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AN APPROACH TO THE ANALYSIS OF OBSERVABILITY AND CONTROLLABILITY IN NONLINEAR SYSTEMS VIA LINEAR METHODS

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The paper is devoted to the problem of observability and controllability analysis in nonlinear dynamic systems. Both continuous- and discrete-time systems described by nonlinear differential or difference equations, respectively, are considered. A new approach is developed to solve this problem whose features include (i) consideration of systems with non-differentiable nonlinearities and (ii) the use of relatively simple linear methods which may be supported by existing programming systems, e.g., Matlab. Sufficient conditions are given for nonlinear unobservability/uncontrollability analysis. To apply these conditions, one isolates the linear part of the system which is checked to be unobservable/uncontrollable and, if the answer is positive, it is examined whether or not existing nonlinear terms violate the unobservability/uncontrollability property.

Keywords: nonlinear dynamic systems, observability, controllability, linear systems, decomposition.

1. Introduction

Observability and controllability are fundamental system theoretical properties intensively studied in many books and papers (e.g., Kalman *et al.*, 1969; Kwakernaak and Sivan, 1972; Sussmann, 1979; van der Schaft, 1982; Isidori, 1989). The reason for investigation of above system properties is to conclude if the available model of the system can be applied to solve the corresponding control problem.

The problem of observability analysis has been successfully solved for a wide class of dynamic systems linear, linearly-analytical, polynomial, etc. (see, e.g., Kalman *et al.*, 1969; Herman and Krener, 1977; Klamka, 1973; 1975; 2002; Sontag, 1979; Nijmeijer, 1982; Isidori, 1989; Jakubczyk and Sontag, 1990; Jank, 2002; Albertini and D'Alessandro, 2002; Guerman *et al.*, 2008; Zhirabok and Shumsky, 2008; Kawano and Ohtsuka, 2010). Various aspects of observability and controllability have also been investigated (Kotta and Schlacher, 2008; Krener and Ide, 2009; Kang and Xu, 2009; Kang, 2010). All of the above works dealing with nonlinear systems assume that the system under consideration contains only smooth nonlinearities.

There are only few papers considering the properties of observability and controllability for non-smooth systems (Koplon and Sontag, 1993; Murphey and Burdick, 2002; Mincheko and Sirotko, 2002). Koplon and Sontag (1993) investigated linear continuous-time and discrete-time systems with a non-smooth output function and gave a number of observability criteria. Local controllability analysis for continuous-time systems performed by Murphey and Burdick (2002) can be interpreted as a non-smooth extension of Chow's theorem, formulated in terms of generalized differential quotients. The same analysis by Mincheko and Sirotko (2002) is based on Mordukhovch's non-smooth constructions, in particular, subdifferentials and coderivatives.

In the present paper, a new approach to analyze the observability and controllability properties for nonlinear dynamic systems that may contain non-smooth nonlinearities is suggested. This approach is based on the so-called logic-dynamic approach proposed first by Zhirabok and Usoltsev (2002) and then developed by Zhirabok (2010) as well as Zhirabok and Shumsky (2010). The main idea of the approach is to replace the nonlinear system under consideration by a certain linear one, solving the problem for this linear system involving linear methods and, finally, taking into account nonlinear terms to correct the obtained solution. Contrary to the cited papers, the present

work investigates global rather than local properties of dynamic systems. The main features of the proposed approach are as follows: (i) it considers systems with non-smooth nonlinearities in dynamics, (ii) it involves known linear methods that result in a possibility to support the observability and controllability analysis by existing programming systems without using symbolic software.

The conference version of this paper was presented by Zhirabok and Shumsky (2010). The present paper contains the following additions:

(i) systems with nonlinear output function are considered,

(ii) more sophisticated analysis in the case of several types of nonlinearities is used.

Both allow extending the class of systems to which the suggested method can be applied.

Consider the class of nonlinear dynamic systems described by the difference equations

$$x'(t + 1) = f'(x'(t), u(t)), \quad y(t) = h'(x'(t)), \quad (1)$$

in the discrete-time case, or differential equations

$$\dot{x}'(t) = f'(x'(t), u(t)), \quad y(t) = h'(x'(t)), \quad (2)$$

in the continuous-time case, where $x' \in X \subseteq \mathbb{R}^n$, $u \in U \subseteq \mathbb{R}^m$, $y \in Y \subseteq \mathbb{R}^l$ are vectors of state, control and output, respectively, f' and h' are nonlinear vector functions. The function f' may be non-smooth and the function h' is assumed to be smooth. Suppose that the function h' satisfies the condition

$$\operatorname{rank}\left(\frac{\partial h'}{\partial x'}\right) = l \quad (3)$$

for all $x' \in \mathbb{R}^n$ except on a set of measure zero.

Typical non-smooth nonlinearities in technical systems are saturation, the Coulomb friction, backlash and hysteresis. The method suggested in the paper is applicable not only for such types but for others, too.

The rest part of the paper is organized as follows. Several important system transformations are introduced in Section 2. Some preliminary results are given in Section 3. The problems of observability and controllability analysis are considered in Sections 4 and 5, respectively. Generalizations are given in Section 6. Some system issues are considered in Section 7. An illustrative example is given in Section 8, while Section 9 concludes the paper.

2. System representations

Consider several transformations of the system (1). Notice under this that appropriate transformations of the model (2) are similar to those discussed below for the model (1) and, as a result, are omitted. Notice also that

these transformations are the starting point for application of the logic-dynamic approach.

For the system (1), a special coordinate transformation

$$x = \Psi(x')$$
$$= (\ h'_1(x') \quad \ldots \quad h'_l(x') \quad x'_{i_1} \quad \ldots \quad x'_{i_{n-l}} \ )^T$$

is given by a function Ψ satisfying the condition

$$\operatorname{rank}\left(\frac{\partial \Psi}{\partial x'}\right) = n \quad (4)$$

for all $x' \in \mathbb{R}^n$ except on a set of measure zero where $x'_{i_1}, \ldots, x'_{i_{n-l}}$ are some variables, h'_i is the i-th component of the function h'. In new coordinates, the system takes the form

$$
\begin{aligned}
x(t + 1) &= \Psi(f'(x'(t), u(t))) \\
&= \Psi(f'(\Psi^{-1}(x(t)), u(t))) \\
&= f(x(t), u(t)), \\
y(t) &= h'(\Psi^{-1}(x(t))) = Hx(t)
\end{aligned}
\quad (5)
$$

with $H = (I_{l \times l} \quad 0)$, where $I_{l \times l}$ is the unit matrix of appropriate dimensions.

Example 1. Let $y_1 = x'_1 x'_2$, $y_2 = \exp(-x'_3)$, $n = 4$, $l = 2$. Under this, $\operatorname{rank}(\partial h'/\partial x') = 2$ except $x'_1 = 0$ or $x'_2 = 0$. The functions Ψ and Ψ^{-1} specify the following transformations:

$$
\begin{aligned}
x = \Psi(x') &= (\ x'_1 x'_2 \quad \exp(-x'_3) \quad x'_2 \quad x'_4 \ )^T, \\
x' = \Psi^{-1}(x) &= (\ x_1/x_3 \quad x_3 \quad -\ln(x_2) \quad x_4 \ )^T.
\end{aligned}
$$

It can be shown that $\operatorname{rank}(\partial \Psi/\partial x') = 4$ except $x'_1 = 0$ or $x'_2 = 0$. As a result,

$$
\begin{aligned}
y_1 &= (x_1/x_3)x_3 = x_1, \\
y_2 &= \exp(-(-\ln(x_2))) = x_2, \\
H &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.
\end{aligned}
$$

♦

Remark 1. Note that a violation of the conditions $\operatorname{rank}(\partial h'/\partial x') = l$ and $\operatorname{rank}(\partial \Psi/\partial x') = n$ on a set of measure zero is acceptable because only generic properties of dynamic systems are analyzed.

Remark 2. The special case is when the function h' in (1) is a linear one, i.e., the appropriate equation has the form $y(t) = Hx'(t)$ for some matrix H. In this case, the model of the form (5) is obtained under the identity function Ψ that corresponds to $x = x'$ and $f(x, u) = f'(x', u)$.

Remark 3. Clearly, the described coordinate transformation is not unique but this does not affect the result presented below because controllability and observability are intrinsic properties independent of such a transformation.

To apply the logic-dynamic approach, the function $f(x, u)$ is transformed via separating linear terms from nonlinear ones. Such a transformation results in a system of the form

$$x(t+1) = Fx(t) + Gu(t)$$
$$+ C \begin{pmatrix} \varphi_1(A_1 x(t), u(t)) \\ \vdots \\ \varphi_p(A_p x(t), u(t)) \end{pmatrix}, \qquad (6)$$
$$y(t) = Hx(t)$$

where F and G are the matrices of appropriate dimensions describing the linear part of the system; C is an $n \times p$ constant matrix constructed as follows: If the right-hand side of the equation from (5) related to the component $x_i(t+1)$ contains the nonlinearity $\varphi_j(A_j x, u)$, then $C(i, j) \neq 0$, otherwise $C(i, j) = 0$. Generally, the function φ_j may contain several terms in the form $A_j x$.

Example 2. Consider a system described by the equations

$$x_1(t+1) = \ln|x_1(t)| - x_1(t) + u_2(t)$$
$$+ x_2(t) + x_2(t)u_1^2(t),$$
$$x_2(t+1) = -x_2(t) - \ln|x_1(t)| - u_1(t).$$

In this case, the matrices and functions used in (6) are as follows:

$$F = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}, \qquad G = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$
$$C = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix},$$
$$\varphi_1(x, u) = \ln|A_1 x|, \qquad \varphi_2(x, u) = (A_2 x)u_1^2,$$
$$A_1 = (\,1 \quad 0\,), \qquad A_2 = (\,0 \quad 1\,).$$

In some cases such a transformation leads to an erroneous decisions because the main conclusion is made based on a linear part of the system. ♦

Example 3. Consider the system

$$x_1(t+1) = \text{sign}(x_2(t)) + u(t),$$
$$x_2(t+1) = x_3(t) - x_2(t),$$
$$x_3(t+1) = x_1(t) - \text{sign}(x_2(t)) - x_3(t) - 2u(t),$$
$$y(t) = x_1(t).$$

A matrix description is as follows:

$$F = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & -1 \end{pmatrix}, \qquad G = \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix},$$
$$H = (\,1 \quad 0 \quad 0\,), \qquad C = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix},$$
$$\varphi(x, u) = \text{sign}(Ax), \qquad A = (\,0 \quad 1 \quad 0\,).$$

Clearly, $HF = 0$, and therefore, an analysis of observability is impossible. To overcome this difficult, the formal term $x_2 - x_2$ is added to the first equation, then the variable x_2 supplements the linear part, and the nonlinear term becomes $\text{sign}(x_2) - x_2$. The matrix description is changed as follows:

$$F = \begin{pmatrix} 0 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & -1 \end{pmatrix}, \qquad C = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & -1 \end{pmatrix},$$
$$\varphi_1(x, u) = \text{sign}(A_1 x) - A_1 x,$$
$$\varphi_2(x, u) = \text{sign}(A_2 x),$$
$$A_1 = A_2 = (\,0 \quad 1 \quad 0\,).$$

Generally, to analyze observability, the initial system can be corrected as follows:

$$x(t+1) = (F + CA)x(t) + Gu(t)$$
$$+ C \begin{pmatrix} \varphi_1(A_1 x(t), u(t)) \\ \vdots \\ \varphi_p(A_p x(t), u(t)) \end{pmatrix} - CAx(t).$$
$$(7)$$

To analyze controllability, a correction is made by analogy:

$$x(t+1) = (F + CA)x(t) + (G + CB)u(t)$$
$$+ C \begin{pmatrix} \varphi_1(A_1 x(t), B_1 u(t)) \\ \vdots \\ \varphi_p(A_p x(t), B_p u(t)) \end{pmatrix}$$
$$- CAx(t) - CBu(t) \qquad (8)$$

with

$$A = \begin{pmatrix} A_1 \\ \vdots \\ A_p \end{pmatrix}, \qquad B = \begin{pmatrix} B_1 \\ \vdots \\ B_p \end{pmatrix}.$$

Note that such corrections make the linear part more complicated but allow taking into account all possible nonlinear terms. In the specific case, one may use partial correction to avoid zero rows or columns in the matrix F.

Reasoning along similar lines, it is easily to transform (2) into

$$
\dot{x}(t) = Fx(t) + Gu(t) + C \begin{pmatrix} \varphi_1(A_1 x(t), u(t)) \\ \vdots \\ \varphi_p(A_p x(t), u(t)) \end{pmatrix},
$$

$$
y(t) = Hx(t).
$$

$$(9)$$

Denote by the quintuple $\Sigma = (F, G, H, C, A)$ the system (6) or (9). The model (6) or (9) without the nonlinear term will be named a linear part of the system Σ, denoted by the triple $\Lambda = (F, G, H)$. In order to avoid the repetition, interpretation of the main results is given below only for the discrete-time case; the same results for the continuous-time case are obvious and can be obtained by analogy.

3. Preliminary results

There are many different kinds of observability (Herman and Krener, 1977; Sontag, 1979): single- and multi-experiment observability, local observability, local weak observability, global observability. This paper considers global observability defined as follows: The system (1) or (2) is said to be *globally observable* if for every pair of states $x(t_0)$ and $x'(t_0)$ a control $u(t)$, $t_0 \leq t < \infty$, exists such that $H(x(t_0), u(t)) \neq H(x'(t_0), u(t))$, where $H(x(t_0), u(t))$ $(H(x'(t_0), u(t)))$ is the output sequence produced by the system in the initial state $x(t_0)$ $(x'(t_0))$ under the control $u(t)$. Thus, multi-experiment observability is studied.

There are also many kinds of controllability (Herman and Krener, 1977): local controllability, global controllability, local weak controllability. This paper is concentrated on global controllability defined as follows: The system (1) or (2) is said to be *globally controllable* if for every pair of states $x(t_0)$ and $x'(t_1)$ a control $u(t)$, $t_0 \leq t \leq t_1$, exists which transfer the state from $x(t_0)$ to $x'(t_1)$ in a finite time interval $[t_0, t_1]$.

The main idea of the suggested approach is as follows: Observability (controllability) of a linear part is checked and, if it is unobservable (uncontrollable), one examines whether or not the nonlinear term violates this conclusion. This examination is based on well-known canonical forms of unobservable (uncontrollable) linear systems (Kwakernaak and Sivan, 1972) and on the analysis whether or not the nonlinear term changes the existing links between observable and unobservable (controllable and uncontrollable) subsystems in these forms. If these links are not changed, a conclusion is made that the initial nonlinear system is unobservable (uncontrollable). Appropriate changes of the links mean that the conclusion about unobservability (uncontrollability) becomes impossible.

To solve the problem under consideration, well-known criteria of observability and controllability for linear systems are used: the system Λ is observable if and only if $\operatorname{rank}(V) = n$, where

$$
V = \begin{pmatrix} H \\ HF \\ \vdots \\ HF^{n-1} \end{pmatrix};
$$

the system Λ is controllable if and only if $\operatorname{rank}(W) = n$, where

$$
W = (\ G \quad FG \quad \dots \quad F^{n-1}G\ ).
$$

Recall that the matrices V and W are known as observability and controllability matrices of the system Λ, respectively.

Consider first the special case when the system has only one nonlinearity of the form $C\varphi(Ax, u)$ with a column matrix C and a row matrix A. Because canonical decompositions of unobservable and uncontrollable linear systems play an important part in our analysis, consider linear coordinate transformation given by the matrix $T : \Sigma \Rightarrow \Sigma_* = (F_*, G_*, H_*, C_*, A_*)$ according to

$$
x_*(t) = Tx(t), \quad \forall t. \qquad (10)
$$

A discrete-time description of the system Σ_* is given by the equations

$$
\begin{aligned}
x_*(t+1) &= F_* x_*(t) + G_* u(t) \\
&\quad + C_* \varphi(A_* x_*(t), u(t)), \\
y(t) &= H_* x_*(t), \qquad\qquad (11)
\end{aligned}
$$

where x_* is a state vector, F_*, G_*, H_*, C_*, and A_* are matrices of appropriate dimensions. To be specific, consider in detail only the discrete-time case.

Consider the relationships between matrices of the systems Σ and Σ_*. Rewriting (10) with $t+1$ and involving (6) and (11), we obtain

$$
\begin{aligned}
F_* x_* + G_* u &+ C_* \varphi(A_* x_*, u) \\
&= TFx + TGu + TC\varphi(Ax, u).
\end{aligned}
$$

Notice that to obtain the same relationship for the continuous-time case, time differentiation of both the sides of (10) should be used instead of the time shift. If the nonlinear term $C\varphi(Ax, u)$ is absent, then $C = 0$ and from the above well-known relationships we get

$$
F_* T = TF, \quad G_* = TG, \quad H = H_* T. \qquad (12)
$$

This, with (10), results in

$$
C_* \varphi(A_* x_*, u) = C_* \varphi(A_* Tx, u) = TC\varphi(Ax, u).
$$

If $\varphi(*) = const$, then

$$C_* = TC. \tag{13}$$

If $\varphi(Ax, u) = Ax$ and C is a nonsingular matrix, then $\varphi(A_* x_*, u) = A_* Tx$ and

$$A = A_* T. \tag{14}$$

When the function φ has a particular form, the relationships between matrices may have another (corresponding particular) form. Therefore, (12)–(14) are sufficient (but not necessary) conditions for transformation $\Sigma \Rightarrow \Sigma_*$. It can be shown that (13) and (14) are valid when several nonlinearities are in the system and C is an $n \times p$ matrix, A is a $p \times n$ matrix.

Clearly, the relations (12)–(14) are true for the continuous-time case.

4. Observability analysis

Suppose that

$$\mathrm{rank}(V) = s < n, \tag{15}$$

i.e., the system Λ is unobservable. Denote by T' the matrix of maximal rank, containing all linearly independent rows of the matrix V and let

$$T = \begin{pmatrix} T' \\ T'' \end{pmatrix},$$

where the matrix T'' satisfies the condition $\mathrm{rank}(T) = n$. This structure of the matrix T corresponds to the canonical decomposition (Fig. 1) of the linear part Λ which contains observable and unobservable subsystems denoted as Λ_{*1} and Λ_{*2}, respectively (Kwakernaak and Sivan, 1972). The nonlinear term $C\varphi(Ax, u)$ is also transformed by the matrix T and added to the decomposition as an additional links between subsystems. Consider the following three cases.
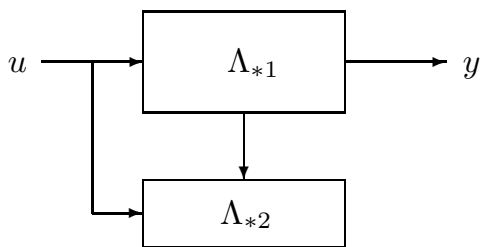

Fig. 1. Canonical decomposition of an unobservable system.

It is assumed in the *first case* that an argument of the nonlinear term is formed on the basis only of the state vector x_*^1 of the subsystem Λ_{*1} and is independent of state vector of the the subsystem Λ_{*2}, i.e., it has the form $C_* \varphi(A_{*1} x_*^1, u)$ with an appropriate matrix C_*, the matrix $A_* = \begin{pmatrix} A_{*1} & A_{*2} \end{pmatrix}$ and $x_*^1 = T'x$. Then, the above nonlinear term is added into the decomposition as a link from the first subsystem to the second one, i.e., the structure of links in this decomposition remains and, hence, the added nonlinearity does not violate the unobservability property.

Consider the condition for this realization. Notice first that the equality (14) results in

$$Ax = A_* Tx = \begin{pmatrix} A_{*1} & A_{*2} \end{pmatrix} \begin{pmatrix} T' \\ T'' \end{pmatrix} x$$
$$= A_{*1} T'x + A_{*2} T''x = A_{*1} x_*^1 + A_{*2} x_*^2.$$

Therefore, the nonlinear term has the form $C_* \varphi(A_{*1} x_*^1, u)$ under the conditions $A_{*2} T'' = 0$ and $A = A_{*1} T'$ (i.e., the row matrix A should be a linear combination of rows of the matrix T'). The last one is equivalent to the condition

$$\mathrm{rank}(T') = \mathrm{rank}\begin{pmatrix} T' \\ A \end{pmatrix}.$$

Finally, since $\mathrm{rank}(V) = \mathrm{rank}(T')$, the above condition can be rewritten as follows:

$$\mathrm{rank}(V) = \mathrm{rank}\begin{pmatrix} V \\ A \end{pmatrix}. \tag{16}$$

So, the condition (16) describes the situation when the nonlinear term does not violate the unobservability property.

Example 4. Consider a system described by the equations

$$\begin{aligned}
x_1(t+1) &= x_2(t) - x_3(t) + u_1(t), \\
x_2(t+1) &= x_1(t) + x_4(t), \\
x_3(t+1) &= x_4(t) - u_1(t)\ln|x_2(t) - x_3(t)|, \\
x_4(t+1) &= x_3(t) + u_2(t), \\
y_1(t) &= x_1(t), \\
y_2(t) &= x_2(t) - x_3(t).
\end{aligned}$$

The third (nonlinear) equation can be rewritten in the equivalent form according to (7):

$$x_3(t+1) = x_4(t) - x_2(t) + x_3(t) \\
- u_1(t)\ln|x_2(t) - x_3(t)| + x_2(t) - x_3(t),$$

which allows taking

$$F = \begin{pmatrix} 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad G = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix},$$

$$H = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 0 \\ 0 \\ -1 \\ 0 \end{pmatrix},$$

$$\varphi(x,u) = u_1 \ln|Ax| + Ax, \quad A = (\; 0 \quad 1 \quad -1 \quad 0 \;).$$

One can see that

$$\text{rank}(V) = \text{rank} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 1 & -1 & 0 \\ 1 & 1 & -1 & 0 \\ 1 & 2 & -2 & 0 \end{pmatrix} = 2 < 4,$$

i.e., a linear part of the system is unobservable. Since

$$\text{rank}\begin{pmatrix} V \\ A \end{pmatrix} = 2,$$

the condition (16) holds, and the system under consideration is unobservable as well. One can check that if the third equation is not subject to transformations according to (7), the final result is the same. ◆

In the *second case*, it is assumed that the condition (16) fails. Consequently, an additional link does not violate the existing connections if it is a member of the unobservable subsystem Λ_{*2} only. According to (13), a contribution of the nonlinear term into the subsystem Λ_{*1} is specified by the matrix $C_{*1} = T'C$ because

$$C_* = \begin{pmatrix} C_{*1} \\ C_{*2} \end{pmatrix} = \begin{pmatrix} T' \\ T'' \end{pmatrix} C.$$

Therefore, the condition mentioned above holds if $C_{*1} = T'C = 0$, or

$$VC = 0. \tag{17}$$

Example 5. Consider the system from Example 3 with the following modifications: the nonlinear term is removed from the third equation and is added to the fourth one in the form $u_1(t)\ln|x_2(t) + x_3(t)|$. Taking into account (7), this equation must be rewritten as

$$x_4(t+1) = 2x_2(t) + u_2(t) + u_1(t)\ln|x_2(t) + x_3(t)| - x_2(t) - x_3(t).$$

Therefore a matrix description of the system is as follows:

$$F = \begin{pmatrix} 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 2 & 0 & 0 \end{pmatrix}, \qquad G = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix},$$

$$H = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \end{pmatrix}, \qquad C = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix},$$

$$\varphi(x,u) = u_1 \ln|Ax| - Ax, \quad A = (\; 0 \quad 1 \quad 1 \quad 0 \;).$$

Then one has

$$\text{rank}(V) = \text{rank}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \end{pmatrix} = 2 < 4,$$

i.e., a linear part of the system is unobservable. Since

$$\text{rank}\begin{pmatrix} V \\ A \end{pmatrix} = 3,$$

the condition (16) is violated but, due to $VC = 0$, the condition (17) holds. Therefore, the initial nonlinear system is unobservable. It can be shown that, if the fourth equation is not transformed according to (7), the conclusion about unobservability does not change. ◆

In the *third case*, it is assumed that the conditions (16) and (17) do not hold. This means that, firstly, an argument of the nonlinear term is formed not only on the basis of the state vector of the subsystem Λ_{*1} and, secondly, this function is contained not only in the subsystem Λ_{*2}. In this case, suppose that the subsystem Λ_{*2} can be presented in the decomposed form shown in Fig. 2 and an argument of the nonlinear term is a combination of the state vectors x_*^1 and x_*^{21} of the subsystems Λ_{*1} and Λ_{*21}, respectively. The latter is equivalent to the condition

$$\text{rank}\begin{pmatrix} T' \\ T_1'' \end{pmatrix} = \text{rank}\begin{pmatrix} T' \\ T_1'' \\ A \end{pmatrix}, \tag{18}$$

where the matrix T_1'' determines the vector x_*^{21} according to the equality $x_*^{21} = T_1''x$. The possibility of such a de-
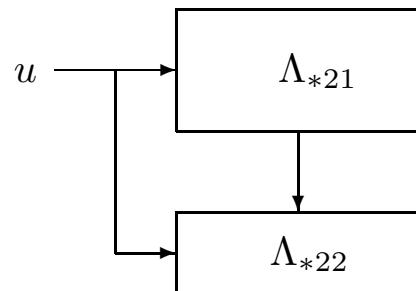

Fig. 2. Decomposition of an unobservable subsystem.

composition is established as follows.

Notice that the subsystem Λ_{*1} estimates the output vector y and is constructed on the basis of the observability matrix V. By analogy, one can say that the complex system $(\Lambda_{*1}, \Lambda_{*21})$ estimates the actual output vector

$y = Hx$ and, according to (18), the fictitious output vector $y_f = Ax$ on the basis of the extended observability matrix

$$V_\Sigma = \begin{pmatrix} H_\Sigma \\ H_\Sigma F \\ \vdots \\ H_\Sigma F^{n-1} \end{pmatrix},$$

where

$$H_\Sigma = \begin{pmatrix} H \\ A \end{pmatrix}.$$

From the structure of the matrices V_Σ and H_Σ it follows that, if we set $V = V_\Sigma$ in the condition (16), then it is fulfilled, and hence in the case of

$$\text{rank}(V_\Sigma) < n \qquad (19)$$

the desired decomposition is possible and the system is not observable.

Example 6. Consider a system described by the equations

$$\begin{aligned}
x_1(t+1) &= x_2(t) - x_5(t) + u_1(t), \\
x_2(t+1) &= x_1(t) + 2x_3(t) - x_2(t), \\
x_3(t+1) &= \text{sign}(x_4(t)) - x_3(t) + x_1(t), \\
x_4(t+1) &= x_3(t) + u_2(t) - \text{sign}(x_4(t)), \\
x_5(t+1) &= x_1(t) + x_3(t) - x_5(t), \\
y_1(t) &= x_1(t), \\
y_2(t) &= x_3(t).
\end{aligned}$$

A matrix description of the system is as follows:

$$F = \begin{pmatrix} 0 & 1 & 0 & 0 & -1 \\ 1 & -1 & 2 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & -1 \end{pmatrix},$$

$$G = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 0 \end{pmatrix},$$

$$H = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix},$$

$$\varphi(x, u) = \text{sign}(Ax),$$

$$A = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

One can see that

$$\text{rank}(V)$$

$$= \text{rank} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 \\ 1 & 0 & -1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 1 \\ -1 & 1 & 1 & 0 & -1 \end{pmatrix} = 3 < 5,$$

i.e., a linear part of the system is unobservable. Since

$$\text{rank} \begin{pmatrix} V \\ A \end{pmatrix} = 4$$

and $VC \neq 0$, the conditions (16) and (17) are violated. To check the condition (19), find the extended output matrix H_Σ and the rank of the observability matrix V_Σ:

$$H_\Sigma = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix},$$

$$\text{rank}(V_\Sigma) = \text{rank} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & -1 \\ 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

$$= 4 < 5.$$

Thus, the system is unobservable according to (19).

Taking into account (7), one obtains

$$F = \begin{pmatrix} 0 & 1 & 0 & 0 & -1 \\ 1 & -1 & 2 & 0 & 0 \\ 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 1 & 0 & 1 & 0 & -1 \end{pmatrix},$$

$$\text{rank}(V)$$

$$= \text{rank} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 \\ 1 & 0 & -1 & 1 & 0 \\ 0 & -1 & 1 & 0 & 1 \\ -1 & 1 & 2 & -2 & -1 \end{pmatrix} = 4 < 5.$$

One can easily check that the condition (16) holds, i.e. the system is unobservable. $\qquad \blacklozenge$

All of the above results are summarized below in the form of a theorem.

Theorem 1. *The system Σ is unobservable if at least one of the three following situations takes place:*

(i) the relations (15) and (16) hold,

(ii) the relations (15) and (17) hold,

(iii) the relation (19) holds.

Because the criteria of observability for linear systems are the same for the discrete-time and continuous-time cases, Theorem 1 is true for continuous-time systems.

5. Controllability analysis

To analyze the controllability of the system Σ, the duality principle for linear systems (Kwakernaak and Sivan, 1972) is involved. According to this principle, a linear system given by the matrices F, G, and H is controllable if and only if the dual system given by the matrices F^T, H^T, and G^T is observable, where the superscript T denotes matrix transposition. Thus, to analyze the controllability of the system Σ, it is necessary, first of all, to construct the observability matrix

$$V_d = \begin{pmatrix} G^T \\ G^T F^T \\ \vdots \\ G^T (F^T)^{n-1} \end{pmatrix}$$
$$= \begin{pmatrix} G & FG & \ldots & F^{n-1}G \end{pmatrix}^T = W^T$$

for the dual system. Suppose that

$$\operatorname{rank}(V_d) = \operatorname{rank}(W) = r < n, \qquad (20)$$

i.e., a linear part Λ of the system Σ is uncontrollable. Denote by Φ' the matrix of the maximal rank containing all linear independent rows of the matrix V_d (transposed columns of the matrix W).

Let

$$\Phi = \begin{pmatrix} \Phi' \\ \Phi'' \end{pmatrix},$$

where the matrix Φ'' satisfies the condition $\operatorname{rank}(\Phi) = n$. The structure of the matrix Φ corresponds to the canonical decomposition (Fig. 3) of the linear part Λ, containing controllable and uncontrollable subsystems Λ_{01} and Λ_{02}, respectively (Kwakernaak and Sivan, 1972).
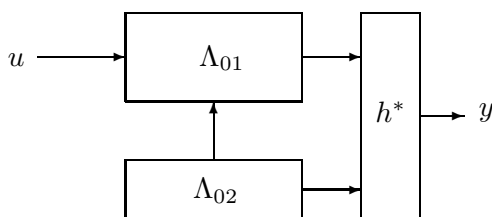

Fig. 3. Canonical decomposition of an uncontrollable system.

Compared with observability analysis, that of controllability seems to be more complicated. This is caused by the fact that the added nonlinearity should be analyzed involving the uncontrollable subsystem Λ_{02} and the matrix Φ''. For this reason, the matrix Φ'' should be constructed in a special way. Below, the matrix Φ'' is chosen from the condition $\Phi'' \Phi'^T = 0$ (i.e., rows of the matrix Φ'' should be orthogonal to the rows of the matrix Φ'). Notice, that such a choice is always possible. The last condition allows obtaining a canonical decomposition of the above form by means of the matrix Φ. Then the transformation method in use differs from the conventional one (Kwakernaak and Sivan, 1972).

Indeed, let the inverse matrix of Φ have the form $\Phi^{-1} = \begin{pmatrix} R_1 & R_2 \end{pmatrix}$. Then $\Phi'' R_1 = 0$, and, hence, with regard to the equation $\Phi'' \Phi'^T = 0$, one obtains $R_1 = \Phi'^T$. The dynamics of the transformed system are determined by the product

$$\begin{pmatrix} \Phi' \\ \Phi'' \end{pmatrix} F \begin{pmatrix} R_1 & R_2 \end{pmatrix} = \begin{pmatrix} \Phi' F R_1 & \Phi' F R_2 \\ \Phi'' F R_1 & \Phi'' F R_2 \end{pmatrix}.$$

Consider the equation $\Phi'' F R_1 = \Phi'' F \Phi'^T$. By definition, the columns of the matrix Φ'^T consist of all linearly independent columns of the matrix W, the columns of the matrix $F \Phi'^T$ are linear combinations of the columns of this matrix, i.e., $F \Phi'^T = \Phi'^T B$ holds for some matrix B. Hence, one can write $\Phi'' F R_1 = \Phi'' F \Phi'^T = \Phi'' \Phi'^T B = 0$, which means the absence of links from the subsystems Λ_{01} into Λ_{02}. Besides,

$$\Phi G = \begin{pmatrix} \Phi' G \\ \Phi'' G \end{pmatrix} = \begin{pmatrix} \Phi' G \\ 0 \end{pmatrix}$$

because of the condition $\Phi'' \Phi'^T = 0$, i.e., the subsystem Λ_{02} has no input.

As above, the nonlinear term $C\varphi(Ax, u)$ after its transformation given by the matrix Φ is also added to the decomposition as an additional link between the subsystems. As under observability analysis, consider the following three cases.

The *first case* assumes that an additional link does not violate the existing connections when it is a member of the uncontrollable subsystem Λ_{02} only. According to (13) with

$$T = \Phi = \begin{pmatrix} \Phi' \\ \Phi'' \end{pmatrix},$$

a contribution of the nonlinear term into the subsystem Λ_{02} is specified by the matrix $C_{02} = \Phi'' C$. Therefore, the condition of the link from the subsystem Λ_{02} into the subsystem Λ_{01} holds if

$$\Phi'' C = 0. \qquad (21)$$

Then from $\Phi'' \Phi'^T = 0$ and the condition (19) one can find that the columns of the matrix C must depend

linearly on the columns of the matrix Φ'^T, i.e., for some matrix C_0, the equality

$$C = \Phi'^T C_0 \qquad (22)$$

has to hold. This is equivalent to the rank condition

$$\text{rank}(\Phi'^T) = \text{rank}(\ \Phi'^T \quad C \ ).$$

Since the columns of the matrix W are linear combinations of the matrix Φ'^T columns, the latter condition can be written as

$$\text{rank}(W) = \text{rank}(\ W \quad C \ ). \qquad (23)$$

Example 7. Consider a system described by the equations

$$x_1(t+1) = x_2(t) - x_1(t) + u_1(t),$$
$$x_2(t+1) = x_1(t) + x_2(t) + x_3(t) + u_2(t)x_4(t),$$
$$x_3(t+1) = x_4(t) - x_1(t) - 2x_2(t) - 2x_3(t)$$
$$- u_2(t)x_4(t),$$
$$x_4(t+1) = x_2(t) + x_3(t).$$

The following matrix description corresponds to this system:

$$F = \begin{pmatrix} -1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ -1 & -2 & -2 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix},$$

$$G = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix},$$

$$\varphi(x, u) = BuAx,$$
$$A = (\ 0 \ \ 0 \ \ 0 \ \ 1 \ ),$$
$$B = (\ 0 \ \ 1 \ ).$$

Then

$$\text{rank}(W)$$
$$= \text{rank} \begin{pmatrix} 1 & 0 & -1 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} = 2 < 4.$$

Therefore, a linear part of the system is uncontrollable. As soon as for the matrices W and C the condition (23) holds, the initial nonlinear system is uncontrollable as well.

If the system is subjected to the transformation according to (8), i.e.,

$$x_2(t+1) = x_1(t) + x_2(t) + x_3(t) + x_4(t)$$
$$+ u_2(t) + u_2(t)x_4(t) - x_4(t) - u_2(t),$$
$$x_3(t+1) = -x_1(t) - 2x_2(t) - 2x_3(t) - u_2(t)$$
$$- u_2(t)x_4(t) + x_4(t) + u_2(t),$$

one obtains the following results:

$$F = \begin{pmatrix} -1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ -1 & -2 & -2 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix}, \quad G = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & -1 \\ 0 & 0 \end{pmatrix},$$

$$\text{rank}(W) = \text{rank} \begin{pmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = 2,$$

i.e., the conclusion concerning controllability does not change. $\blacklozenge$

In the *second case*, violation of (23) is assumed. Consider the particular case when the nonlinear term has the form $C\varphi(Ax, u) = C\varphi(Ax)$, i.e., it does not depend on the control u. Suppose that an argument of the nonlinear term depends only on the state vector of the uncontrollable subsystem Λ_{02} and does not depend on the state vector of the controllable subsystem Λ_{01}. In this case, the nonlinear term added to the decomposition preserves the structure of the links existing in this decomposition, and hence it does not affect the property of uncontrollability.

The above condition is fulfilled if the row matrix A, which the argument depends on, is a linear combination of the matrix Φ'' rows, i.e., for some matrix A_0 it holds that

$$A = A_0\Phi'',$$

and then

$$A\Phi'^T = A_0\Phi''\Phi'^T = 0 \qquad (24)$$

or

$$AW = 0. \qquad (25)$$

Example 8. Consider the system

$$x_1(t+1) = x_2(t) - x_1(t) + |x_1(t) + x_4(t)|$$
$$+ u_1(t),$$
$$x_2(t+1) = x_3(t),$$
$$x_3(t+1) = 2x_1(t) - x_3(t) + |x_1(t) + x_4(t)|$$
$$+ u_2(t),$$
$$x_4(t+1) = -x_2(t) - x_4(t) - u_1(t).$$

A matrix description of the system is as follows:

$$F = \begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 2 & 0 & -1 & 0 \\ 0 & -1 & 0 & -1 \end{pmatrix},$$

$$G = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ -1 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix},$$

$$\varphi(x, u) = |Ax|, \quad A = (\ 1 \ \ 0 \ \ 0 \ \ 1 \ ).$$

Computations without system transformation give

$$\text{rank}(W)$$

$$= \text{rank} \begin{pmatrix} 1 & 0 & -1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 2 & -1 \\ 0 & 1 & 2 & -1 & -4 & 1 \\ -1 & 0 & 1 & 0 & -1 & -1 \end{pmatrix} = 3 < 4.$$

Therefore, a linear part of the system is uncontrollable. As soon as $\text{rank}(\ W\quad C\ ) = 4$, the condition (23) does not hold but $AW = 0$. Therefore, the initial nonlinear system is uncontrollable as well.

One can transform the initial system according to (8) as follows:

$$x_1(t + 1) = x_2(t) + x_4(t) + |x_1(t) + x_4(t)|$$
$$+ u_1(t) - (x_1(t) + x_4(t)),$$
$$x_2(t + 1) = x_3(t),$$
$$x_3(t + 1) = 3x_1(t) - x_3(t) + x_4(t) + |x_1(t)$$
$$+ x_4(t)| + u_2(t) - (x_1(t) + x_4(t)),$$
$$x_4(t + 1) = -x_2(t) - x_4(t) - u_1(t)$$

but it can be shown that the matrix W coincides with the one given above, and the conclusion about uncontrollability of the initial nonlinear system does not change. ♦

In the *third case*, it is assumed that the conditions (23) and (25) do not hold or the function φ depends on the control u. The subsequent analysis is based on the decomposition of the uncontrollable subsystem Λ_{02} (Fig. 4) under the assumptions that an argument of the nonlinear term is formed on the basis of the state vectors of the subsystems Λ_{01} and Λ_{021} and that the subsystem Λ_{022} remains uncontrollable. The possibility of a such decomposition is established by introducing the fictitious input vector $u_f = \varphi(Ax, u)$ with the matrix C and constructing the extended controllability matrix W_Σ based on the matrix $G_\Sigma = (\ G\quad C\ )$:

$$W_\Sigma = (\ G_\Sigma\quad FG_\Sigma\quad \ldots\quad F^{n-1}G_\Sigma\ ). \qquad (26)$$

Then, if

$$\text{rank}(W_\Sigma) < n, \qquad (27)$$

the conclusion is made that the desired decomposition is possible and the system Σ is uncontrollable, since the condition (23) holds for $W = W_\Sigma$ in this case.

Example 9. Consider a system described by the equations

$$x_1(t + 1) = x_2(t) - x_1(t) + u_1(t),$$
$$x_2(t + 1) = x_3(t) - x_2(t),$$
$$x_3(t + 1) = x_4(t) - x_3(t) + x_1(t) + u_2(t),$$
$$x_4(t + 1) = x_5(t) - x_1(t) - x_4(t)$$
$$+ \ln|x_1(t) + x_2(t)|,$$
$$x_5(t + 1) = x_2(t) - x_5(t) + u_1(t).$$

A matrix description of the system is as follows:

$$F = \begin{pmatrix} -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 1 & 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & 0 & -1 \end{pmatrix},$$

$$G = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix},$$

$$\varphi(x, u) = \ln|Ax|, \quad A = (\ 1\quad 1\quad 0\quad 0\quad 0\ ).$$

One can see that

$$\text{rank}(W)$$

$$= \text{rank} \begin{pmatrix} 1 & 0 & -1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & -2 \\ 0 & 1 & 1 & -1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 1 & 1 \end{pmatrix} = 3 < 5,$$

i.e., a linear part of the system is uncontrollable. Since $\text{rank}(\ W\quad C\ ) = 4$ and $AW \neq 0$, the conditions (23) and (25) are violated. To check the condition (27), find the extended output matrix G_Σ and rank of the controllability matrix W_Σ:

$$G_\Sigma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix},$$

$$\text{rank}(W_\Sigma)$$

$$= \text{rank} \begin{pmatrix} 1 & 0 & 0 & -1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & -2 & 1 \\ 0 & 1 & 0 & 1 & -1 & 1 & -2 & 1 & -2 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 & 0 & 0 & 1 & 1 & 0 \end{pmatrix}$$
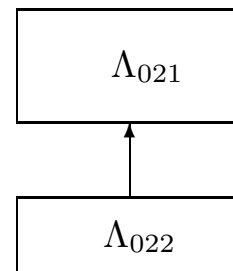
$$= 4 < 5.$$

Fig. 4. Decomposition of an uncontrollable subsystem.

Therefore, the system is unobservable according to (27).

Taking into account (8) with $B = 0$, we obtain

$$F = \begin{pmatrix} -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 1 & 0 & 0 & -1 \end{pmatrix},$$

$$\begin{aligned} \mathrm{rank}(W) \\ = \mathrm{rank} \begin{pmatrix} 1 & 0 & -1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & -2 \\ 0 & 1 & 1 & -1 & -1 & 1 \\ 0 & 0 & 1 & 0 & -2 & 1 \\ 1 & 0 & -1 & 0 & 1 & 1 \end{pmatrix} \\ = 4 < 5. \end{aligned}$$

One can easily to check that the condition (23) holds, i.e., the system is uncontrollable. ♦

All of the above results can be summarized in the form of a theorem.

Theorem 2. *The system Σ is uncontrollable if at least one of the two following situations takes place:*

  (i) both relations (20) and (23) hold,

  (ii) both relations (20) and (25) hold,

(iii) relation (27) holds.

As in the case of observability analysis, Theorem 2 is true for continuous-time systems.

6. Generalizations

Consider a number of generalizations assuming that conditions (15) and (20) hold.

Generalization 1. The nonlinear function φ contains several arguments of the form Ax:

$$\varphi(A_1x, A_2x, \ldots, A_rx, u) \tag{28}$$

where A_i is a row matrix, $i = 1, 2, \ldots, r$; for example, $\varphi(x, u) = \sin(x_1)\ln|x_2u|$. In this case, it is required to form the matrix

$$A_\Sigma = \begin{pmatrix} A_1 \\ A_2 \\ \vdots \\ A_r \end{pmatrix}$$

and to use it as described in Sections 4 and 5.

Generalization 2. The nonlinear function φ has the general form used in (6) and (9). In this case, the analysis is more sophisticated. Consider the property of observability first based on the following algorithm.

Algorithm 1.

Step 1. If the output function of the original system is nonlinear and the condition (3) holds, transform the coordinates to obtain the output function in the form $y = Hx$. Otherwise go to Step 2.

Step 2. Transform the system into the form (6); if some rows or columns of the matrix F are zero, make a correction to obtain the form (7) in the general case.

Step 3. Construct the matrix T' containing all linearly independent rows of the matrix V.

Step 4. Calculate the product

$$T'C \begin{pmatrix} \varphi_1(A_1x, u) \\ \vdots \\ \varphi_p(A_px, u) \end{pmatrix}$$

and combine similar terms. For example, the sum $A_ixu_k + A_jxu_k$ takes a form $A_{ij}xu_k$, where $A_{ij} = A_i + A_j$. Collect all row matrices of the form $A_{j_1}, A_{j_2}, \ldots, A_{j_d}$ contained in this product and construct the matrix

$$A_\Sigma = \begin{pmatrix} A_{j_1} \\ A_{j_2} \\ \vdots \\ A_{j_d} \end{pmatrix}. \tag{29}$$

Step 5. Check the condition (16); if it holds for $A = A_\Sigma$, then the system Σ is unobservable.

Step 6. If the condition (16) does not hold for some row matrices $A_{i_1}, A_{i_2}, \ldots, A_{i_c}$ from A_Σ, then the condition (17) is checked for columns of the matrix C with numbers $i_1, i_2, \ldots, i_c$. If it holds for all columns, a conclusion is made that the system Σ is unobservable.

Step 7. Assume that the condition (16) does not hold for some row matrices $A_{i_1}, A_{i_2}, \ldots, A_{i_c}$ from A_Σ and the condition (17) does not hold for columns of the matrix C with numbers $k_1, k_2, \ldots, k_b$, $k_i \in \{i_1, i_2, \ldots, i_c\}$. In this case, construct the matrix

$$A' = \begin{pmatrix} A_{k_1} \\ A_{k_2} \\ \vdots \\ A_{k_b} \end{pmatrix}.$$

Step 8. The extended observability matrix V_Σ is constructed on the basis of the matrix

$$H_\Sigma = \begin{pmatrix} H \\ A' \end{pmatrix}.$$

If $\mathrm{rank}(V_\Sigma) < n$, then the system Σ is unobservable.

The analysis of controllability is based on the following algorithm.

Algorithm 2.

Step 1. Transform the system into the form (6); if some rows or columns of the matrix F are zero, make a correction to obtain the form (7) in the general case.

Step 2. Construct the matrix Φ', containing all linearly independent rows of the matrix W^T.

Step 3. Calculate the product

$$\psi(x,u) = \Phi'C \begin{pmatrix} \varphi_1(A_1 x, u) \\ \vdots \\ \varphi_p(A_p x, u) \end{pmatrix},$$

combine similar terms and construct the matrix A_Σ as in (29) (we use the same notations for simplicity). Consider three cases.

Step 4. Check the condition (25); if it holds for $A = A_\Sigma$ and the function $\psi(x,u)$ does not contain the control u, then the system Σ is uncontrollable.

Step 5. If the condition (25) does not hold for some row matrices A_{i_1}, A_{i_2},..., A_{i_c} from A_Σ or the functions φ_{i_1}, φ_{i_2}, ...,φ_{i_c} contain the control u, then the condition (23) has to be checked for columns of the matrix C with numbers $i_1, i_2, \ldots, i_c$; if it holds for all columns, a conclusion is made that the system Σ is unobservable.

Step 6. Assume that the condition (25) does not hold for some row matrices A_{i_1}, A_{i_2},..., A_{i_c} from A_Σ or the functions φ_{i_1}, φ_{i_2}, ...,φ_{i_c} the contain the control u and condition (23) does not hold for columns of the matrix C with numbers $k_1, k_2, \ldots, k_b$, $k_i \in \{i_1, i_2, \ldots, i_c\}$. In this case, construct the matrix

$$C' = (\ C^{(k_1)} \quad C^{(k_2)} \quad \ldots \quad C^{(k_b)}\ ),$$

where $C^{(k_i)}$ is the k_i-th column of the matrix C.

Step 7. The extended controllability matrix W_Σ is constructed on the basis of the matrix $G_\Sigma = (\ G \quad C'\ )$. If $\mathrm{rank}(W_\Sigma) < n$, then the system Σ is uncontrollable.

Generalization 3. Some matrices in the general form of the nonlinear function φ have the form (28). In this case, all operations from the Generalization 2 take place. It is only required to replace the term "row matrix" by the term "matrix".

7. System issues

Suppose that the system Σ is unobservable according to the criteria (16) and the matrix T' is constructed on the basis of the observability matrix V. Then, the equality $A = A_* T'$ holds for some matrix A_*, and it follows from Wohnam (1985) that the equalities in (12) are valid. Define the matrix C_* as $C_* = CT'$ and represent the equalities in (12) and the last two equations in the form of the commutative diagrams

(30)

(31)
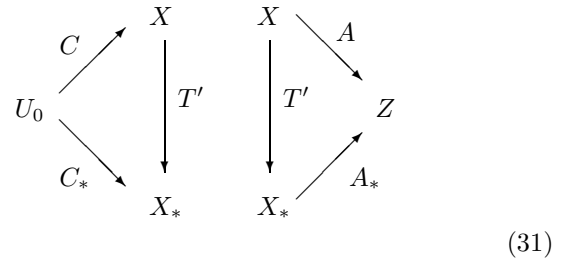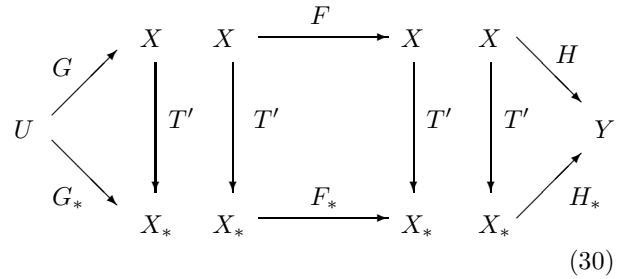
Here X_* is the state space of the system Σ_*, U_0 is the range of the nonlinear function $\varphi(Ax, u)$, and Z is the set of the argument Ax values.

The system Σ_* may be referred to as an image of the original system. Its dimension is less than that of Σ. The matrices describing this system (in particular, F_*, H_*, and A_*) are solutions to (12) and (14), respectively. Let us find out how to obtain these solutions. From the definition of the matrix T' it follows that rows of the matrices TF and H are linear combinations of those of the matrix T'. therefore, the first equation in the above system of three equations is always solvable in matrices F_* and H_*. The second equation has solutions if the condition (16) holds. If the condition (16) is violated, Eqn. (14) is not solvable. However, this does not matter, since the equality (17) holds. According to it, $C_* = 0$, and the system Σ_* is linear, which means that the condition (14) is not important and the diagrams (31) can be omitted.

If the system is unobservable according to (19), the matrix T' is constructed on the basis of the matrix V_Σ and the equality $A = A_* T'$ holds in this case.

The matrices considered can be defined as follows. Let T'^{-R} be the right inverse matrix of T', i.e., $T'T'^{-R} = I_{s \times s}$. Then from (12)–(14) we obtain

$$F_* = T'FT'^{-R}, \quad G_* = T'G, \quad H_* = HT'^{-R},$$
$$C_* = T'C, \qquad A_* = AT'^{-R}.$$

Similarly to the image of an unobservable system and according to Wohnam (1985), one can construct an image of an uncontrollable system. However, according to the duality principle for nonlinear systems (Zhirabok, 1998a;

1998b), it is more interesting to construct an inverse image of such a system, which can be done as follows. It follows from the duality principle for linear systems that for the unobservable dual system (uncontrollable original system) and the matrix Φ' constructed on the basis of the matrix W^T, the condition (12) and

$$F_0^T \Phi' = \Phi' F^T, \quad H_0^T = \Phi' H^T, \quad G^T = G_0 \Phi'$$

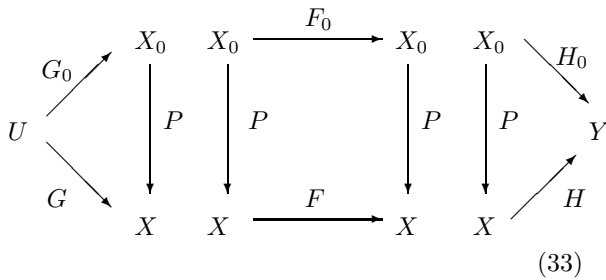hold for some matrices F_0, H_0, and G_0.

Write

$$P = \Phi'^T.$$

Notice that the matrix P is composed the maximum number of linearly independent columns of the controllability matrix W. With regard to this, rewrite the above equations as

$$PF_0 = FP, \quad H_0 = HP, \quad G = PG_0. \quad (32)$$

The obtained equations describe the homomorphism of linear systems and can be represented by the commutative diagrams

(33)

If the condition (23) holds, then the matrix Φ' satisfies the equation $C = \Phi'^T C_0 = PC_0$. Since the product $A\Phi'^T = AP$ in (20) is not generally equal to zero, denote it by $A_0 = AP$. Represent the last two equations by the commutative diagrams

(34)
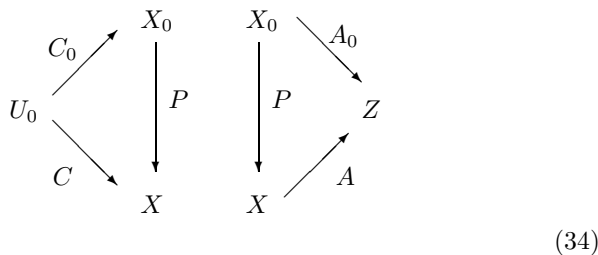
Suppose that the condition (25) holds, then $C\varphi(Ax) = C\varphi(0)$ for states accessible from the initial zero state and, if $\varphi(0) = 0$, the diagrams (34) can be omitted. If the system is uncontrollable according to (27), the

matrix P is constructed on the basis of the matrix W_Σ and the equality $C = PC_0$ holds.

Comparing the diagrams (30) and (33), as well as (31) and (34), one can conclude that they are dual. It is not difficult to see that the relations (16) and (23), as well as (17) and (25), are also dual. These dual relations supplement the well known duality principle for linear systems; they may also be viewed as particular cases of general duality principles for nonlinear systems (Herman and Krener, 1977; Zhirabok, 1998a; 1998b).

Matrices describing the inverse image $\Sigma_0 = (F_0, G_0, H_0, C_0, A_0)$ of the uncontrollable system are solutions to (12) and $C = PC_0$. Also $A_0 = AP$. The possibility of their solution can be analyzed in the same way as in the case of observability. They can be found as follows. Let P^{-L} be the left inverse matrix of P, i.e., $P^{-L}P = I_{r \times r}$. Then, based on (32), equations $C = PC_0$ and $A_0 = AP$, we obtain

$$F_0 = P^{-L}FP, \qquad G_0 = P^{-L}G, \qquad H_0 = HP,$$
$$C_0 = P^{-L}C, \qquad A_0 = AP.$$

Let us show that a linear part of the system Σ_* is observable. It is easy to see that from (12) the equalities $H = H_*T'$ and $HF = H_*T'F = H_*F_*T'$ follow. It can be shown that the latter relation is valid for arbitrary powers of the matrices F and F_*: $HF^i = H_*F_*^iT'$, $i = 2, 3, \dots$ Then, taking into account the form of the observability matrix for the system Σ, we have

$$V = \begin{pmatrix} H \\ HF \\ \vdots \\ HF^{n-1} \end{pmatrix} = \begin{pmatrix} H_*T' \\ H_*F_*T' \\ \vdots \\ H_*F_*^{n-1}T' \end{pmatrix} = V_*T'$$

where V_* is the observability matrix of the system Σ_*. From the matrix theory, it is known that

$$\text{rank}(V) = \text{rank}(V_*T')$$
$$\leq \min(\text{rank}(V_*), \text{rank}(T')). \quad (35)$$

Let the system Σ be unobservable and $\text{rank}(V) = \text{rank}(T') = s < n$. Then it follows from the above rank inequality that $\text{rank}(V_*) = s$. Since the system Σ_* is s-dimensional, its linear part is observable. As noted earlier, this generally does not imply that the entire system Σ_* is observable.

To obtain the dual result, i.e., the controllability of a linear part of the system Σ_0, the relations (32) are used, which yields $G = PG_0$ and $FG = FPG_0 = PF_0G_0$. The latter relation is true also for arbitrary powers of the matrices F and F_0: $F^iG = PF_0^iG_0$, $i = 2, 3, \dots$ Then, taking into account the form of the controllability matrix, we have

$$W = \begin{pmatrix} G & FG & \dots & F^{n-1}G \end{pmatrix}$$
$$= \begin{pmatrix} PG_0 & PF_0G_0 & \dots & PF_0^{n-1}G_0 \end{pmatrix} = PW_0,$$

where W_0 is the controllability matrix of the system Σ_0.

Let the system Σ be uncontrollable and $\mathrm{rank}(W) = \mathrm{rank}(P) = r < n$. Then, it follows from the rank inequality (like (35)) that $\mathrm{rank}(W_0) = r$. Since the system Σ_0 is r-dimensional, its linear part is controllable.

8. Illustrative example

Consider the system given by the equations

$$x_1(t+1) = x_2(t) - x_1(t) + u_1(t),$$
$$x_2(t+1) = 2x_3(t) + 2x_4(t) + |x_1(t)|u_2(t),$$
$$x_3(t+1) = x_2(t) - x_1(t) - x_3(t) - x_5(t)$$
$$\qquad + x_6(t) - \ln|x_5(t)| - u_2(t),$$
$$x_4(t+1) = x_5(t) - x_4(t) + \ln|x_5(t)| + u_2(t),$$
$$x_5(t+1) = x_1(t) - x_5(t) + \cos(x_4(t)),$$
$$x_6(t+1) = x_2(t) - x_6(t) + u_1(t),$$
$$y_1(t) = x_1(t) + x_3(t),$$
$$y_2(t) = x_4(t) - x_2(t).$$

A linear part of this system and nonlinearities are described by the matrices

$$F = \begin{pmatrix} -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 2 & 0 & 0 \\ -1 & 1 & -1 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 \end{pmatrix},$$

$$G = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & -1 \\ 0 & 1 \\ 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$

$$H = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{pmatrix},$$

$$\varphi(x,u) = \begin{pmatrix} \ln|A_1 x| \\ |A_2 x| \\ \cos(A_3 x) \end{pmatrix},$$

$$A_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix},$$
$$A_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$
$$A_3 = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

Analysis shows that we may consider only the first three blocks of the observability matrix:

$$V = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & -1 & -1 & 0 & 1 \\ 1 & -1 & 2 & 2 & 0 & 0 \\ 2 & -1 & 3 & 3 & 0 & -2 \end{pmatrix}.$$

Since $\mathrm{rank}(V) = 4$, a linear part of the system is not observable. Verification of the condition (16) shows that it is fulfilled for the matrix A_2 and is not fulfilled for A_1 and A_3. However, since $VC^{(1)} = VC^{(3)} = 0$, the original nonlinear system is also unobservable.

The controllability matrix is found to be

$$W = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & -1 & -1 & 3 & 1 \\ 0 & 1 & 0 & -1 & 1 & 1 & -3 & -1 \\ 0 & 0 & 1 & 0 & -2 & 0 & 3 & 0 \\ 1 & 0 & -1 & 0 & 1 & 0 & -1 & 0 \end{pmatrix}.$$

As $\mathrm{rank}(W) = 3$, a linear part is unobservable. One can easily check that the condition (23) holds for $C^{(1)}$ and $C^{(3)}$ except for $C^{(2)}$ under $A_2 W \neq 0$. Therefore, we have to find the extended matrix W_Σ based on the matrix $H_\Sigma = (\; H \quad C^{(2)} \;)$:

$$W_\Sigma = \begin{pmatrix} 1 & 0 & 0 & -1 & 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & -1 & 0 & 0 & 1 & 1 & -1 & -1 & -1 \\ 0 & 1 & 0 & 0 & -1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -2 & 0 & 1 \\ 1 & 0 & 0 & -1 & 0 & 1 & 1 & 0 & -1 \end{pmatrix}.$$

Because $\mathrm{rank}(W_\Sigma) = 5 < 6$, the initial system is uncontrollable.

Transforming the initial system according to (8), we obtain

$$F = \begin{pmatrix} -1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 2 & 2 & 0 & 0 \\ -1 & 1 & -1 & 0 & -2 & 1 \\ 0 & 0 & 0 & -1 & 2 & 0 \\ 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 \end{pmatrix},$$

$$G = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & -1 \\ 0 & 1 \\ 0 & 0 \\ 1 & 0 \end{pmatrix},$$

$$V = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & -1 & -1 & 0 & 1 \\ 2 & -1 & 2 & 2 & 0 & 0 \\ 3 & -1 & 3 & 3 & 0 & -2 \end{pmatrix},$$

$$W = \begin{pmatrix} 1 & 0 & -1 & 1 & 2 & -1 & -3 & 4 \\ 0 & 1 & 1 & 0 & -1 & 3 & 10 & -3 \\ 0 & -1 & 0 & 2 & -1 & -4 & 9 & -12 \\ 0 & 1 & 0 & -1 & 2 & 3 & -6 & -5 \\ 0 & 0 & 1 & 1 & -2 & -1 & 6 & 3 \\ 1 & 0 & -1 & 1 & 2 & -1 & -3 & 4 \end{pmatrix}.$$

Because $\mathrm{rank}(V) = 4$ and $\mathrm{rank}(W) = 5$, the system is unobservable and uncontrollable.

9. Conclusions

The approach to analyzing observability and controllability of nonlinear dynamic systems proposed in this work is based on known observability and controllability criteria for linear systems and additional conditions that are also linear. The use of linear methods allows obtaining only sufficient conditions for observability and controllability. However, in a number of cases, this is justified by the simplicity of the proposed criteria, which impose almost no constraints on the character of the nonlinearities, and the universality of their use in the sense that they can be applied to both discrete and continuous systems without any modifications. Moreover, the proposed criteria demonstrate new facets of unobservability and uncontrollability duality features of nonlinear dynamic systems.

Examples 6 and 9 and an illustrative example show that, if the system unobservable (uncontrollable) according to condition (19) (the condition (25)) is preliminarily transformed according to the relation (7) (relation (8)), then it is unobservable (or uncontrollable) according to the condition (16) or (17) (the condition (21) or (25)). One can suppose that these examples reflect some general property.

To compare the suggested approach with existing mathematical packages using the symbolic software, consider the following possibilities:

(i) The original system is smooth and the output function is linear.

(ii) The original system is smooth and the output function is nonlinear.

(iii) The original system is non-smooth.

In the cases (i) and (iii), an advantage of the suggested approach in analysis of observability is evident because of using linear methods with the possible coordinate transformation in the case (iii). In the case (ii), packages using symbolic software like Maple or Mathematica are preferable. Clearly, if controllability is analyzed, coordinate transformation is not required and the suggested approach is preferable.

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