R}$ is a global number of erythrocytes in blood circulation, σ denotes the destruction rate of cells, ρ is oxygen demand, γ is the coefficient describing system excitation and h is the delay time representing the time of maturation of erythrocytes.

The contribution of parameter s to a biomedical interpretation can be found in the work of Mitkowski (2011). According to the authors' knowledge, the biomedical meaning of this parameter has not been explained in the literature yet. The production function of blood cells in Eqn. (11) (which can be interpreted as a feedback) has the form of the so-called unimodal function. Briefly speaking, it is a function with one smooth maximum. Because of such a form of the feedback, Eqn. (11) may display very complicated dynamics including chaos (see Ważewska-Czyżewska, 1983; Mackey, 2007; Liz and Rost, 2009; Mitkowski, 2011). Biological delay models with unimodal nonlinearities were considered also by Mackey and Glass (1977) as well as Gurney *et al.* (1980), who described experimental data of Nicholson (1954). However, the nonlinearity in Eqn. (11) is more "flexible" and gives stronger possibilities for applications (for a detailed discussion of this problem, see Mitkowski (2011).

3.1. Conjecture of Lasota.
Lasota (1977, p. 248) formulated a conjecture concerning ergodic properties of Eqn. (11), i.e., let C_h be the space of continuous functions $v : [-h, 0] \to \mathbb{R}$ with the supremum norm topology. For some positive values of parameters ρ, h, s and σ, there exists a continuous measure on C_h which is ergodic and invariant with respect to Eqn. (11). By a continuous measure we understand here a measure which vanishes at points (see Lasota, 1977; Lasota and Yorke, 1977) and in this sense the measure is nontrivial. Thus, according to our previous discussion, the conjecture concerns the chaotic behaviour of Eqn. (11). It might be very difficult to solve this problem using only mathematical tools. In general, according to the authors' knowledge, there are very few results where chaos for delay differential equations was proved using only mathematical tools. One of such results was given by Walther (1981). Our aim is to investigate Eqn. (11) numerically in order to check if it exhibits nontrivial ergodic properties.

There is also an interesting historical context of Lasota's hypothesis. Ulam (1960, p. 74) (see also Myjak, 2008) posed the problem of the existence of nontrivial invariant measures for transformations of the unit interval into itself defined by a sufficiently "simple" function (e.g., a piecewise linear function or a polynomial) whose graph does not cross the line $y = x$ with a slope in an absolute value less than 1. Later Lasota and Yorke (1973) solved the problem. The conjecture of Lasota for Eqn. (11) looks like a generalization of Ulam's conjecture to first order differential delay equations. This association comes up during numerical investigations of Lasota's delay equation, where we search for a proper "shape" of unimodal feedback to find nontrivial ergodic properties (see Fig. 7).

3.2. Calculations.
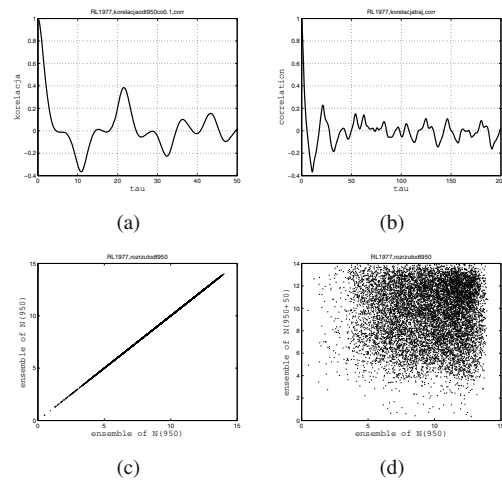Numerical investigations show that Eqn. (11) exhibits nontrivial ergodic properties for $\rho \in$

Fig. 6. Rapid decrease in the correlation for Eqn. (11) for an ensemble of 10^4 trajectories (a), correlation for a single trajectory and its time shift (b), spread of points of the ensemble for $\tau = 0$, i.e., correlation of the "initial" ensemble with itself (c), spread of points of the ensemble for $\tau = 50$ (d).
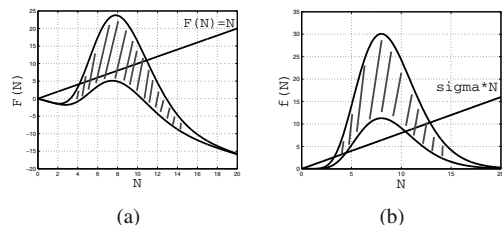

Fig. 7. Range of parameters (shaded area) which generate the nontrivial ergodic behaviour of the right-hand side of (11) (a), unimodal feedback function in reference to the linear destruction rate of the red blood cells (b). In both the cases the lower bound corresponds to $\rho = 0.46$ and the upper to $\rho = 0.52$.

[0.46, 0.52], $\sigma = 0.8$, $s = 8$, $\gamma = 1$ and a delay of $h > 9$. In Fig. 7(a), we can see the range of the right-hand side $F(N)$ of (11), with the line $F(N) = N$. The lower bound of the shaded area corresponds to $\rho = 0.46$ and the upper bound to $\rho = 0.52$. In Fig. 7(b) there is the same range of parameters but presented in the form of the unimodal feedback function in reference to the linear destruction rate of red blood cells. Ergodic properties sustain for large values of h (like $h = 50$); however, the more h increases, the more trajectory is attracted to 0 and ergodic properties decay.

We will show now some numerical experiments indicating ergodic properties of Eqn. (11). We choose $\rho = 0.46$, $\sigma = 0.8$, $s = 8$, $\gamma = 1$, i.e., the lower bounds from Fig. 7(a) and (b). Equation (11) is solved using the MATLAB solver de23 (see Shampine *et al.*, 2002).

Many important aspects concerning numerical investigations of probabilistic properties of delay differential equations were presented by Taylor (2004). Useful directions for computational analysis of ergodic properties were presented by Lasota and Mackey (1994), Kudrewicz (1991; 1993, 2007) as well as Ott (1993).

It is obvious that numerically we cannot show ergodic properties on the whole infinite dimensional space. We want to show that on some subspaces, Eqn. (11) has a smooth invariant density, which for a large ensemble (see Fig. 9) of trajectories is equal to the average along all single trajectories. That would indicate that the system exhibits basic ergodic properties. After that, using correlation techniques and examining the unstability of trajectories, we want to investigate mixing properties. As the state of Eqn. (11) we will consider a function of an interval of length h (delay) (see Fig. 8(a)). We will analyze its behaviour in subspaces of an infinite dimensional space of its values. A graphical example of such a subspace is presented in Fig. 8 (b). It is a six-dimensional space constructed by taking six arbitrary points of the functional state of Eqn. (11). Another solution is to equip the space C_h with a proper norm; however, in this article apart, from one exception (see Fig. 5), we shall not consider this case. Results of computational analysis of Eqn. (11) in such spaces can be found in the work of Mitkowski (2011).

3.3. Ergodicity of the flow. Consider Fig. 9, showing a bunch of trajectories of Eqn. (11). First they evolve quite regularly but after some time the flow becomes very irregular, we could even say turbulent. Additionally, trajectories are bounded. Let us take two arbitrary subspaces form the infinite dimensional space we have introduced previously, e.g., the most natural space of values $N(t) \in \mathbb{R}$ and the space $N(t) \times N(t-h)$ (which is often used for delay differential equations). In Fig. 10 we can observe chosen moments of evolution of 10^4 constant initial functions of Eqn. (11) distributed exponentially on some interval. Figure 10 (a),(c),(e),(g) shows the evolution on the space

of $N(t) \in \mathbb{R}$ and Fig. 10(b), (d), (f), (h) on the space $N(t) \times N(t-h)$. After some time the normalized (to the probability density) histograms (counting the number of points of the ensemble in the subintervals of the space) tend to invariant histograms, i.e., some time after simulations they almost do not change their shape. This may indicate that we have reached some invariant density.

In order to check if this density tends to be smooth, we could calculate a significantly larger ensemble of tra-
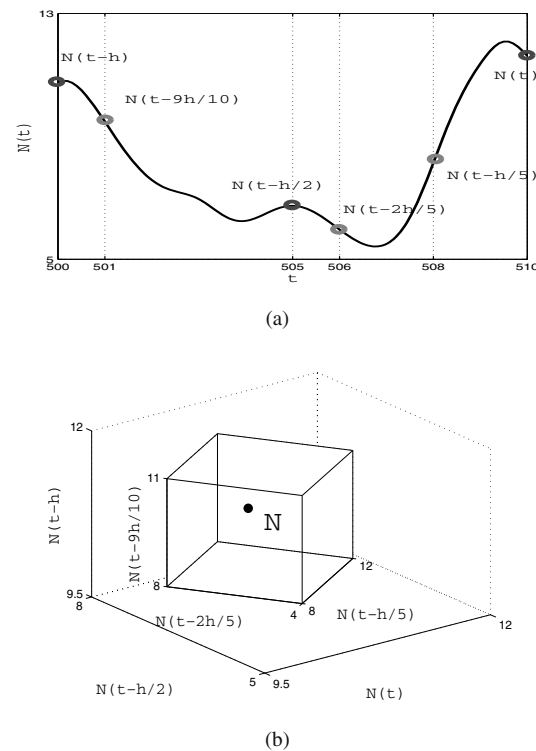

(a)

(b)

Fig. 8. Geometrical representation of state evolution given by Eqn. (11): an arbitrary state (a), an example of its representation in the six-dimensional space $N(t) \times N(t-h/2) \times N(t-h) \times N(t-h/5) \times N(t-2h/5) \times N(t-9h/10)$ (b).
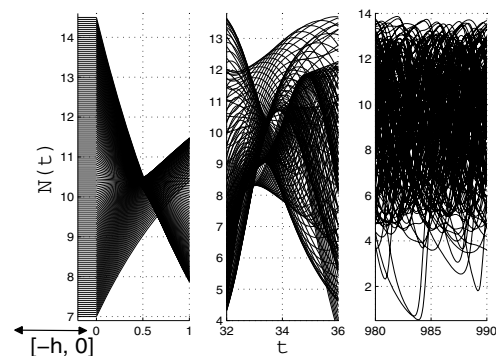

Fig. 9. Bunch of trajectories of Eqn. (11). First the flow is regular, then it becomes turbulent.

jectories, but then numerical calculations take a lot of time and become useless. However, we can examine if the flow exhibits the ergodic property, i.e., if histograms for single trajectories are similar to that of the ensemble. If that were true we could construct a histogram for a very long single trajectory and that would reflect also the average over the ensemble (see Theorem 1). Indeed, numerical simulations indicate that Eqn. (11) exhibits this typical property of ergodic flows; in Figs. 11(a), (b), we have more accurate histograms for single trajectories. We can see that the higher the accuracy the smoother the histograms. Each trajectory is also irregular (see Figs. 11(c), (d)), which is in accordance with the theory discussed in previous sections. The behaviour of the ensemble on the space $N(t) \times N(t - h)$ (see Figs. 10(b), (d), (f), (h)) as well as that of the single trajectory on this space (see Figs. 11(b),(d)) may suggest

that there exists an attractor, which has a significant "volume", supporting the invariant ergodic measure.

3.4. Mixing properties of the flow. The flow generated by Eqn. (11) exhibits also properties typical for mixing systems. Numerical simulations indicate that each trajectory is unstable. In Fig. 5(a) we can see that the absolute value of the distance between the values $N(t)$ of two trajectories starting from very close initial functions is fluctuating irregularly. We have marked before that we will not consider any specific norm in the space, but here we will make an exception, because the unstability for Eqn. (11) is much better visible when we equip the space with the supremum or L^1 norm (see Fig. 5(b),(c)). Additionally, the correlation for the ensemble and for the single trajectory and its time shifts decreases rapidly (see Fig. 6), which is characteristic for mixing systems (see Section 2.2). The lack of correlation suggests that the attractor does not have a simple structure. It may also indicate that each trajectory is turbulent in the sense of Bass (Bass, 1974; Rudnicki, 2004; 1988). Computational results concerning the problem of turbulence for Eqn. (11) can be found in the work of Mitkowski (2011).

4. Concluding remarks

We have presented numerical computations suggesting that the delay differential equation (11) possesses an attractor of a nonsimple structure, supporting an invariant mixing measure. This verifies the conjecture of Lasota which, using the language of ergodic theory, poses the problem of the chaotic behaviour of Eqn. (11).

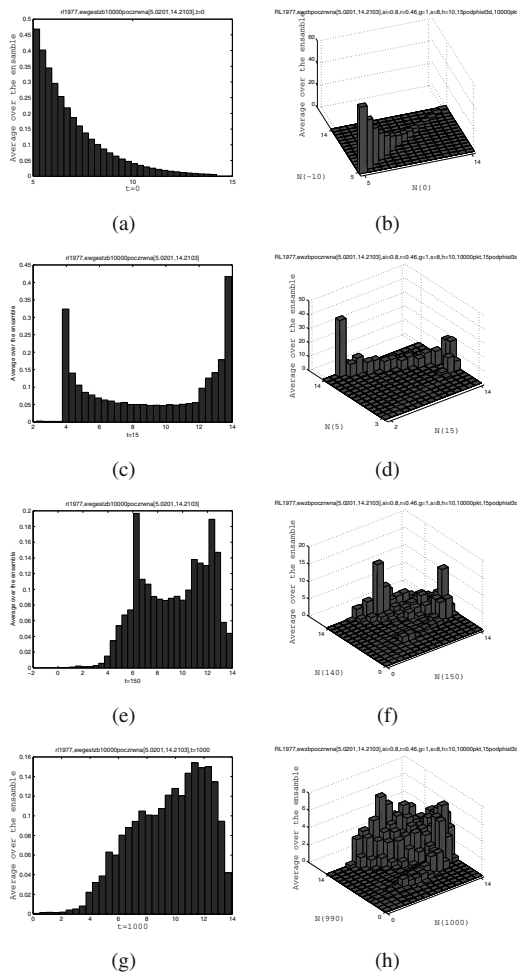More computational analysis concerning ergodic properties of Eqn. (11) as well as new contributions to its

Fig. 10. Evolution of the initial exponential distribution of 10^4 initial constant functions on $[-h, 0]$ in the space of $N(t)$ at time $t = 0$ (a), time $t = 15$ (c), time $t = 150$ (e), time $t = 1000$ and in the space $N(t) \times N(t - h)$ (g), time $t = 0$ (b), time $t = 15$ (d), time $t = 150$ (f), time $t = 1000$ (h).
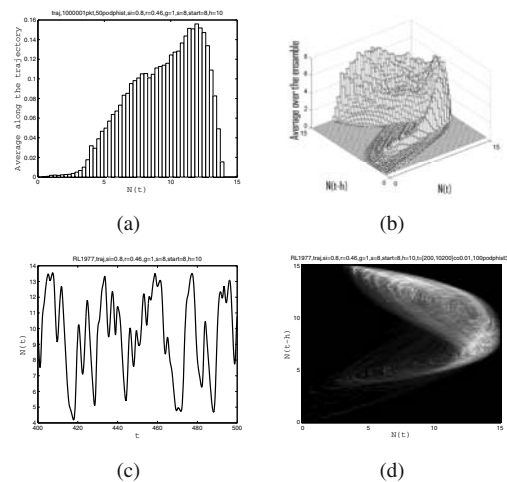
Fig. 11. Average along a single trajectory in the space of $N(t)$ (a), in the space $N(t) \times N(t - h)$ (b). Time evolution of a single trajectory (c), the projection onto the space $N(t) \times N(t - h)$ (d).

biological meaning can be found in the work of Mitkowski (2011).

Acknowledgment

References

Anosov, D.V. (1963). Ergodic properties of geodesic flows on closed Riemanian manifolds of negative curvature, *Soviet Mathematics—Doklady* **4**: 1153–1156.

Arnold, V.I. (1989). *Mathematical Methods of Classical Mechanics*, 2nd Edn., Springer-Verlag, New York, NY, (translation from Russian).

Auslander, J. and Yorke, J.A. (1980). Interval maps, factors of maps and chaos. *Tohoku Mathematical Journal. II. Series* **32**: 177–188.

Bass, J. (1974). Stationary functions and their applications to the theory of turbulence, *Journal of Mathematical Analysis and Applications* **47**: 354–399.

Birkhoff, G.D. (1931a). Proof of a recurrence theorem for strongly transitive systems, *Proceedings of the National Academy of Sciences of the United States of America* **17**: 650–655.

Birkhoff, G.D. (1931b). Proof of the ergodic theorem, *Proceedings of the National Academy of Sciences of the United States of America* **17**: 656–660.

Birkhoff, G.D. and Koopman, B.O. (1932). Recent contributions to the ergodic theory, *Mathematics: Proceedings of the National Academy of Sciences* **18**: 279–282.

Bronsztejn, I.N., Siemiendiajew, K.A., Musiol, G. and Muhlig, H. (2004). *Modern Compendium of Mathematics*, PWN, Warsaw, (in Polish, translation from German).

Dawidowicz, A.L. (1992). On invariant measures supported on the compact sets II, *Universitatis Iagellonicae Acta Mathematica* **29**: 25–28.

Dawidowicz, A.L. (1992). A method of construction of an invariant measure, *Annales Polonici Mathematici* **LVII**(3): 205–208.

Dawidowicz, A.L. (2007). On the Avez method and its generalizations, *Matematyka Stosowana* **8**: 46–55, (in Polish).

Dawidowicz, A.L., Haribash, N. and Poskrobko, A. (2007). On the invariant measure for the quasi-linear Lasota equation. *Mathematical Methods in the Applied Sciences* **30**: 779–787.

Devaney, R.L. (1987). *An Introduction to Chaotic Dynamical Systems*, Addison-Wesley Publishing Company, New York, NY.

Dorfman, J.R. (2001). *Introduction to Chaos in Nonequilibrium Statistical Mechanics*, PWN, Warsaw, (in Polish, translation from English).

Foias, C. (1973). Statistical study of Navier–Stokes equations II, *Rendiconti del Seminario Matematico della Universita di Padova* **49**: 9–123.

Fomin, S.W., Kornfeld, I.P. and Sinaj, J.G. (1987). *Ergodic Theory*, PWN, Warsaw, (in Polish, translation from Russian).

Górnicki, J. (2001). Fundamentals of nonlinear ergodic theory, *Wiadomosci Matematyczne* **37**: 5–16, (in Polish).

Gurney, W.S.C., Blythe, S.P. and Nisbet, R.M. (1980). Nicholson's blowflies revisited, *Nature* **287**: 17–21.

Kudrewicz, J. (1991). *Dynamics of Phase-Locked Loops*, WNT, Warsaw, (in Polish).

Kudrewicz, J. (1993, 2007). *Fractals and Chaos*, WNT, Warsaw, (in Polish).

Landau, L.D., Lifszyc, J.M. (2007). *Mechanics*, PWN, Warsaw, (in Polish, translation from Russian).

de Larminat, P. and Thomas, Y. (1983). *Automatic Control—Linear Systems. Vol. 1: Signals and Systems*, WNT, Warsaw, (in Polish, translation from French).

Lasota, A. (1977). Ergodic problems in biology, *Société Mathematique de France, Asterisque* **50**: 239–250.

Lasota, A. (1979). Invariant measures and a linear model of turbulence, *Rediconti del Seminario Matematico della Universita di Padova* **61**: 39–48.

Lasota, A. (1981). Stable and chaotic solutions of a first order partial differential equation, *Nonlinear Analysis Theory, Methods & Applications* **5**(11): 1181–1193.

Lasota, A. and Mackey, M.C. (1994). *Chaos, Fractals, and Noise Stochastic Aspects of Dynamics*, Springer-Verlag, New York, NY.

Lasota, A., Mackey, M.C. and Wazewska-Czyzewska, M. (1981). Minimazing theraupetically induced anemia, *Journal of Mathematical Biology* **13**: 149–158.

Lasota, A. and Myjak, J. (2002). On a dimension of measures, *Bulletin of the Polish Academy of Sciences: Mathematics* **50**(2): 221–235.

Lasota, A. and Szarek, T. (2002). Dimension of measures invariant with respect to the Wazewska partial differential equation, *Journal of Differential Equations* **196**: 448–465.

Lasota, A. and Yorke, J.A. (1973). On the existence of invariant measures for piecewise monotonic transformations, *Transactions of the American Mathematical Society* **186**: 481–488.

Lasota, A., and Yorke, J.A. (1977). On the existence of invariant measures for transformations with strictly turbulent trajectories, *Bulletin of the Polish Academy of Sciences: Mathematics, Astronomy and Physics* **65**(3): 233–238.

Lebowitz, J.L. and Penrose, O. (1973). Modern ergodic theory, *Physics Today* **26**: 155–175.

Liz, E.and Rost, G. (2009). On the global attractor of delay differential equations with unimodal feedback, *Discrete and Continuous Dynamical Systems* **24**(4): 1215–1224.

Mackey, M.C. (2007). Adventures in Poland: Having fun and doing research with Andrzej Lasota, *Matematyka Stosowana* **8**: 5–32.

Mackey, M.C. and Glass, L. (1977). Oscillations and chaos in physiological control systems, *Science, New Series* **197**(4300): 287–289.

Mitkowski, P.J. (2010). Numerical analysis of existence of invariant and ergodic measure in the model of dynamics of red blood cell's production system, *Proceedings of the 4th European Conference on Computational Mechanics, Paris, France*, pp. 1–2.

Mitkowski, P.J. (2011). *Chaos in the Ergodic Theory Approach in the Model of Disturbed Erythropoiesis*, Ph.D. thesis, AGH University of Science and Technology, Cracow.

Mitkowski, W. (2010). Chaos in linear systems, *Pomiary, Automatyka, Kontrola* **56**(5): 381–384, (in Polish).

Mitkowski, P.J. and Ogorzałek, M.J. (2010). Ergodic properties of the model of dynamics of blood-forming system, *3rd International Conference on Dynamics, Vibration and Control, Shanghai-Hangzhou, China*, pp. 71–74.

Myjak, J. (2008). Andrzej Lasota's selected results. *Opuscula Mathematica* **28**(4): 363–394.

Myjak, J. and Rudnicki, R. (2002). Stability versus chaos for a partial differential equation, *Chaos Solitons & Fractals* **14**: 607–612.

Nadzieja, T. (1996). Individual ergodic theorem from the topological point of view, *Wiadomości Matematyczne* **32**: 27–36, (in Polish).

Nicholson, A.J. (1954). An outline of the dynamic of animal population, *Australian Journal of Zoology* **2**: 9–65.

Ott, E. (1993). *Chaos in Dynamic Systems*, WNT, Warsaw, (in Polish, translation from English).

Prodi, G. (1960). *Teoremi Ergodici per le Equazioni della Idrodinamica*, C.I.M.E., Rome.

Rudnicki, R. (1985a). Invariant measures for the flow of a first order partial differential equation, *Ergodic Theory & Dynamical Systems* **5**: 437–443.

Rudnicki, R. (1985b). Ergodic properties of hyperbolic systems of partial differential equations, *Bulletin of the Polish Academy of Sciences: Mathematics* **33**(11–12): 595–599.

Rudnicki, R. (1988). Strong ergodic properties of a first-order partial differential equation, *Journal of Mathematical Analysis and Applications* **133**: 14–26.

Rudnicki, R. (2004). Chaos for some infinite-dimensional dynamical systems, *Mathematical Methods in the Applied Sciences* **27**: 723–738.

Rudnicki, R. (2009). Chaoticity of the blood cell production system, *Chaos: An Interdisciplinary Journal of Nonlinear Science* **19**(043112): 1–6.

Shampine, L.F., Thompson, S. and Kierzenka, J. (2002). Solving delay differential equations with dde23, available at `www.mathworks.com/dde_tutorial`.

Silva, C.E. (2010). Lecture on dynamical systems, *Spring School of Dynamical Systems, Będlewo, Poland*.

Szlenk, W. (1982). *Introduction to the Theory of Smooth Dynamical Systems*, PWN, Warsaw, (in Polish).

Taylor, S.R. (2004), *Probabilistic Properties of Delay Differential Equations*, Ph.D. thesis, University of Waterloo, Ontario, Canada.

Tucker, W. (1999). The Lorenz attractor exists, *Comptes Rendus. Mathématique. Académie des Sciences, Paris* **328**(I): 1197–1202.

Ulam, S.M. (1960), *A Collection of Mathematical Problems*, Interscience Publishers, New York, NY/London.

Walther, H.O. (1981). Homoclinic solution and chaos in $\dot{x}(t) = f(x(t-1))$, *Nonlinear Analysis: Theory, Methods & Applications* **5**(7): 775–788.

Ważewska-Czyżewska, M. (1983). *Erythrokinetics. Radioisotopic Methods of Investigation and Mathematical Approach*, National Center for Scientific, Technical and Economic Information, Warsaw.

Ważewska-Czyżewska, M. and Lasota, A. (1976). Mathematical problems of blood cells dynamics system, *Matematyka Stosowana* **6**: 23–40, (in Polish).

Paweł J. Mitkowski received his M.Sc. in electrical engineering in 2005 from the Faculty of Electrical Engineering, Automatics, Computer Science and Electronics of the AGH University in Cracow, Poland. In 2011 he obtained a Ph.D. degree upon completing doctoral studies at the same university. His research is now mainly focused on applications of mathematics in biology and medicine, as well as chaotic dynamics and numerical analysis of dynamical systems.

Wojciech Mitkowski received his M. Sc. degree in electrical engineering in 1970 at the Faculty of Electrical Engineering of the AGH University in Cracow, where he currently works in the Department of Automatics. At the same faculty in 1974 he obtained a Ph.D. degree and in 1984 a D.Sc. degree in the field of automatic control. In 1992, he received the title of a professor of technical sciences. In the years 1990–96 he was the dean of the faculty. Since 1988 he has been a member of the Committee on Electrical Engineering, Computer Science and Control of the Polish Academy of Sciences, Cracow Branch, and since 1996 a member of the Committee on Automatic Control and Robotics of the Polish Academy of Sciences. In the years 2005–2010 he was the president of the Cracow Branch of the Polish Mathematical Society. His main scientific interests are automatic control and robotics, control theory, optimal control, dynamic systems, theory of electrical circuits, numerical methods and applications of mathematics. He has published 9 books and 197 scientific papers.