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THE CONTROLLABILITY OF NONLINEAR IMPLICIT FRACTIONAL DELAY DYNAMICAL SYSTEMS

RAJAGOPAL JOICE NIRMALA [a,*], KRISHNAN BALACHANDRAN [a]

[a] Department of Mathematics
Bharathiar University, Coimbatore 641 046, India
e-mail: joys.maths.bu@gmail.com

This paper is concerned with the controllability of nonlinear fractional delay dynamical systems with implicit fractional derivatives for multiple delays and distributed delays in control variables. Sufficient conditions are obtained by using the Darbo fixed point theorem. Further, examples are given to illustrate the theory.

Keywords: controllability, fractional delay differential equation, Mittag-Leffler function, fixed point theorem.

1. Introduction

Controllability is one of the fundamental concepts in mathematical control theory. The fixed point technique is the most powerful method for the controllability problem for nonlinear dynamical systems. Several fixed point theorems are explicitly used to derive the controllability conditions depending on the nonlinear functions involved. The notion of a fractional derivative dates back two centuries and several authors have published books on this subject (see Kilbas *et al*., 2006; Kaczorek, 2011). Fractional derivatives and integrals in control theory lead to better results than integer order ones.

Controllability and observability for fractional control systems were studied by Shamardan and Moubarak (1999). Adams and Hartley (2008) obtained the finite time controllability of fractional order systems. Bettayeb and Djennoune (2008) established new results on the controllability and observability of fractional dynamical systems. The controllability of nonlinear fractional dynamical systems was studied by Balachandran *et al.* (2012b) using the fixed point argument. Fractional order differential equations with delay in the state variable have recently proved to be a valuable tool in the modeling of many phenomena in various fields. The delay differential equation was extensively studied by Bellman and Cooke (1963) or Hale (1977). Wiess (1967) examined the controllability of delayed differential systems. The controllability of nonlinear delay systems with a fixed point technique was discussed by Dauer and Gahl (1977).

Based on the measure of noncompactness of a set, Dacka (1980) introduced a new method of analysis to study the controllability of nonlinear systems with implicit derivatives through the Darbo fixed point theorem. Balachandran (1988; 1989) extended the technique to a larger class of nonlinear dynamical systems with implicit derivatives. Klamka (1976a; 1976b; 2000) published several papers regarding the controllability of nonlinear systems with various types of delays. The relative controllability of perturbed nonlinear systems with delay in control and time varying delays were discussed by Dacka (1982). The results on the controllability of nonlinear systems with distributed delays in the control variable and delay depending on the state variable were studied by Balachandran and Somasundaram (1983; 1986). Yi *et al*. (2008) obtained the controllability and observability of a linear delay system by constructing a Lambert W function. The controllability of differential equations with delayed and advanced arguments was given by Manzanilla *et al*. (2010) and the numerical method was established by Wang (2013).

Many authors generalize results from the integer order case to the fractional order case to obtain better results. Controllability criteria for linear fractional differential systems with a state delay and impulse were studied by Zhang *et al*. (2013). Several results

*Corresponding author

on the controllability of nonlinear fractional dynamical systems with multiple delays and distributed delays in control were derived by Balachandran and Kokila (2012; 2014) as well as Balachandran *et al.* (2012a; 2012c). Recently, Balachandran and Divya (2014) studied the controllability of nonlinear implicit fractional integrodifferential systems. Analysis and numerical methods for a fractional differential equation with delay were studied by Morgado *et al*. (2013). The application of this equation was discussed by Bhalekar and Gejji (2010) or Bhalekar *et al*. (2011). But there is no work reported on the control problem of nonlinear fractional delay systems governed by fractional delay differential equations with implicit fractional derivatives.

In this paper, we make an attempt to study the controllability problem for fractional delay systems with implicit fractional derivative through the notion of a measure of noncompactness of a set and the Darbo fixed point theorem. Further controllabilty results for multiple delays and distributed delays in control variables are also discussed using the Darbo fixed point theorem. Examples are provided to illustrate the theoretical results.

2. Preliminaries

In this section, we give some basic definitions required for this paper (Kilbas *et al*., 2006). The *Caputo fractional derivative* of order $\alpha > 0$, for $n-1 < \alpha < n$, is defined as

$$^{C}D^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} f^{(n)}(s)\,\mathrm{d}s,$$

where the function $f(t)$ has absolutely continuous derivatives up to order $n-1$.

The Laplace transform of the Caputo derivative for $n-1 < \alpha < n$ is

$$L[D^{\alpha}x(t)](s) = s^{\alpha} L[x(t)](s) - \sum_{k=0}^{n-1} x^k(0) s^{\alpha-1-k}.$$

The *Mittag-Leffler functions* of various types are defined by

$$E_{\alpha}(\lambda t^{\alpha}) = \sum_{k=0}^{\infty} \frac{\lambda^k t^{\alpha k}}{\Gamma(\alpha k + 1)}, \quad t \in \mathbb{C}, \tag{1}$$

$$E_{\alpha,\beta}(\lambda t^{\alpha}) = \sum_{k=0}^{\infty} \frac{\lambda^k t^{\alpha k}}{\Gamma(\alpha k + \beta)}, \quad t, \beta \in \mathbb{C}, \tag{2}$$

$$E^{(\gamma)}_{\alpha,\beta}(\lambda t^{\alpha}) = \sum_{k=0}^{\infty} \frac{(\gamma)_k (\lambda)^k}{k!\Gamma(\alpha k + \beta)} t^{\alpha k}, \tag{3}$$

where $(\gamma)_n$ is a Pochhamer symbol which is defined as $\gamma(\gamma+1)\cdots(\gamma+n-1)$ and

$$(\gamma)_n = \frac{\Gamma(\gamma+n)}{\Gamma(\gamma)}.$$

The relation between (1) and (3) is given as

$$\left(\frac{\partial}{\partial \lambda}\right)^{\nu} E_{\alpha}(\lambda t^{\alpha}) = \nu! t^{\alpha\nu} E^{\nu+1}_{\alpha,\alpha\nu+1}(\lambda t^{\alpha}). \tag{4}$$

The Laplace transforms of the Mittag-Leffler functions (1)–(3) are defined respectively as

$$L[E_{\alpha}(\pm\lambda t^{\alpha})](s) = \frac{s^{\alpha-1}}{(s^{\alpha} \pm \lambda)}, \tag{5}$$

$$L[t^{\beta-1}E_{\alpha,\beta}(\pm\lambda t^{\alpha})](s) = \frac{s^{\alpha-\beta}}{(s^{\alpha} \pm \lambda)}, \quad \mathrm{Re}(\alpha) > 0, \tag{6}$$

$$L[t^{\beta-1}E^{(\gamma)}_{\alpha,\beta}(\pm\lambda t^{\alpha})](s) = \frac{s^{\alpha\gamma-\beta}}{(s^{\alpha} \pm \lambda)^{\gamma}}, \quad |\lambda s^{-\alpha}| < 1. \tag{7}$$

If $F(s) = L[f(t)](s)$ for $\mathrm{Re}(s) > 0$, then

$$F(s-a) = L[e^{at} f(t)](s),$$

and

$$L[u_a(t) f(t-a)](s) = e^{-as} F(s), \quad a \geq 0,$$

and we also have

$$L^{-1}[e^{-as} F(s)](t) = u_a(t) f(t-a), \tag{8}$$

where the delayed unit step function is defined as

$$u_a(t) = \begin{cases} 1, & t \geq a, \\ 0, & t < a. \end{cases}$$

Consider the fractional delay differential equation of the form

$$\begin{aligned} ^{C}D^{\alpha}x(t) &= Ax(t) + Bx(t-h) + f(t), \\ x(t) &= \phi(t), \quad -h < t \leq 0, \end{aligned} \tag{9}$$

where $0 < \alpha < 1$, $x \in \mathbb{R}^n$, A and B are $n \times n$ matrices, $\phi(t)$ is a continuous function on $[-h, 0]$ and f is a real valued continuous function on $\mathbb{R}^n$. The solution of (9) as in the work of Joice Nirmala *et al*. (2016) is

$$\begin{aligned} x(t) &= X_{\alpha}(t)\phi(0) + B \int_{-h}^{0} (t-s-h)^{\alpha-1} \\ &\quad \times X_{\alpha,\alpha}(t-s-h)\phi(s)\,\mathrm{d}s \\ &\quad + \int_0^t (t-s)^{\alpha-1} X_{\alpha,\alpha}(t-s) f(s)\,\mathrm{d}s, \end{aligned} \tag{10}$$

where

$$X_\alpha(t) = L^{-1}\left[\frac{s^{\alpha-1}}{s^\alpha - A - Be^{-s}}\right](t)$$

and

$$X_{\alpha,\alpha}(t) = t^{1-\alpha}\int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)} X_\alpha(s)\,\mathrm{d}s.$$

Next we summarize the basic facts about the measure of noncompactness and the related fixed point theorem; for more details, we refer the reader to Dacka (1980).

Definition 1. Let $(X, ||\cdot||)$ be a Banach space and E be a bounded subset of X. Then the noncompactness of the set E is defined as

$$\mu(E) = \inf\{\ r > 0\text{: } E \text{ can be covered by a finite number of balls whose radii are smaller than } r\ \}.$$

Theorem 1. (Darbo fixed point theorem) *If S is a nonempty bounded closed convex subset of X and $T : S \to S$ is a continuous mapping such that, for any $E \subset S$, we have*

$$\mu(TE) \le k\mu(E),$$

where k is a constant, $0 \le k < 1$, then T has a fixed point.

2.1. Implicit fractional delay dynamical systems. Consider the fractional delay dynamical equation with an implicit fractional derivative of the form

$$\begin{aligned} {}^C D^\alpha x(t) &= Ax(t) + Bx(t-h) + Cu(t) \\ &\quad + f(t, x(t), x(t-h), {}^C D^\alpha x(t), u(t)), \end{aligned} \tag{11}$$

with $x(t) = \phi(t)$, on $[-h, 0]$, where $t \in J = [0, t_1]$, $x \in \mathbb{R}^n$, A, B are $n\times n$ matrices and C is an $n\times m$ matrix, $u(t) \in \mathbb{R}^n$ is the control function, ϕ is a continuous function on $[-h, 0]$ and the nonlinear function $f : J \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is continuous. We will need the following terminology:

(i) Define the space of continuous functions $C_n(J)$ with norm $||x|| = \max\{|x_i(t)|, i = 1, 2, \dots, n, t \in J\}$. Then the measure of noncompactness of a set S is given by

$$\mu(S) = \frac{1}{2}\omega_0(S) = \frac{1}{2}\lim_{\delta\to 0}\omega(S, \delta), \tag{12}$$

where $\omega(S, \delta) = \sup_{x\in S}[\sup|x(t) - x(s)| : |t - s| \le \delta]$ is the common modulus of continuity of the functions which belong to the set S.

(ii) Introduce the space $C_n^\alpha(J) = \{x : {}^C D^\alpha x \in C_n \text{ and } x \in C_n\}$ with the norm

$$||x||_{C_n^\alpha} = ||x||_{C_n} + ||{}^C D^\alpha x||_{C_n}.$$

Then the measure of noncompactness of a set S is given by

$$\mu(S) = \frac{1}{2}\omega_0({}^C D^\alpha S), \tag{13}$$

where

$${}^C D^\alpha S = \{{}^C D^\alpha x : x \in S\}.$$

(iii) For the space being the Cartesian product $C_{n+m}^\alpha(J) = C_n^\alpha(J) \times C_m(J)$ with the norm

$$||(x, u)||_{C_{n+m}^\alpha} = \max\{||x||_{C_n^\alpha}, ||u||_{C_m}\},$$

the measure of noncompactness of any bounded set S in $C_{n+m}^\alpha(J)$ is given by

$$\mu(S) = [\mu(S_1), \mu(S_2)], \tag{14}$$

where S_1 and S_2 are natural projections of the set S onto the spaces $C_n^\alpha(J)$ and $C_m(J)$, respectively.

(iv) Assume that there exist positive real constants K_1 and k with $0 \le k < 1$ such that

$$|f(t, x, y, z, u)| \le K_1, \tag{15}$$

$$|f(t, x, y, z, u) - f(t, x, y, \bar{z}, u)| \le k(|z - \bar{z}|) \tag{16}$$

for all $x, y, z \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$.

The solution of the system (11) with the initial function $x(t) = \phi(t)$ is given by

$$\begin{aligned} x(t) &= x_L(t;\phi) \\ &\quad + \int_0^t (t-s)^{\alpha-1} X_{\alpha,\alpha}(t-s) Cu(s)\,\mathrm{d}s \\ &\quad + \int_0^t (t-s)^{\alpha-1} X_{\alpha,\alpha}(t-s) \\ &\quad \times f(s, x(s), x(s-h), {}^C D^\alpha x(s), u(s))\,\mathrm{d}s, \end{aligned} \tag{17}$$

where

$$\begin{aligned} x_L(t;\phi) &= X_\alpha(t)\phi(0) + \int_{-h}^0 (t-s-h)^{\alpha-1} \\ &\quad \times X_{\alpha,\alpha}(t-s-h)\phi(s)\,\mathrm{d}s. \end{aligned} \tag{18}$$

The controllability of the fractional delay dynamical system is defined as follows.

Definition 2. The system (11) is said to be *controllable* on $[0, t_1]$, if for every $\phi \in C_n[-h, 0]$ and every $x_1 \in \mathbb{R}^n$ there exists a control function u defined on $[0, t_1]$ such that the solution of (11) satisfies $x(t_1) = x_1$.

Define the *controllability Gramian* matrix W by

$$W = \int_0^{t_1} [X_{\alpha,\alpha}(t_1 - s)C][X_{\alpha,\alpha}(t_1 - s)C]^*\,\mathrm{d}s, \tag{19}$$

where * denotes the matrix transpose.

Theorem 2. *If the controllability Gramian matrix W is positive definite and the nonlinear function f satisfies the conditions (15) and (16), then the nonlinear system (11) is controllable on J.*

Proof. Let the initial function ϕ be continuous on $[-h, 0]$ and let $x_1 \in \mathbb{R}^n$. Define the following nonlinear mapping on the space $C^\alpha_{n+m}(J)$:

$$(z, v) = \Phi([x, u])(t) = (\Phi_1([x, u])(t), \Phi_2([x, u])(t)),$$

where the pair of operators Φ_1 and Φ_2 is defined by

$$\begin{aligned}\Phi_2([x,u])(t) &\qquad (20)\\ = (t-s)^{1-\alpha} C^* X^*_{\alpha,\alpha}(t-s) W^{-1} \Big[x_1 - x_L(t_1;\phi) \\ - \int_0^{t_1} (t_1-s)^{\alpha-1} X_{\alpha,\alpha}(t_1-s) \\ \times f(s, x(s), x(s-h), {}^C D^\alpha x(s), u(s))\,\mathrm{d}s \Big]\end{aligned}$$

and

$$\begin{aligned}\Phi_1([x,u])(t) \\ = x_L(t;\phi) &\qquad (21)\\ + \int_0^t (t-s)^{\alpha-1} X_{\alpha,\alpha}(t-s) C v(s)\,\mathrm{d}s \\ + \int_0^t (t-s)^{\alpha-1} X_{\alpha,\alpha}(t-s) \\ \times f(s, x(s), x(s-h), {}^C D^\alpha x(s), v(s))\,\mathrm{d}s.\end{aligned}$$

Since all the functions involved in the definition of the operator Φ are continuous, this mapping is continuous and maps the space $C^\alpha_{n+m}(J)$ into itself. Consider the closed convex subset S of $C^\alpha_{n+m}(J)$ defined by

$$S = \{[x,u] : ||u|| \le L_1, ||x|| \le L_2, \; ||{}^C D^\alpha x|| \le L_3\}, \qquad (22)$$

where the positive real constants L_1, L_2 and L_3 are defined by

$$\begin{aligned}L_1 &= c_1[|x_1| + c_2],\\ L_2 &= b_1 + b_2||C||L_1 t_1^\alpha \alpha^{-1} + a_2 K_1 t_1^\alpha \alpha^{-1},\\ L_3 &= (||A|| + ||B||)L_2 + ||C||L_1 + K_1,\end{aligned}$$

with

$$\begin{aligned}b_1 &= \sup ||x_L(t;\phi)||, & b_2 &= \sup ||X_{\alpha,\alpha}(t-s)||,\\ c_1 &= ||C^*|| b_2 ||W^{-1}||, & c_2 &= b_1 + b_2 t_1^\alpha \alpha^{-1} K_1.\end{aligned}$$

The set S is bounded, closed and convex in $C^\alpha_{n+m}(J)$ and the operator Φ transforms S into S. It is easily seen that, for each pair $[x, u] \in S$, we have

$$\omega(\Phi_2([x,u],\delta)) \le \omega(\kappa_1, \delta) e_1, \qquad (23)$$

where $\kappa_1(t_1, s) = (t_1 - s)^{\alpha-1} C^* X_{\alpha,\alpha}(t_1 - s)$ and

$$\begin{aligned}e_1 = \sup_{[x,u]\in S} ||W^{-1}|| \Big[|x_1| + ||x_L(t_1;\phi)|| \\ + \int_0^{t_1} (t_1-s)^{\alpha-1} ||X_{\alpha,\alpha}(t_1-s)|| \\ \times ||f(s, x(s), x(s-h), {}^C D^\alpha x(s), u(s))||\,\mathrm{d}s \Big].\end{aligned}$$

Since the function κ_1 does not depend on the choice of the points in S, all the functions $\Phi_2([x, u])(t)$ have a uniform bounded modulus of continuity. Therefore, they are equicontinuous. All the functions $\Phi_1([x, u])(t)$ are also equicontinuous, since they have uniformly bounded derivatives.

Let us consider the moduli of continuity of the functions ${}^C D^\alpha \Phi_1([x, u])(t)$. We have

$$\begin{aligned}&|{}^C D^\alpha \Phi_1([x,u])(t) - {}^C D^\alpha \Phi_1([x,u])(s)|\\ &\le |A\Phi_1([x,u])(t) - A\Phi_1([x,u])(s)|\\ &\quad + |B\Phi_1([x,u])(t-h) - B\Phi_1([x,u])(t-h)|\\ &\quad + |C\Phi_2([x,u])(t) - C\Phi_2([x,u])(s)|\\ &\quad + \big| f(t, x(t), x(t-h), {}^C D^\alpha x(t), u(t))\\ &\quad - f(s, x(s), x(s-h), {}^C D^\alpha x(s), u(s)) \big|\\ &\le |A\Phi_1([x,u])(t) - A\Phi_1([x,u])(s)|\\ &\quad + |B\Phi_1([x,u])(t-h) - B\Phi_1([x,u])(s-h)|\\ &\quad + |C\Phi_2([x,u])(t) - C\Phi_2([x,u])(s)|\\ &\quad + \big| f(t, x(t), x(t-h), {}^C D^\alpha x(t), u(t))\\ &\quad - f(t, x(t), x(t-h), {}^C D^\alpha x(s), u(t)) \big|\\ &\quad + \big| f(t, x(t), x(t-h), {}^C D^\alpha x(s), u(t))\\ &\quad - f(s, x(s), x(s-h), {}^C D^\alpha x(s), u(s)) \big| \qquad (24)\\ &\le \beta_0(|t-s|) + \beta_1(|t-s|)\\ &\quad + k|{}^C D^\alpha (x(t) - x(s))|.\end{aligned}$$

For the first three terms on the right hand side of (24) we may give an upper bound $\beta_0(|t - s|)$, where β_0 is a nonnegative function such that $\lim_{\delta\to 0+} \beta_0(\delta) = 0$. In the same manner, we find that the last term on the right hand side of (24) can be bounded from above by $k|{}^C D^\alpha x(t) - {}^C D^\alpha x(s)| + \beta(|t - s|)$. Setting $\beta = \beta_0 + \beta_1$, we finally obtain

$$\begin{aligned}&|{}^C D^\alpha \Phi_1([x,u])(t) - {}^C D^\alpha \Phi_1([x,u])(s)|\\ &\qquad \le k|{}^C D^\alpha x(t) - {}^C D^\alpha x(s)| + \beta(|t-s|).\end{aligned}$$

Therefore,

$$\omega({}^C D^\alpha \Phi_1([x,u]), \delta) \le k\omega({}^C D^\alpha x, \delta) + \beta(\delta). \qquad (25)$$

Hence, by using (12)–(14), we have, for any set $S \subset C^\alpha_{n+m}(J)$,

$$\omega_0(\Phi_2 S) \text{ and } \omega_0(\Phi_1 S) \le k\omega_0({}^C D^\alpha S_1),$$

where S_1 is the normal projection of the set S on the space $C_n^\alpha(J)$. Hence it follows that

$$\mu(\Phi S) \le k\mu(S).$$

By the Darbo fixed point theorem, the mapping Φ has at least one fixed point. Therefore there exist functions $z \in C_n^\alpha(J)$ and $v \in C_m(J)$ such that

$$(x, u) = (z, v) = [\Phi_1([x, u])(t), \Phi_2([x, u])(t)].$$

This shows that $x(t)$ is the solution of (11) for the control

$$\begin{aligned} u(t) &\qquad (26)\\ &= (t-s)^{1-\alpha} C^* X^*_{\alpha,\alpha}(t-s) \times W^{-1}\Big[x_1 - x_L(t_1, \phi)\\ &\quad - \int_0^{t_1} (t_1 - s)^\alpha X_{\alpha,\alpha}(t_1 - s)\\ &\quad \times f(s, x(s), x(s-h), {}^C D^\alpha x(s), u(s))\,\mathrm{d}s\Big]. \end{aligned}$$

By using (26) in the solution

$$\begin{aligned} x(t) = x_L(t;\phi) &\qquad (27)\\ &+ \int_0^t (t-s)^{\alpha-1} X_{\alpha,\alpha}(t-s) C u(s)\,\mathrm{d}s\\ &+ \int_0^t (t-s)^{\alpha-1} X_{\alpha,\alpha}(t-s)\\ &\times f(s, x(s), x(s-h), {}^C D^\alpha x(s), u(s))\,\mathrm{d}s, \end{aligned}$$

at $t = t_1$, we have

$$x(t_1) = x_1.$$

Hence the system (11) is controllable on J. ■

3. System with multiple delays in control

Consider the implicit fractional delay dynamic system with time varying multiple delays in control of the form

$$\begin{aligned} {}^C D^\alpha x(t) &= Ax(t) + Bx(t-h) + \sum_{i=0}^{M} C_i u(\sigma_i(t))\\ &\quad + f(t, x(t), x(t-h), {}^C D^\alpha x(t), u(t)),\\ &\qquad t \in J,\\ x(t) &= \phi(t), \quad -h < t \le 0, \end{aligned} \tag{28}$$

where $0 < \alpha < 1$, $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ and A, B are $n \times n$ matrices, C_i for $i = 0, 1, \dots M$ are $n \times m$ matrices, $\phi(t)$ is a continuous function on $[-h, 0]$ and $f : J \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is a continuous function.

(H1) The functions $\sigma_i : J \to \mathbb{R}, i = 0, 1, \dots, M,$ are twice continuously differentiable and strictly increasing in J. Moreover $\sigma_i(t) \le t, i = 0, 1, \dots, M,$ for $t \in J$.

(H2) Introduce the time lead functions $r_i(t) : [\sigma_i(0), \sigma_i(t_1)] \to [0, t_1],\ i = 0, 1, \dots, M,$ such that $r_i(\sigma_i(t)) = t$ for $t \in J$. Further, $\sigma_0(t) = t$ and for $t = t_1$ the following inequality holds:

$$\begin{aligned} \sigma_M(t_1) \le \sigma_{M_1}(t_1) \le \cdots \sigma_{m+1}(t_1) \le 0 = \sigma_m(t_1)\\ < \sigma_{m-1}(t_1) = \cdots = \sigma_1(t_1) = \sigma_0(t_1) = t_1. \end{aligned} \tag{29}$$

(H3) Given $\sigma > 0$, for functions $u : [-\sigma, t_1] \to \mathbb{R}^m$ and $t \in t_1$, we use the symbol u_t to denote the function on $[-\sigma, 0]$ defined by $u_t(s) = u(t+s)$ for $s \in [-\sigma, 0)$.

The set $y(t) = \{x(t), u_t\}$ is the complete state of the system at time t. The solution of the system (28) is given by

$$\begin{aligned} x(t) &= x_L(t;\phi)\\ &\quad + \int_0^t (t-s)^{\alpha-1} X_{\alpha,\alpha}(t-s) \sum_{i=0}^{M} C_i u_i(\sigma_i(s))\,\mathrm{d}s\\ &\quad + \int_0^t (t-s)^{\alpha-1} X_{\alpha,\alpha}(t-s)\\ &\quad \times f(s, x(s), x(s-h), {}^C D^\alpha x(s), u(s))\,\mathrm{d}s, \end{aligned} \tag{30}$$

where $x_L(t;\phi)$ is defined as in (18). Using the time lead function $r_i(t)$, the solution can be written as

$$\begin{aligned} x(t) &= x_L(t;\phi) + \sum_{i=0}^{M} \int_{\sigma_i(0)}^{\sigma_i(t)} (t - r_i(s))^{\alpha-1}\\ &\quad \times X_{\alpha,\alpha}(t - r_i(s)) C_i \dot{r}_i(s) u(s)\,\mathrm{d}s \qquad (31)\\ &\quad + \int_0^t (t-s)^{\alpha-1} X_{\alpha,\alpha}(t-s)\\ &\quad \times f(s, x(s), x(s-h), {}^C D^\alpha x(s), u(s))\,\mathrm{d}s. \end{aligned}$$

By using the inequality (29), we get

$$\begin{aligned} x(t) &= x_L(t;\phi) + \sum_{i=0}^{m} \int_{\sigma_i(0)}^{0} (t - r_i(s))^{\alpha-1}\\ &\quad \times X_{\alpha,\alpha}(t - r_i(s)) C_i \dot{r}_i(s) u_0(s)\,\mathrm{d}s\\ &\quad + \sum_{i=0}^{m} \int_0^t (t - r_i(s))^{\alpha-1}\\ &\quad \times X_{\alpha,\alpha}(t - r_i(s)) C_i \dot{r}_i(s) u(s)\,\mathrm{d}s\\ &\quad + \sum_{i=m+1}^{M} \int_{\sigma_i(0)}^{\sigma_i(t)} (t - r_i(s))^{\alpha-1}\\ &\quad \times X_{\alpha,\alpha}(t - r_i(s)) C_i \dot{r}_i(s) u_0(s)\,\mathrm{d}s\\ &\quad + \int_0^t (t-s)^{\alpha-1} X_{\alpha,\alpha}(t-s) \end{aligned}$$

$$\times f(s, x(s), x(s-h), {}^C D^\alpha x(s), u(s))\, \mathrm{d}s.$$

For simplicity, let us write the solution as

$$\begin{aligned} x(t) = x_L(t;\phi) + H(t) + \sum_{i=0}^{m} \int_0^t (t - r_i(s))^{\alpha-1} \\ \times X_{\alpha,\alpha}(t - r_i(s)) C_i \dot{r}_i(s) u(s) \mathrm{d}s \\ + \int_0^t (t-s)^{\alpha-1} X_{\alpha,\alpha}(t-s) \\ \times f(s, x(s), x(s-h), {}^C D^\alpha x(s), u(s))\, \mathrm{d}s, \end{aligned} \tag{32}$$

where

$$\begin{aligned} H(t) = \sum_{i=0}^{m} \int_{\sigma_i(0)}^{0} (t - r_i(s))^{\alpha-1} \\ X_{\alpha,\alpha}(t - r_i(s)) C_i \dot{r}_i(s) u_0(s)\, \mathrm{d}s \\ + \sum_{i=m+1}^{M} \int_{\sigma_i(0)}^{\sigma_i(t)} (t - r_i(s))^{\alpha-1} \\ X_{\alpha,\alpha}(t - r_i(s)) C_i \dot{r}_i(s) u_0(s)\, \mathrm{d}s. \end{aligned}$$

Now let us define the controllability Gramian matrix by

$$\begin{aligned} W = \sum_{i=0}^{m} \int_0^{t_1} (X_{\alpha,\alpha}(t_1 - r_i(s)) C_i \dot{r}_i(s)) \\ \times (X_{\alpha,\alpha}(t_1 - r_i(s)) C_i \dot{r}_i(s))^* \mathrm{d}s, \end{aligned}$$

where the complete state $y(0)$ and the vector $x_1 \in \mathbb{R}^n$ are chosen arbitrarily and $*$ denotes the matrix transpose.

Theorem 3. *Assume that the hypotheses (H1)–(H3) hold. Further assume that the nonlinear function satisfies the conditions (15) and (16) and suppose that* $\det W > 0$. *Then the nonlinear system (28) is relatively controllable on* J.

Proof. Let the initial function ϕ be continuous on $[-h, 0]$ and let $x_1 \in \mathbb{R}^n$. Define the following nonlinear mapping on the space $C^\alpha_{n+m}(J)$:

$$\Psi(x, u) = (z, v) = (\Psi_1([x,u])(t), \Psi_2([x,u])(t)),$$

where the pair of operators Ψ_1 and Ψ_2 is defined by

$$\begin{aligned} &\Psi_2([x,u])(t) \\ &= (t_1 - r_i(t))^{1-\alpha} (X_{\alpha,\alpha}(t_1 - r_i(t)) C_i^* \dot{r}_i(t))^* \\ &\times W^{-1} \Big[x_1 - x_L(t_1;\phi) - \sum_{i=0}^{m} \int_{\sigma_i(0)}^{0} (t_1 - r_i(s))^{\alpha-1} \\ &\times X_{\alpha,\alpha}(t_1 - r_i(s)) C_i \dot{r}_i(s) u_0(s)\, \mathrm{d}s \\ &- \sum_{i=m+1}^{M} \int_0^{t_1} (t_1 - r_i(s))^{\alpha-1} \\ &\times X_{\alpha,\alpha}(t_1 - r_i(s)) C_i \dot{r}_i(s) u_0(s)\, \mathrm{d}s \\ &- \int_0^{t_1} (t_1 - s)^{\alpha-1} X_{\alpha,\alpha}(t_1 - s) \\ &\times f(s, x(s), x(s-h), {}^C D^\alpha x(s), u(s))\, \mathrm{d}s \Big] \end{aligned}$$

and

$$\begin{aligned} &\Psi_1([x,u])(t) \\ &= x_L(t;\phi) + \sum_{i=0}^{m} \int_{\sigma_i(0)}^{0} (t - r_i(s))^{\alpha-1} \\ &\times X_{\alpha,\alpha}(t - r_i(s)) C_i \dot{r}_i(s) u_0(s)\, \mathrm{d}s \\ &+ \sum_{i=0}^{m} \int_0^t (t - r_i(s))^{\alpha-1} \\ &\times X_{\alpha,\alpha}(t - r_i(s)) C_i \dot{r}_i(s) v(s)\, \mathrm{d}s \\ &+ \sum_{i=m+1}^{M} \int_{\sigma_i(0)}^{\sigma_i(t)} (t - r_i(s))^{\alpha-1} \\ &\times X_{\alpha,\alpha}(t - r_i(s)) C_i \dot{r}_i(s) u_0(s)\, \mathrm{d}s \\ &+ \int_0^t (t-s)^{\alpha-1} X_{\alpha,\alpha}(t-s) \\ &\times f(s, x(s), x(s-h), {}^C D^\alpha x(s), v(s))\, \mathrm{d}s. \end{aligned}$$

Since all the functions involved in the definition of the operator Ψ are continuous, this mapping is continuous and maps the space $C^\alpha_{n+m}(J)$ into itself. Consider the closed convex subset S of $C^\alpha_{n+m}(J)$ defined by

$$S = \{[x,u] : ||u|| \le P_1, ||x|| \le P_2, ||{}^C D^\alpha x|| \le P_3\}, \tag{33}$$

where the positive real constants L_1, L_2 and L_3 are defined by

$$\begin{aligned} P_1 &= d_1[|x_1| + d_2], \\ P_2 &= \beta + \mu + l_i a_i ||C_i|| b_i L_1 \\ &\quad + t_1^\alpha \alpha^{-1} K_1 \nu, \quad i = 0, 1, \dots, M, \\ P_3 &= (||A|| + ||B||) L_2 + ||C_i|| L_1 + K_1, \\ &\qquad i = 0, 1, \dots, M, \end{aligned}$$

where

$$\begin{aligned} a_i &= \sup ||X_{\alpha,\alpha}(t_1 - r_i(s))||, \quad b_i = \sup ||r_i(s)||, \\ \nu &= \sup ||X_{\alpha,\alpha}(t_1 - s)||, \quad \rho = \sup ||u_0(s)||, \\ \mu &= \sum_{i=0}^{m} a_i b_i \rho ||C_i|| N_i + \sum_{i=m+1}^{M} a_i b_i \rho_3 M_i, \\ N_i &= \int_{\sigma_i(0)}^{0} (t_1 - r_i(s))^{\alpha-1}\, \mathrm{d}s, \\ M_i &= \int_{\sigma_i(0)}^{\sigma_i(t_1)} (t_1 - r_i(s))^{\alpha-1}\, \mathrm{d}s, \end{aligned}$$

$$\begin{aligned}
l_i &= \int_0^{t_1} (t_1 - r_i(s))^{\alpha-1}\,\mathrm{d}s,\\
\beta &= \sup ||x_L(t;\phi)||,\\
d_1 &= a_i b_i ||C_i|| ||W^{-1}||,\\
d_2 &= \beta + \mu + \nu K_1 t_1^{\alpha} \alpha^{-1}.
\end{aligned}$$

Hence the set S is bounded, closed and compact in $C^{\alpha}_{n+m}(J)$ and the operator Ψ transforms S into S. It is easily seen that, for each pair $(x, u) \in S$, we have

$$\omega(\Psi_2([x,u]),\delta) \le \omega(\kappa_2,\delta)e_2,$$

where

$$\kappa_2 = \max\{(t_1 - r_i(t))^{1-\alpha}(X_{\alpha,\alpha}(t_1 - r_i(t))C_i^{*}\dot{r}_i(t))^{*},\\ i = 0,1,\dots,M\}$$

and

$$\begin{aligned}
e_2 = \sup_{(x,u)\in S} ||W||^{-1}\Bigg[&|x_1| + ||x_L(t_1;\phi)||\\
&+ \sum_{i=0}^{m} \int_{\sigma_i(0)}^{0} (t_1 - r_i(s))^{\alpha-1}\\
&\times ||X_{\alpha,\alpha}(t_1 - r_i(s))||\, ||C_i||\, ||\dot{r}_i(s)||\, u_0(s)\,\mathrm{d}s\\
&- \sum_{i=m+1}^{M} \int_{0}^{t_1} (t_1 - r_i(s))^{\alpha-1}\\
&\times ||X_{\alpha,\alpha}(t_1 - r_i(s))||\, ||C_i||\, ||\dot{r}_i(s)||\, u_0(s)\,\mathrm{d}s\\
&- \int_0^{t_1} (t_1 - s)^{\alpha-1} ||X_{\alpha,\alpha}(t_1 - s)||\\
&\times ||f(s, x(s), x(s-h), {}^{C}D^{\alpha}x(s), u(s))||\,\mathrm{d}s\Bigg].
\end{aligned}$$

Since the function κ_2 does not depend on the choice of the points in S, all the functions $\Psi_2([x,u])(t)$ have a uniform bounded modulus of continuity. Hence they are equicontinuous. All the functions $\Psi_1([x,u])(t)$ are also equicontinuous, since they have uniformly bounded derivatives. Let us consider the moduli of continuity of the functions ${}^{C}D^{\alpha}\Psi_1([x,u])(t)$. We have

$$\begin{aligned}
|{}^{C}D^{\alpha}\Psi_1([x,u])(t) - {}^{C}D^{\alpha}\Psi_1([x,u])(s)|&\\
\le k|{}^{C}D^{\alpha}x(t) - {}^{C}D^{\alpha}x(s)| + \beta(t-s).&
\end{aligned}$$

$$\omega({}^{C}D^{\alpha}\Psi_1([x,u]),\delta) \le k\omega({}^{C}D^{\alpha}x,\delta) + \beta(\delta), \tag{34}$$

where β is a nonnegative function such that $\lim_{\delta\to 0+}\beta(\delta) = 0$. Hence, from (12)–(14), we come to the conclusion that, for any set S in $C^{\alpha}_{n+m}(J)$,

$$\omega_0(\Psi_2 S) = 0, \quad \omega_0(\Psi_1 S) \le k\omega_0({}^{C}D^{\alpha}S_1),$$

where S_1 is the normal projection of the set S on the space $C^{\alpha}_{n}(J)$. Hence it follows that

$$\mu(\Psi S) \le k\mu(S).$$

By the Darbo fixed point theorem, the mapping Ψ has at least one fixed point. Therefore there exist functions $z \in C^{\alpha}_{n}(J)$ and $v \in C_m(J)$ such that

$$(x,u) = (z,v) = [\Psi_1([x,u])(t), \Psi_2([x,u])(t)].$$

This shows that $x(t)$ is the solution of (28) for the control $u(t)$. It is easy to check that the control $u(t)$ steers the system from x_0 to x_1 at time t_1 on the interval J. Hence the system (28) is controllable on J. ∎

4. System with distributed delays in control

Consider the fractional delay dynamical system with distributed delays in control of the form

$$\begin{aligned}
{}^{C}D^{\alpha}x(t) &= Ax(t) + Bx(t-h)\\
&\quad + \int_{-h}^{0} \mathrm{d}_s H(t,s)u(t+s)\\
&\quad + f(t, x(t), x(t-h), {}^{C}D^{\alpha}x(t), u(t)),\\
x(t) &= \phi(t), \quad -h < t \le 0,
\end{aligned} \tag{35}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, A and B are $n \times n$ matrices, $H(t,s)$ is an $n \times m$ matrix, continuous in t for fixed s and of bounded variation in s on $[-h, 0]$ for each $t \in [0, t_1]$. The nonlinear function $f : J \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is continuous. The symbol d_s denotes the integrals in the Lebesgue–Stieltjes sense. The solution of the above system (35) is given by

$$\begin{aligned}
x_L(t) = x_L(t;\phi) &+ \int_0^t (t-s)^{\alpha-1} X_{\alpha,\alpha}(t-s)\\
&\times \int_{-h}^{0} \mathrm{d}_\tau H(s,\tau)u(s+\tau)\,\mathrm{d}s\\
&+ \int_0^t (t-s)^{\alpha-1} X_{\alpha,\alpha}(t-s)\\
&\times f(s, x(s), x(s-h), {}^{C}D^{\alpha}x(s), u(s))\,\mathrm{d}s,
\end{aligned}$$

where $x_L(t;\phi)$ is defined as in (18). Now, using the well known result of the unsymmetric Fubini theorem and a change of the order of integration in the last term, we have

$$\begin{aligned}
x_L(t) = x_L(t;\phi) &+ \int_{-h}^{0} \mathrm{d}H_\tau \int_{\tau}^{0} (t-(s-\tau))^{\alpha-1}\\
&\times X_{\alpha,\alpha}(t-(s-\tau))H(s-\tau,\tau)u_0(s)\,\mathrm{d}s\\
&+ \int_0^t \int_{-h}^{0} (t-(s-\tau))^{\alpha-1}\\
&\times X_{\alpha,\alpha}(t-(s-\tau))\,\mathrm{d}_\tau H_t(s-\tau,\tau)u(s)\,\mathrm{d}s
\end{aligned}$$

$$+\int_0^t (t-s)^{\alpha-1}X_{\alpha,\alpha}(t-s) \times f(s,x(s),x(s-h),{}^C D^\alpha x(s),u(s))\,\mathrm{d}s, \tag{36}$$

where

$$H_t(s,\tau)=\begin{cases}H(s,\tau), & s\le t,\\ 0, & s>t,\end{cases}$$

and $\mathrm{d}H_\tau$ denotes the integration in the Lebesgue–Stietljes sense with respect to the variable τ in the function $H(t,\tau)$. For our convenience, we take

$$q(t,u_0)=\int_{-h}^0 \mathrm{d}H_\tau \int_\tau^0 (t-(s-\tau))^{\alpha-1} \times X_{\alpha,\alpha}(t-(s-\tau))H(s-\tau,\tau)u_0(s)\,\mathrm{d}s$$

and

$$S(t,s)=\int_{-h}^0 (t-(s-\tau))^{\alpha-1} \times X_{\alpha,\alpha}(t-(s-\tau))\mathrm{d}_\tau H_t(s-\tau,\tau).$$

Hence the solution is of the form

$$x_L(t)=x_L(t;\phi)+q(t;u_0)+\int_0^t S(t,s)u(s)\,\mathrm{d}s + \int_0^t (t-s)^{\alpha-1}X_{\alpha,\alpha}(t-s) \times f(s,x(s),x(s-h),{}^C D^\alpha x(s),u(s))\,\mathrm{d}s, \tag{37}$$

for $t\in J$ and $x(t)=\phi(t)$ for $t\in[-h,0]$. The Gramian matrix is defined by

$$W=\int_0^{t_1} S(t_1,s)S^*(t_1,s)\,\mathrm{d}s. \tag{38}$$

Theorem 4. *Assume that the nonlinear function f satisfies the conditions (15) and (16) and suppose that* $\det W>0$. *Then the nonlinear system (35) is relatively controllable on J.*

Proof. Let the initial function ϕ be continuous on $[-h,0]$ and let $x_1\in\mathbb{R}^n$. Define the following nonlinear mapping on the space $C^\alpha_{n+m}(J)$:

$$\Gamma(x,u)=(z,v)=(\Gamma_1([x,u])(t),\Gamma_2([x,u])(t)),$$

where

$$\begin{aligned}&\Gamma_2(x,u)(t)\\ &=S^*(T,t)W^{-1}\big[x_1-x_L(t_1;\phi)-q(t_1;u_0)\\ &\quad-\int_0^{t_1}(t_1-s)^{\alpha-1}X_{\alpha,\alpha}(t_1-s)\\ &\quad\times f(s,x(s),x(s-h),{}^C D^\alpha x(s),u(s))\,\mathrm{d}s\big]\end{aligned}$$

and

$$\begin{aligned}&\Gamma_1(x,u)(t)\\ &=x_L(t;\phi)+q(t;u_0)+\int_0^t S(t,s)v(s)\,\mathrm{d}s\\ &\quad-\int_0^t (t-s)^{\alpha-1}X_{\alpha,\alpha}(t-s)\\ &\quad\times f(s,x(s),x(s-h),{}^C D^\alpha x(s),v(s))\,\mathrm{d}s.\end{aligned}$$

Since all the functions involved in the definition of the operator Γ are continuous, this mapping is continuous and maps the space $C^\alpha_{n+m}(J)$ into itself. Consider the closed convex subset S of $C^\alpha_{n+m}(J)$ defined by

$$S=\Big\{[x,u]:||u||\le N_1,||x||\le N_2,||{}^C D^\alpha x||\le N_3\Big\}, \tag{39}$$

where the positive real constants N_1,N_2 and N_3 are defined by

$$\begin{aligned}N_1&=k_1||W^{-1}||[|x_1|+a_1+k_2+a_2K_1t_1^\alpha\alpha^{-1}],\\ N_2&=a_1+k_2+k_1N_1t_1+a_2K_1t_1^\alpha\alpha^{-1},\\ N_3&=||A||N_2+||B||N_2+b_1b_2N_1+K_1,\end{aligned}$$

with

$$\begin{aligned}a_1&=\sup||x_L(t_1;\phi)||,\\ a_2&=\sup||X_{\alpha,\alpha}(t_1-s)||,\\ b_1&=||H(t,s)||,\\ b_2&=\max_{0\le t\le t_1}\mathrm{var}_{s\in[-h,0]}||H(t,s)||,\\ k_1&=\max_{0\le\tau\le t_1}||S(t,\tau)||,\\ k_2&=\sup||q(t;u_0)||.\end{aligned}$$

Hence the set S is closed, bounded and convex in $C^\alpha_{n+m}(J)$ and the operator Γ maps S into S. It is easily seen that, for each pair $[x,u]\in S$, we have

$$\omega(\Gamma([x,u])(t),\delta)\le\omega(\kappa_3,\delta)e_3,$$

where

$$\begin{aligned}\kappa_3&=S^*(t_1,s),\\ e_3&=\sup_{[x,u]\in S}||W^{-1}||\big[|x_1|+||x_L(t_1;\phi)||\\ &\quad+||q(t_1;u_0)||-\int_0^{t_1}(t_1-s)^{\alpha-1}X_{\alpha,\alpha}(t_1-s)\\ &\quad\times f(s,x(s),x(s-h),{}^C D^\alpha x(s),u(s))\,\mathrm{d}s\big].\end{aligned}$$

Since the function κ_3 does not depend on the choice of the points in S, all the functions $\Gamma_2([x,u])(t)$ have a uniform bounded modulus of continuity. Hence they are equicontinuous. All the functions $\Gamma_1([x,u])(t)$ are also equicontinuous, since they have uniformly bounded derivatives. Let us consider the moduli of continuity of the functions ${}^C D^\alpha \Gamma_1([x,u])(t)$. We have

$$\begin{aligned}|{}^C D^\alpha \Gamma_1([x,u])(t) - {}^C D^\alpha \Gamma_1([x,u])(s)| \\ \le k|{}^C D^\alpha x(t) - {}^C D^\alpha x(s)| + \beta(t-s),\end{aligned}$$

$$\omega({}^C D^\alpha \Gamma_1([x,u]), \delta) \le k\omega({}^C D^\alpha x, \delta) + \beta(\delta), \tag{40}$$

where β is a nonnegative function such that $\lim_{\delta\to 0+}\beta(\delta) = 0$. Hence, from (12)–(14), we come to the conclusion that, for any set S in $C^\alpha_{n+m}(J)$,

$$\omega_0(\Gamma_2 S)\omega_0(\Gamma_1 S) \le k\omega_0({}^C D^\alpha S_1),$$

where S_1 is the normal projection of the set S on the space $C^\alpha_n(J)$. Hence it follows that

$$\mu(\Gamma S) \le k\mu(S).$$

By the Darbo fixed point theorem, the mapping Γ has at least one fixed point. Therefore there exist functions $z \in C^\alpha_n(J)$ and $v \in C_m(J)$ such that

$$(x,u) = (z,v) = [\Gamma_1([x,u])(t), \Gamma_2([x,u])(t)].$$

This shows that $x(t)$ is the solution of (35) for the control $u(t)$. It is easy to check that the control $u(t)$ steers the system from x_0 to x_1 at time t_1 on the interval J. Hence the system (35) is controllable on J. ■

Remark 1. Control functions defined in Theorems 2–4 are not constrained ones. We can restrict the control function $u(t)$ inside a closed and convex cone with a nonempty interior and a vertex at zero. The constrained controllability criteria for both linear and nonlinear fractional delay dynamic systems are obtained by constructing the reachable set by following the work of Klamka (2001).

Example 1. Consider the nonlinear fractional delay dynamic system of the form

$$\begin{aligned}&{}^C D^{\frac12} x(t)\\ &= \begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix} x(t) \\ &\quad + \begin{pmatrix} 0 & 1 \\ 0 & 0\end{pmatrix} x(t-1) + \begin{pmatrix} 0 \\ 1\end{pmatrix} u(t) \\ &\quad + \begin{pmatrix} 0 \\ \frac{x_1(t)+x_2(t-1)}{1+x_1^2+x_2^2(t-1)} + \sin({}^C D^{\frac12} x_1(t))\cos({}^C D^{\frac12} x_1(t))\end{pmatrix},\end{aligned} \tag{41}$$

where $\alpha = 1/2$, $x(t) = \phi(t) \in \mathbb{R}^2$,

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix},$$

$$B = \begin{pmatrix} 0 & 1 \\ 0 & 0\end{pmatrix},$$

$$C = \begin{pmatrix} 0 \\ 1\end{pmatrix},$$

$h = 1$ and

$$\begin{aligned}&f(t, x(t), x(t-1), {}^C D^\alpha x(t), u(t)) \\ &= \begin{pmatrix} 0 \\ \frac{x_1(t)+x_2(t-1)}{1+x_1^2+x_2^2(t-1)} + \sin({}^C D^{\frac12} x_1(t))\cos({}^C D^{\frac12} x_1(t))\end{pmatrix}.\end{aligned}$$

The solution of the system (41), using the Laplace transform, is

$$\begin{aligned}x(t) &= \sum_{n=0}^{[t]} B^n (t-n)^{\frac12 n} \\ &\quad \times E^{(n+1)}_{\frac12, \frac12 n+1}(A(t-n)^{\frac12}) \\ &\quad + \sum_{n=0}^{[t]} B^{n+1} \int_{-1}^{0} (t-s-n-1)^{\frac12 n - \frac12} \\ &\quad \times E^{(n+1)}_{\frac12, \frac12 (n+1)}(A(t-s-n-1)^{\frac12})\phi(s)\,\mathrm{d}s \\ &\quad + \sum_{n=0}^{[t]} \int_0^{t-n} B^n (t-s-n)^{\frac12 n - \frac12} \\ &\quad \times E^{(n+1)}_{\frac12, \frac12 n + \frac12}(A(t-s-n)^{\frac12}) u(s)\,\mathrm{d}s \\ &\quad + \sum_{n=0}^{[t]} \int_0^{t-n} B^n (t-s-n)^{\frac12 n - \frac12} \\ &\quad \times E^{(n+1)}_{\frac12, \frac12 n + \frac12}(A(t-s-n)^{\frac12}) \\ &\quad \times f(s, x(s), x(s-1), {}^C D^{\frac12} x(s), u(s))\,\mathrm{d}s,\end{aligned}$$

where $[\,\cdot\,]$ is the floor function, i.e., the greatest integer that is less than or equal to its argument.

Now consider the controllability on $[0,1]$. Here $[t] = 0$. Therefore the solution of (41) on $[0,1]$ is

$$\begin{aligned}x(t) &= E_{\frac12}(At^{\frac12}) + B\int_{-1}^{0} (t-s-1)^{-\frac12} \\ &\quad \times E_{\frac12,\frac12}(A(t-s-1)^{\frac12})\phi(s)\,\mathrm{d}s \\ &\quad + \int_0^t (t-s)^{-\frac12} E_{\frac12,\frac12}(A(t-s)^{\frac12}) u(s)\,\mathrm{d}s \\ &\quad + \int_0^t (t-s)^{-\frac12} E_{\frac12,\frac12}(A(t-s)^{\frac12}) \\ &\quad \times f(s, x(s), x(s-1), {}^C D^\alpha x(s), u(s))\,\mathrm{d}s,\end{aligned}$$

where

$$E_{\frac12}(At^{\frac12}) = \begin{pmatrix} E_1(-t) & t^{\frac12} E_{1,\frac32}(-t) \\ -t^{\frac12} E_{1,\frac32}(-t) & E_1(-t)\end{pmatrix}, \tag{42}$$

and

$$E_{\frac{1}{2},\frac{1}{2}}(A(t-s)^{\frac{1}{2}}) = \begin{pmatrix} (t-s)^{-\frac{1}{2}}E_{1,\frac{1}{2}}(-t) & E_{1,1}(-t) \\ -E_{1,1}(-t) & (t-s)^{-\frac{1}{2}}E_{1,\frac{1}{2}}(-t) \end{pmatrix}. \tag{43}$$

The Gramian matrix is defined by

$$W = \int_0^1 [E_{\frac{1}{2},\frac{1}{2}}(A(1-s)^{\frac{1}{2}})C][E_{\frac{1}{2},\frac{1}{2}}(A(1-s)^{\frac{1}{2}})C]^* \,\mathrm{d}s.$$

After some algebra, we get

$$W = \begin{pmatrix} 0.1494 & 0.0636 \\ 0.0636 & 0.0661 \end{pmatrix},$$

so that $\det(W) = 0.0058 \geq 0$. Therefore the linear system is controllable. The nonlinear function satisfies

$$|f(t,x,y,z,u) - f(t,x,y,\overline{z},u)| \leq |\sin z \cos z - \sin \overline{z} \cos \overline{z}| \leq |z - \overline{z}|.$$

Since the function f is continuous and bounded, and satisfies the Lipschitz condition with respect to ${}^C D^{\frac{1}{2}} x(t)$ with the constant $k = 1$, the hypotheses of Theorem 2 are satisfied. Take the control function $u(t)$ as

$$\begin{aligned} u(t) &= (t-s)^{\frac{1}{2}}[E_{\frac{1}{2},\frac{1}{2}}(A(1-s)^{\frac{1}{2}})B]^* W^{-1} \\ &\quad \times \Big[E_{\frac{1}{2}}(AT^{\frac{1}{2}}) + B\int_{-1}^0 (T-s-1)^{-\frac{1}{2}} \\ &\quad \times E_{\frac{1}{2},\frac{1}{2}}(A(T-s-1)^{\frac{1}{2}})\phi(s)\,\mathrm{d}s \\ &\quad + \int_0^T (T-s)^{-\frac{1}{2}} E_{\frac{1}{2},\frac{1}{2}}(A(T-s)^{\frac{1}{2}}) \\ &\quad \times f(s, x(s), x(s-1), {}^C D^{\alpha} x(s), u(s))\,\mathrm{d}s\Big], \end{aligned}$$

which steers the system (41) from x_0 and x_1 and hence the system (41) is controllable on $[0,1]$. ♦

Example 2. Consider the fractional delay dynamical system with multiple delays in control in the form

$$\begin{aligned} {}^C D^{\frac{1}{2}} x(t) &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} x(t) \\ &\quad + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x(t-1) \\ &\quad + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u(t) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u(t-1) \\ &\quad + \begin{pmatrix} \frac{x_1(t)+x_2(t-1)}{x_1(t)^2+x_2(t-1)^2+u(t)} \\ \frac{1}{4}\sin({}^C D^{\frac{1}{2}} x_1(t) + {}^C D^{\frac{1}{2}} x_2(t)) \end{pmatrix}, \end{aligned} \tag{44}$$

where $x(t) = \phi(t) \in \mathbb{R}^2$, $h = 1$, $\sigma = 1$,

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

$$B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

$$C_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

$$C_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The solution of the system (44) by using the Laplace transform is of the form

$$\begin{aligned} x(t) &= \sum_{n=0}^{[t]} B^n (t-n)^{\frac{1}{2}n} E_{\frac{1}{2},\frac{1}{2}n+1}(A(t-n)^{\frac{1}{2}}) \\ &\quad + B\sum_{n=0}^{[t]} B^n \int_{-1}^0 (t-s-n-1)^{\frac{1}{2}n+\frac{1}{2}-1} \\ &\quad \times E_{\frac{1}{2},\frac{1}{2}(n+1)}(A(t-s-n-1)^{\frac{1}{2}})\phi(s)\,\mathrm{d}s \\ &\quad + \sum_{n=0}^{[t]}\sum_{i=0}^{1} B^n C_i \int_0^{t-n} (t-r_i(s)-n)^{\frac{1}{2}n-\frac{1}{2}} \\ &\quad \times E^{(n+1)}_{\frac{1}{2},\frac{1}{2}n+\frac{1}{2}}(A(t-r_i(s)-n))^{\frac{1}{2}}\dot{r}_i(s)u(s)\,\mathrm{d}s \\ &\quad + \sum_{n=0}^{[t]} \int_0^{t-n} B^n (t-s-n)^{\frac{1}{2}n-\frac{1}{2}} \\ &\quad \times E^{(n+1)}_{\frac{1}{2},\frac{1}{2}n+\frac{1}{2}}(A(t-s-n)^{\frac{1}{2}}) \\ &\quad \times f(s, x(s), x(s-1), {}^C D^{\frac{1}{2}} x(s), u(s))\,\mathrm{d}s. \end{aligned}$$

Now consider the controllability on $[0,1]$, where $[t] = 0$.

$$\begin{aligned} x(t) &= E_{\frac{1}{2}}(At^{\frac{1}{2}}) + B\int_{-1}^0 (t-s-1)^{-\frac{1}{2}} \\ &\quad \times E_{\frac{1}{2},\frac{1}{2}}(A(t-s-1)^{\frac{1}{2}})\phi(s)\,\mathrm{d}s \\ &\quad + \sum_{i=0}^{1} C_i \int_0^t (t-r_i(s))^{-\frac{1}{2}} \\ &\quad \times E_{\frac{1}{2},\frac{1}{2}}(A(t-r_i(s))^{\frac{1}{2}})\dot{r}_i(s)u(s)\,\mathrm{d}s \\ &\quad + \int_0^t (t-s)^{-\frac{1}{2}} E_{\frac{1}{2},\frac{1}{2}}(A(t-s)^{\alpha}) \\ &\quad \times f(s, x(s), x(s-1), {}^C D^{\frac{1}{2}} x(s), u(s))\,\mathrm{d}s, \end{aligned}$$

with $E_{\frac{1}{2}}(At^{\frac{1}{2}})$ and $E_{\frac{1}{2},\frac{1}{2}}(A(1-s)^{\frac{1}{2}})$ defined as in (42) and (43). The Gramian matrix is defined by

$$\begin{aligned} W &= \sum_{i=0}^{1} \int_0^1 [C_i E_{\frac{1}{2},\frac{1}{2}}(A(1-r_i(s))^{\frac{1}{2}})\dot{r}_i(s)] \\ &\quad \times [C_i E_{\frac{1}{2},\frac{1}{2}}(A(1-r_i(s))^{\frac{1}{2}})\dot{r}_i(s)]^* \,\mathrm{d}s, \end{aligned}$$

where $r_i(s)$ is a time lead function and it is defined as $r_0(s) = s$ and $r_1(s) = s - 1$. Then the Gramian matrix can be written as

$$W = \int_0^1 [C_0 E_{\frac{1}{2},\frac{1}{2}}(A(1-s))^{\frac{1}{2}})] \times [C_0 E_{\frac{1}{2},\frac{1}{2}}(A(1-s))^{\frac{1}{2}})]^* \,\mathrm{d}s + \int_0^1 [C_1 E_{\frac{1}{2},\frac{1}{2}}(A(1-(s-1)))^{\frac{1}{2}})] \times [C_1 E_{\frac{1}{2},\frac{1}{2}}(A(1-(s-1))^{\frac{1}{2}})]^* \,\mathrm{d}s.$$

Evaluating it, we get

$$W = \begin{pmatrix} 681.8463 & 229.7418 \\ 229.7418 & 78.6800 \end{pmatrix},$$

so that $\det(W) = 866.3722 \geq 0$. Therefore the linear system is controllable. Since the function f is continuous, bounded and satisfies the Lipschitz condition with respect to ${}^C D^{1/2}x(t)$ with the constant $k = \frac{1}{2}$, the hypotheses of Theorem 3 are satisfied. Hence the system (44) is controllable on [0, 1]. ♦

Example 3. Consider the nonlinear fractional delay dynamic system with distributed delays in control of the form

$$ {}^C D^{\frac{1}{2}} x(t) = Ax(t) + Bx(t-1) + \int_{-1}^0 \mathrm{d}_s H(t,s)u(t+s) + f(t, x(t), x(t-1), {}^C D^\alpha x(t), u(t)), \tag{45}$$

where $\alpha = 1/2, x(t) = \phi(t) \in \mathbb{R}^2$, $h = 1$, A, B are defined as in Example 1,

$$H(t,\tau) = \begin{pmatrix} e^{\tau+t} & e^\tau \sin t \\ -e^\tau \sin t & e^{\tau+t} \end{pmatrix}$$

and

$$f(t, x(t), x(t-1), {}^C D^{\frac{1}{2}} x(t), u(t)) = \begin{pmatrix} \dfrac{x_1(t)}{1 + x_1^2(t-1) + u(t)} \\ \dfrac{x_2}{1+x_2^2} + |{}^C D^{\frac{1}{2}} x_2(t) - \cos {}^C D^{\frac{1}{2}} x_2(t)| \end{pmatrix}.$$

We consider the controllability on $[0, 1]$. The solution of the system (45) can be written in the form

$$x(t) = E_{\frac{1}{2},1}(At^{\frac{1}{2}}) + B\int_{-1}^0 (t-s-1)^{-\frac{1}{2}} \times E_{\frac{1}{2},\frac{1}{2}}(A(t-s-1)^{\frac{1}{2}})\phi(s)\,\mathrm{d}s + \int_{-1}^0 \mathrm{d}H_\tau \int_\tau^0 (t-(s-\tau))^{-\frac{1}{2}} \times E_{\frac{1}{2},\frac{1}{2}}[A(t-(s-\tau))^{\frac{1}{2}}]H(s-\tau,\tau)u_0(s)\,\mathrm{d}s + \int_0^t \int_{-1}^0 (t-(s-\tau))^{-\frac{1}{2}} \times E_{\frac{1}{2},\frac{1}{2}}[A(t-(s-\tau))^{\frac{1}{2}}]\mathrm{d}_\tau H_t(s-\tau)u(s)\,\mathrm{d}s + \int_0^t (t-s)^{\frac{1}{2}} E_{\frac{1}{2},\frac{1}{2}}[A(t-s)^{\frac{1}{2}}] \times f(s, x(s), x(s-1), {}^C D^{\frac{1}{2}} x(s), u(s))\,\mathrm{d}s.$$

The Mittag-Leffler functions of the matrices $E_{\frac{1}{2}}(At^{\frac{1}{2}})$ and $E_{\frac{1}{2},\frac{1}{2}}(A(1-(s-\tau))^{\frac{1}{2}})$ are defined as in (42) and (43), where

$$(1-(s-\tau))^{-\frac{1}{2}} E_{\frac{1}{2},\frac{1}{2}}(A(1-(s-\tau))^{\frac{1}{2}}) = \begin{pmatrix} a(\tau) & b(\tau) \\ -b(\tau) & a(\tau) \end{pmatrix},$$

with $a(\tau) = (1-(s-\tau))^{-1}E_{1,\frac{1}{2}}(1-(s-\tau))$ and $b(\tau) = (1-(s-\tau))^{-\frac{1}{2}}E_{1,1}(1-(s-\tau))$. Then

$$S(1,s) = \int_{-1}^0 (1-(s-\tau))^{-\frac{1}{2}} E_{\frac{1}{2},\frac{1}{2}}(A(1-(s-\tau))^{\frac{1}{2}}) \,\mathrm{d}_\tau H_1(s-\tau,\tau).$$

Hence

$$S(1,s) = \begin{pmatrix} L(s) & K(s) \\ -K(s) & L(s) \end{pmatrix},$$

where

$$L(s) = \int_{-1}^0 b(\tau)e^\tau \cos(s-\tau) - b(\tau)e^\tau \sin(s-\tau)\,\mathrm{d}\tau,$$

$$K(s) = \int_{-1}^0 a(\tau)e^\tau \sin(s-\tau) - a(\tau)e^\tau \cos(s-\tau)\,\mathrm{d}\tau.$$

Now, evaluating the Gramian matrix, we have

$$W = \int_0^1 S(1,s)S^*(1,s)\,\mathrm{d}s, = \int_0^1 L^2(s) + K^2(s) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mathrm{d}s,$$

which means that is positive definite. Further, if the nonlinear function f is continuous, bounded and satisfies the Lipschitz condition with constant $k = 1$, then by Theorem 4, the given nonlinear system is controllable on [0, 1]. ♦

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Rajagopal Joice Nirmala received the BSc degree in mathematics from Bharathiar University, Coimbatore, India, in 2010. She obtained her MSc, MPhil, and PhD degrees in mathematics in 2012, 2013 and 2016, respectively, from the same university under the guidance of Prof. K. Balachandran. Her research area is the controllability of fractional delay dynamical systems.

Krishnan Balachandran is a UGC BSR faculty fellow at the Department of Mathematics, Bharathiar University, Coimbatore, India. He has published more than 400 technical articles in well reputed journals. His major research areas are control theory, abstract integrodifferential equations, stochastic differential equations, fractional differential equations and partial differential equations. He is acting as a referee and an associate editor for various international journals. He is a member of the AMS, IMS, CMS and ISCA. He is also a fellow of the Academy of Sciences, Chennai.

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