Int. J. Appl. Math. Comput. Sci., 2022, Vol. 32, No. 2, 229–240
DOI: 10.34768/amcs-2022-0017

ON SOME WAYS TO IMPLEMENT STATE–MULTIPLICATIVE FAULT DETECTION IN DISCRETE–TIME LINEAR SYSTEMS

DUŠAN KROKAVEC [a,*], ANNA FILASOVÁ [a]

[a] Department of Cybernetics and Artificial Intelligence
Technical University of Košice
Letná, 04200 Košice, Slovakia
e-mail: {dusan.krokavec,anna.filasova}@tuke.sk

New design conditions on the observer based residual filter design for the linear discrete-time linear systems with zoned system parameter faults are presented. With respect to time evolution of residual signals and with a guarantee of their robustness, the design task is stated in terms of linear matrix inequalities, while the recursive implementation of algorithms is motivated by the platform existence for real-time processing. A major objective is to analyze the configuration required and, in particular, a new characterization of the norm boundaries of the multiplicative zonal parametric faults to be projected onto the structure of the set of linear matrix inequalities.

Keywords: linear discrete-time systems, state-multiplicative faults, Luenberger observers, residual filters, linear matrix inequalities.

1. Introduction

The problems of the system state estimation and the model-based fault detection in the presence of disturbances have been essential since their formulation (Luenberger, 1979; Gertler, 1998; Ellis, 2002). The impact of the linear matrix inequalities (LMIs) on observer design tasks has been well recognized to allow for fault detection, and has inspired numerous fault detection filter structures (Chiang *et al.*, 2001; Baïkeche *et al.*, 2006; Ferdowsi and Jagannathan, 2011; Sun and Yang, 2014). Tools for fault detection and residual filters in discrete-time systems have been scrutinized by Kim and Rew (2013), Gao (2015), Filasová *et al.* (2016) or Krokavec and Filasová (2019a). Modern trends in this field with relation to fault tolerant control (FTC) systems are discussed by Borutzky (2021), Ding (2021) and Hamdi *et al.* (2021).

One from the properties characterizing multiplicative faults is their effect on the system structure and their dependence on the system state variables (Gershon *et al.*, 2005). The treatment of the local plant parametric uncertainty makes fault detection methods different from standard (Korovin and Fomichev, 2009). Although the impact of parametric faults on dynamic systems is ubiquitous, often occurring in the form of a magnitude saturation of a system variable, only a few obtained results concern direct connections of multiplicative faults in diagnosis (Ferdowsi and Jagannathan, 2011; Gao and Duan, 2012). Other support principles can be found in the works of Zhong *et al.* (2006), Gil *et al.* (2006) or Doraiswami and Cheded (2012). A review of the state of the art is given by Park *et al.* (2020) or Huang *et al.* (2021).

If the system is characterizable by a parameter vector $\boldsymbol{\theta}$ and the set of the related measurements from the system represents an asymptotically stationary stochastic process, parametric fault detection can be transformed into monitoring the mean of a Gaussian residual vector (cf. the work of Döhler *et al.* (2020) and the references therein). The main problem remains the construction of the measurement structure for the detection of faults of individual parameters of a fault of a group of such parameters (Wu *et al.*, 2015). In view of the above, the aim of this paper is to formulate such a task using a sector of parameters in the system matrix of the system state description and to find a solution by estimating the corresponding components of the system state. The main objective of this paper is to derive LMI conditions for

*Corresponding author

this problem and to show that the obtained conditions are of the same computational complexities as standard LMI methods for fault detection and isolation (FDI) design.

Some new approaches to the treatment of sector multiplicative faults in the systems, which propose limited state variable measurements related to fault detection, or using zonate fault interpretations, have been proposed by Krokavec and Filadová (2019b; 2020). With reference to these results, and using a more realistic way to reflect the quadratic stability principle for uncertain discrete-time linear systems (Krokavec and Filasová, 2021), a subject of the paper are extended methods of fault residual filter design for discrete-time linear systems with single zoned multiplicative faults. By considering the vertically stripped system matrix element changes in the faulty regime after the occurrence of a single multiplicative fault, the standard (or enhanced) bounded real lemma LMI structure is modified to trace over the residual filter design conditions and the quadratic stability principle is applied to respect the existence of bounds on given stripped sector parameters for which the constrained form in residual filter design is directly applicable. The goal is sufficient flexibility to guarantee dynamic properties of the observer structure, as well as satisfactory residual signal sensitivity and thresholds in the fault detection.

The paper is organized as follows. After a short introduction in Section 1 and the problem formulation in Section 2, the approaches based on the H_∞ norm in the residual filter parameter design are addressed in Section 3. Enabling the internal parameter bound properness and using the quadratic stability idea, the related design method is addressed in Section 4. In the sequel, Section 5 shows the applicability of the method using a simulation example and Section 6 gives some concluding remarks.

Throughout the paper, the following notation is used: $\boldsymbol{x}^{\mathrm{T}}$, $\boldsymbol{X}^{\mathrm{T}}$ denote the transposes of a vector $\boldsymbol{x}$ and a matrix $\boldsymbol{X}$, respectively; for a square matrix the inequality $\boldsymbol{X} \prec 0$ means that $\boldsymbol{X}$ is a negative definite symmetric matrix; the symbol $\boldsymbol{I}_n$ indicates the n-th order identity matrix; $\mathbb{R}_+$ denotes the set of positive real numbers; $\mathbb{R}^n$ and $\mathbb{R}^{n\times r}$ refer to the set of all n-dimensional real vectors and $n \times r$ real matrices, respectively, and $\mathbb{Z}_+$ is the set of positive integers.

2. Problem formulation and description

The discrete-time dynamical systems considered are represented by the space-time description belonging to the following class of equations:

$$\boldsymbol{q}(i+1) = (\boldsymbol{F} + \Delta\boldsymbol{F}(i))\boldsymbol{q}(i) + \boldsymbol{G}\boldsymbol{u}(i)) + \boldsymbol{E}\boldsymbol{d}(i)\,, \quad (1)$$

$$\boldsymbol{z}(i) = \boldsymbol{C}\boldsymbol{q}(i)\,, \quad (2)$$

$$\boldsymbol{y}(i) = \boldsymbol{C}_y\boldsymbol{q}(i)\,, \quad (3)$$

where $\boldsymbol{q}(i) \in \mathbb{R}^n$, $\boldsymbol{u}(i) \in \mathbb{R}^r$, $\boldsymbol{y}(i), \boldsymbol{z}(i) \in \mathbb{R}^s$ stand for the system state, the system control input, the system output and the fault detection support measurement output, respectively, and $\boldsymbol{d}(i) \in \mathbb{R}^p$ is a norm bounded unknown disturbance ($\boldsymbol{d}(i)^{\mathrm{T}}\boldsymbol{d}(i) \leq \overline{\boldsymbol{d}}$). The nominal part of the system is characterized by the finite valued matrices $\boldsymbol{E} \in \mathbb{R}^{n\times p}$, $\boldsymbol{C} \in \mathbb{R}^{s\times n}$, $\boldsymbol{C}_y \in \mathbb{R}^{m\times n}$, $\boldsymbol{F} \in \mathbb{R}^{n\times n}$, $\boldsymbol{G} \in \mathbb{R}^{n\times r}$. The instant $i \in \mathcal{Z}_+$ is used as a representation of the time-instant point $t_i = i\,t_s$, where t_s is the sampling period.

After a system parameter fault occurrence, the system matrix in (1) is considered as $\boldsymbol{F}(i) = \boldsymbol{F} + \Delta\boldsymbol{F}(i)$, where $\Delta\boldsymbol{F}(i) \in \mathbb{R}^{n\times n}$ represents parametric faults. Since the fault parametric bounds are always known as they have clear physical meaning, it is considered in the following that $\Delta\boldsymbol{F}(i)$ is norm-bounded and $\operatorname{rank}\Delta\boldsymbol{F}(i) = s < n$. This implies that there are s linear independent columns in $\Delta\boldsymbol{F}(i)$ associated with parametric faults. For simplicity, it is considered that these columns from $\Delta\boldsymbol{F}(i)$ are arranged in the vertical strip of dimension $n \times s$. In the opposite case, there exists a perturbation matrix to reorder system state variables to yield such a form of vertical strip in $\Delta\boldsymbol{F}(i)$.

Since FTC methods have to be applied when a parametric fault causes system instability, the proposed method is concentrated on detection of parametric faults which do not lead to system instability, but their permanent occurrence is unacceptable. Because of practical reasons, it is assumed that all state variables related with the vertical strip columns of $\Delta\boldsymbol{F}(i)$ are measured, the existence of parametric faults is locally reproducible only in a given vertical strip and the vertical strip is a sparse array.

To cope with the parametric fault residual generation task, the following assumptions should be made throughout (Chen *et al.*, 2011):

(i) $\boldsymbol{q}(i+1) = (\boldsymbol{F} + \Delta\boldsymbol{F}(i))\boldsymbol{q}(i)$ is robust stable for all $\Delta\boldsymbol{F}(i)$, $i \in \mathbb{Z}_+$;

(ii) $\Delta\boldsymbol{F}(i)$ are norm-bounded (the faulty-free regime means $\Delta\boldsymbol{F}(i) = \boldsymbol{0}$);

(iii) the pair $(\boldsymbol{F}, \boldsymbol{C})$ is observable.

Note that we do not exclude a specific principle for solving the problem of the observer-based residual synthesis for linear systems with disturbances if $s > p$ (Korovin and Fomichev (2009)).

Remark 1. A natural key idea is to get a discrete-time state-space representation from the autonomous faulty linear continuous-time system

$$\dot{\boldsymbol{q}}(t) = (\boldsymbol{A} + \Delta\boldsymbol{A}(t))\boldsymbol{q}(t), \quad (4)$$

where $\boldsymbol{q}(t) \in \mathbb{R}^n$, $\boldsymbol{A}$, $\Delta\boldsymbol{A}(t) \in \mathbb{R}^{n\times n}$, $\Delta\boldsymbol{A}(t)$ is norm-bounded, $\operatorname{rank}\Delta\boldsymbol{A}(t) = s < n$ and the linear independent columns in $\Delta\boldsymbol{A}(t)$, associated with parametric faults, are arranged in a vertical strip of dimension $n \times s$. With only several potentially faulty nonzero parameters $\Delta a_{\alpha,\beta}(t),\ \alpha,\beta \in \{1,\dots,n\}$ of the matrix $\Delta\boldsymbol{A}(t) = \{\Delta a_{jl}(t)\}_{j,l}^{n}$ the Euler approximation

$$\left.\frac{\mathrm{d}\boldsymbol{q}(t)}{\mathrm{d}t}\right|_{t=i\,t_s} \approx \frac{\boldsymbol{q}(i+1)-\boldsymbol{q}(i)}{t_s} \tag{5}$$

implies

$$\begin{aligned}\boldsymbol{q}(i+1) &= (\boldsymbol{I}_n + t_s(\boldsymbol{A}+\Delta\boldsymbol{A}(i)))\boldsymbol{q}(i)\\ &= (\boldsymbol{I}_n + t_s\boldsymbol{A})\boldsymbol{q}(i) + t_s\Delta\boldsymbol{A}(i)\boldsymbol{q}(i)\\ &= (\boldsymbol{F}+\Delta\boldsymbol{F}(i))\boldsymbol{q}(i),\end{aligned} \tag{6}$$

where, with $\Delta f_{jl}(i) = t_s\Delta a_{jl}(t)|_{t=i\,t_s}$,

$$\begin{aligned}\boldsymbol{F} &= \boldsymbol{I}_n + t_s\boldsymbol{A},\\ \Delta\boldsymbol{F}(i) &= t_s\Delta\boldsymbol{A}(i) = \{\Delta f_{jl}(i)\}_{j,l}^{n}.\end{aligned} \tag{7}$$

It is an evident choice that such sampling preserves the same faulty element positions $\alpha,\beta \in \{1,\dots,n\}$ in $\Delta\boldsymbol{A}(t)$ and $\Delta\boldsymbol{F}(i)$, while the change in parameters in $\Delta\boldsymbol{F}(i)$ will be t_s-times smaller. The consequence of that sampling which uses the fundamental matrix of a continuous-time system model is that all elements of the $\Delta\boldsymbol{F}(i)$ matrix seem to be faulty and correlated. The disadvantage of the proposed approach is a higher sampling frequency, the advantage is the recurrent algorithm, preserving the system parametric structure.

Remark 2. To make multiplicative fault uncorrelated with those state variables that are not exposed to the vertical strip of multiplicative faults, the block diagonal matrix $\boldsymbol{H}^{\circ}(i) \in \mathbb{R}^{n\times n}$ is defined as

$$\boldsymbol{H}^{\circ}(i) = \operatorname{diag}\begin{bmatrix}\boldsymbol{0} & \boldsymbol{H}(i) & \boldsymbol{0}\end{bmatrix}, \tag{8}$$

where the sub-matrix $\boldsymbol{H}(i) \in \mathbb{R}^{s\times s}$ has to be diagonal and its elements represent the unknown parametric faults, that is

$$\boldsymbol{H}(i) = \operatorname{diag}\begin{bmatrix}h_1(i) & h_2(i) & \cdots & h_s(i)\end{bmatrix}. \tag{9}$$

It is assumed that the component fault terms $h_l(i),\ l = 1,2,\dots,s$ of $\boldsymbol{H}(i)$ are unknown but with known bounds. Naturally, under the above structure, the matrix $\boldsymbol{H}^{\circ}(i)$ is quasi-orthogonal.

Lemma 1. *To express parameter faults as multiplicative uncorrelated sensor faults while the fault model is defined as*

$$\Delta\boldsymbol{F}(i)\boldsymbol{q}(i) = \boldsymbol{F}_{\Delta}^{\circ}\boldsymbol{H}^{\circ}(i)\boldsymbol{q}(i), \tag{10}$$

where

$$\boldsymbol{F}_{\Delta}^{\circ} = \begin{bmatrix}\boldsymbol{0} & \boldsymbol{F}_{\Delta} & \boldsymbol{0}\end{bmatrix}, \quad \boldsymbol{F}_{\Delta}^{\circ} \in \mathbb{R}^{n\times n},\ \boldsymbol{F}_{\Delta} \in \mathbb{R}^{n\times s}, \tag{11}$$

$\boldsymbol{F}_{\Delta}$ is a binary strip matrix (matrix in which each entry is either 0 or 1), define the faulty element positions (Hogben, 2011) so as to get

$$\boldsymbol{C} = \begin{bmatrix}\boldsymbol{0} & \boldsymbol{I}_s & \boldsymbol{0}\end{bmatrix}, \quad \boldsymbol{C} \in \mathbb{R}^{s\times n}. \tag{12}$$

Then

$$\boldsymbol{F}_{\Delta}^{\circ}\boldsymbol{H}^{\circ}(i)\boldsymbol{q}(i) = \boldsymbol{F}_{\Delta}^{\circ}\boldsymbol{Z}^{\circ}(i)\boldsymbol{h}^{\circ}(i), \tag{13}$$

where

$$\boldsymbol{h}^{\circ}(i) = \begin{bmatrix}\boldsymbol{0}\\ \boldsymbol{h}(i)\\ \boldsymbol{0}\end{bmatrix}, \quad \boldsymbol{h}^{\mathrm{T}}(i) = \begin{bmatrix}h_1(i) & \cdots & h_s(i)\end{bmatrix}, \tag{14}$$

$$\boldsymbol{Z}^{\circ}(i) = \operatorname{diag}\begin{bmatrix}\boldsymbol{0} & \boldsymbol{Z}_d(i) & \boldsymbol{0}\end{bmatrix}, \tag{15}$$

$$\boldsymbol{Z}_d(i) = \operatorname{diag}\begin{bmatrix}z_1(i) & z_2(i) & \cdots & z_s(i)\end{bmatrix}. \tag{16}$$

Here $\boldsymbol{Z}^{o}(i)$ is the diagonal matrix selector of the measured strip state variables and $\boldsymbol{Z}_d(i)$ reflects measured variables $\boldsymbol{z}(i)$ at the time instant i.

Proof. For the model (10) we deduce that

$$\boldsymbol{F}_{\Delta}^{\circ}\boldsymbol{H}^{\circ}(i)\boldsymbol{q}(i) = \boldsymbol{F}_{\Delta}^{\circ}\boldsymbol{Q}_d(i)\boldsymbol{h}^{\circ}(i), \tag{17}$$

where $\boldsymbol{Q}_d(i)$ is constructed from the associated system state variables as

$$\boldsymbol{Q}_d(i) = \operatorname{diag}\begin{bmatrix}q_1(i) & \cdots & q_n(i)\end{bmatrix} \tag{18}$$

and the left and right-hand sides of (17) are mutually equivalent.

Define

$$\boldsymbol{C}^{\circ} = \operatorname{diag}\begin{bmatrix}\boldsymbol{0} & \boldsymbol{I}_s & \boldsymbol{0}\end{bmatrix} = \begin{bmatrix}\boldsymbol{0}\\ \boldsymbol{C}\\ \boldsymbol{0}\end{bmatrix}. \tag{19}$$

Due to the intertwining properties, (11) implies

$$\boldsymbol{F}_{\Delta}^{\circ} = \boldsymbol{F}_{\Delta}^{\circ}\boldsymbol{C}^{\circ}, \tag{20}$$

which, using (15) and (16), leads to the parametrization of (17) as

$$\begin{aligned}\boldsymbol{F}_{\Delta}^{\circ}\boldsymbol{Q}_d(i)\boldsymbol{h}^{\circ}(i) &= \boldsymbol{F}_{\Delta}^{\circ}\boldsymbol{C}^{\circ}\boldsymbol{Q}_d(i)\boldsymbol{h}^{\circ}(i)\\ &= \boldsymbol{F}_{\Delta}^{\circ}\boldsymbol{Z}^{\circ}(i)\boldsymbol{h}^{\circ}(i),\end{aligned} \tag{21}$$

where $\boldsymbol{Z}^{\circ}(i) = \boldsymbol{C}^{\circ}\boldsymbol{Q}_d(i)$.

The structure of the above matrix product has the form

$$\boldsymbol{F}_{\Delta}^{\circ}\boldsymbol{H}^{\circ}(i)\boldsymbol{q}(i) = \begin{bmatrix}\boldsymbol{0} & \boldsymbol{F}_{\Delta} & \boldsymbol{0}\end{bmatrix}\begin{bmatrix}\boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{Z}_d(i) & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0}\end{bmatrix}\begin{bmatrix}\boldsymbol{0}\\ \boldsymbol{h}(i)\\ \boldsymbol{0}\end{bmatrix} \tag{22}$$

and implies (13). This concludes the proof. ∎

Remark 3. Assume that the state variables connected with nonzero diagonal components of $\boldsymbol{H}(i)$ are measurable. Then (13) is arranged in zones and inherently supports the H_∞ norm approach in the fault residual design. Independently, the non-overlapping measurement structure can be prescribed by the condition

$$\boldsymbol{C}_y \boldsymbol{C}^{\mathrm{T}} = \boldsymbol{0}\,, \tag{23}$$

if such a structural set of measurements can be technically realizable. Such an implementation allows the system operating point to be set independently of the FTC measurement subsystem, only depending on the measured output variables $\boldsymbol{y}(i)$.

The family of full order state observers of the form

$$\boldsymbol{q}_e(i+1) = \boldsymbol{F}\boldsymbol{q}_e(i) + \boldsymbol{G}\boldsymbol{u}(i) + \boldsymbol{J}(\boldsymbol{z}(i) - \boldsymbol{z}_e(i))\,, \tag{24}$$

$$\boldsymbol{z}_e(i) = \boldsymbol{C}\boldsymbol{q}_e(i) \tag{25}$$

represents a basis to generate fault residuals, where $\boldsymbol{q}_e(i) \in \mathbb{R}^n$ is the observer state vector and $\boldsymbol{J} \in \mathbb{R}^{n \times s}$ is a matrix with entries in the prescribed real matrix space, chosen from the condition that $\boldsymbol{F}_e = \boldsymbol{F} - \boldsymbol{J}\boldsymbol{C}$ is a Hurwitz matrix.

The objective is the solvability of the parameters of (24) subject to (1)–(3), and the design condition that the state variable estimation error

$$\boldsymbol{e}(i) = \boldsymbol{q}(i) - \boldsymbol{q}_e(i)\,, \tag{26}$$

converges in the fault-free regime, while the estimation error is not equal to the zero vector in a fault occurrence.

3. H_∞ norm approach in residual filter design

Residual filters can be written compactly in different forms. For convenience, it is considered below that for the state observer (24) and (25) the fault residual filter

$$\boldsymbol{r}(i) = \boldsymbol{R}\boldsymbol{C}\boldsymbol{e}(i) \tag{27}$$

is built, where $\boldsymbol{R} \in \mathbb{R}^{s \times s}$ is the filter gain matrix, optimized for H_∞ norms upper-bounds δ, γ, of the transfer function matrices $\boldsymbol{L}_d(z)$ and $\boldsymbol{L}_h(z)$, reflecting the mapping

$$\tilde{\boldsymbol{r}}(z) = \boldsymbol{L}_h(z)\tilde{\boldsymbol{h}}^{\bullet}(z)\,, \quad \tilde{\boldsymbol{r}}(z) = \boldsymbol{L}_d(z)\tilde{\boldsymbol{d}}(z)\,, \tag{28}$$

where $\boldsymbol{h}^{\bullet}(i) = \boldsymbol{Z}_d(i)\boldsymbol{h}(i)$, $\tilde{r}(z)$, $\tilde{h}^o(z)$, $\tilde{d}(z)$ are $\mathcal{Z}$-transforms of discrete-time variable sequences $r(i)$, $h^o(i)$ and $d(i)$, respectively.

Since the main point is the need to analyze the effect of the residual relation (27) and the observer state error described by, respectively

$$\boldsymbol{e}(i+1) = \boldsymbol{F}_e\boldsymbol{e}(i) + \boldsymbol{F}^{\circ}_{\Delta}\boldsymbol{h}^{\bullet}(i) + \boldsymbol{E}\boldsymbol{d}(i)\,, \tag{29}$$

where

$$\boldsymbol{F}_e = \boldsymbol{F} - \boldsymbol{J}\boldsymbol{C}\,, \tag{30}$$

(27) and (38) imply

$$\boldsymbol{L}_h(z) = \boldsymbol{R}\boldsymbol{C}(z\boldsymbol{I}_n - \boldsymbol{F}_e)^{-1}\boldsymbol{F}^{\circ}_{\Delta}, \tag{31}$$

$$\boldsymbol{L}_d(z) = \boldsymbol{R}\boldsymbol{C}(z\boldsymbol{I}_n - \boldsymbol{F}_e)^{-1}\boldsymbol{E}\,. \tag{32}$$

In the following, some existence and uniqueness design conditions are presented.

Theorem 1. *The Luenberger observer (24), (25) is quadratically stable if there exist a positive definite symmetric matrix $\boldsymbol{P} \in \mathbb{R}^{n \times n}$, matrices $\boldsymbol{R} \in \mathbb{R}^{s \times s}$, $\boldsymbol{Y} \in \mathbb{R}^{n \times s}$ and positive scalars $\gamma, \delta \in \mathbb{R}$ such that*

$$\boldsymbol{P} = \boldsymbol{P}^T > 0\,, \quad \gamma > 0\,, \quad \delta > 0\,, \tag{33}$$

$$\begin{bmatrix} -\boldsymbol{P} & * & * & * & * & * \\ \boldsymbol{P}\boldsymbol{F} - \boldsymbol{Y}\boldsymbol{C} & -\boldsymbol{P} & * & * & * & * \\ \boldsymbol{0} & \boldsymbol{F}^{\circ\mathrm{T}}_{\Delta}\boldsymbol{P} - \delta\boldsymbol{I}_s & * & * & * \\ \boldsymbol{0} & \boldsymbol{E}^{\mathrm{T}}\boldsymbol{P} & \boldsymbol{0} & -\gamma\boldsymbol{I}_p & * & * \\ \boldsymbol{R}\boldsymbol{C} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & -\gamma\boldsymbol{I}_s & * \\ \boldsymbol{R}\boldsymbol{C} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & -\delta\boldsymbol{I}_s \end{bmatrix} \prec 0\,. \tag{34}$$

Here the star denotes the symmetric item in a symmetric matrix. When the above conditions are satisfied, compute

$$\boldsymbol{J} = \boldsymbol{P}^{-1}\boldsymbol{Y} \tag{35}$$

and the residual generator gain matrix is defined directly by the matrix variable $\boldsymbol{R}$.

Proof. To afford optimization with respect to residuals, consider the scalar function

$$\begin{aligned} v(\boldsymbol{e}(i)) = {} & \boldsymbol{e}^{\mathrm{T}}(i)\boldsymbol{P}\boldsymbol{e}(i) \\ & + \delta^{-1}\sum_{j=0}^{i-1}(\boldsymbol{r}^{\mathrm{T}}(j)\boldsymbol{r}(j) - \delta^2\boldsymbol{h}^{\bullet\mathrm{T}}(j)\boldsymbol{h}^{\bullet}(j)) \\ & + \gamma^{-1}\sum_{j=0}^{i-1}(\boldsymbol{r}^{\mathrm{T}}(j)\boldsymbol{r}(j) - \gamma^2\boldsymbol{d}^{\mathrm{T}}(j)\boldsymbol{d}(j)) \\ & > 0. \end{aligned} \tag{36}$$

The last inequality to be satisfied for a positive definite symmetric matrix $\boldsymbol{P} \in \mathbb{R}^{n \times n}$ and positive scalars $\gamma, \delta \in \mathbb{R}$. Then $\Delta v(\boldsymbol{e}(i))$ is the change in $v(\boldsymbol{e}(i))$ along a trajectory $\boldsymbol{e}(i)$ determined by the observer dynamics and it must yield

$$\begin{aligned} \Delta v(\boldsymbol{e}(i)) = {} & \boldsymbol{e}^{\mathrm{T}}(i+1)\boldsymbol{P}\boldsymbol{e}(i+1) - \boldsymbol{e}^{\mathrm{T}}(i)\boldsymbol{P}\boldsymbol{e}(i) \\ & + \gamma^{-1}\boldsymbol{r}^{\mathrm{T}}(i)\boldsymbol{r}(i) - \gamma\boldsymbol{d}^{\mathrm{T}}(i)\boldsymbol{d}(i) \\ & + \delta^{-1}\boldsymbol{r}^{\mathrm{T}}(i)\boldsymbol{r}(i) - \delta\boldsymbol{h}^{\bullet\mathrm{T}}(i)\boldsymbol{h}^{\bullet}(i) \\ & < 0\,. \end{aligned} \tag{37}$$

Since the main point is the need to analyze the effect of the state error relation (24), (25), substituting

$$e(i+1) = F_e e(i) + F_\Delta^\circ h^\bullet(i) + E d(i)\,, \tag{38}$$

where

$$F_e = F - JC\,, \tag{39}$$

into (37) gives

$$\begin{aligned}\Delta v(e(i)) &= e^{\mathrm{T}}(i)(F_e^{\mathrm{T}} P F_e - P)e(i) \\ &\quad + e^{\mathrm{T}}(i)(\gamma^{-1} C^{\mathrm{T}} R^{\mathrm{T}} R C + \delta^{-1} C^{\mathrm{T}} R^{\mathrm{T}} R C)e(i) \\ &\quad - \gamma d^{\mathrm{T}}(i) d(i) - \delta h^{\bullet\mathrm{T}}(i) h^\bullet(i) \\ &\quad + e^{\mathrm{T}}(i) F_e^{\mathrm{T}} P F_\Delta^\circ h^\bullet(i) + h^{\bullet\mathrm{T}}(i) F_\Delta^{\circ\mathrm{T}} P F_e e(i) \\ &\quad + e^{\mathrm{T}}(i) F_e^{\mathrm{T}} P E d(i) + d^{\mathrm{T}}(i) E^{\mathrm{T}} P F_e e(i) \\ &\quad + h^{\bullet\mathrm{T}}(i) F_\Delta^{\circ\mathrm{T}} P E d(i) + d^{\mathrm{T}}(i) E^{\mathrm{T}} P F_\Delta^\circ h^\bullet(i) \\ &\quad + h^{\bullet\mathrm{T}}(i) F_\Delta^{\circ\mathrm{T}} P F_\Delta^\circ h^\bullet(i) + d^{\mathrm{T}}(i) E^{\mathrm{T}} P E d(i) \\ &< 0\,.\end{aligned} \tag{40}$$

For

$$e_\Sigma^{\mathrm{T}}(i) = \begin{bmatrix} e^{\mathrm{T}}(i) & h^{\bullet\mathrm{T}}(i) & d^{\mathrm{T}}(i)\end{bmatrix}, \tag{41}$$

the inequality given in (40) can be rewritten as

$$e_\Sigma^{\mathrm{T}}(i) P_\Sigma e_\Sigma(i) < 0\,, \tag{42}$$

where the negativity of (40) and (42) implies the negative definiteness of the matrix

$$\begin{aligned} P_\Sigma &= \begin{bmatrix} F_e^{\mathrm{T}} \\ F_\Delta^{\circ\mathrm{T}} \\ E^{\mathrm{T}} \end{bmatrix} P \begin{bmatrix} F_e & F_\Delta^\circ & E \end{bmatrix} \\ &\quad + \begin{bmatrix} \gamma^{-1} C^{\mathrm{T}} R^{\mathrm{T}} R C + \delta^{-1} C^{\mathrm{T}} R^{\mathrm{T}} R C - P & 0 & 0 \\ 0 & -\delta I_n & 0 \\ 0 & 0 & -\gamma I_s \end{bmatrix} \\ &\prec 0. \end{aligned} \tag{43}$$

It is then possible to convert (43) into one LMI using Schur's complement property ($*$ is not used for better visualization of LMIs structures in the proofs of this section)

$$\begin{bmatrix} \Pi_{11} & 0 & 0 & F_e^{\mathrm{T}} P \\ 0 & -\delta I_n & 0 & F_\Delta^{\circ\mathrm{T}} P \\ 0 & 0 & -\gamma I_p & E^{\mathrm{T}} P \\ P F_e & P F_\Delta^\circ & P E & -P \end{bmatrix} \prec 0\,, \tag{44}$$

where

$$\Pi_{11} = \gamma^{-1} C^{\mathrm{T}} R^{\mathrm{T}} R C + \delta^{-1} C^{\mathrm{T}} R^{\mathrm{T}} R C - P\,. \tag{45}$$

When constructing the strict LMI by eliminating the bilinear forms related to matrix variable R in (45), the repeated application of yields the Schur complement

$$\begin{bmatrix} -P & 0 & 0 & F_e^{\mathrm{T}} P & C^{\mathrm{T}} R^{\mathrm{T}} & C^{\mathrm{T}} R^{\mathrm{T}} \\ 0 & -\delta I_n & 0 & F_\Delta^{\circ\mathrm{T}} P & 0 & 0 \\ 0 & 0 & -\gamma I_p & E^{\mathrm{T}} P & 0 & 0 \\ P F_e & P F_\Delta^\circ & P E & -P & 0 & 0 \\ R C & 0 & 0 & 0 & -\gamma I_s & 0 \\ R C & 0 & 0 & 0 & 0 & -\delta I_s \end{bmatrix} \prec 0\,. \tag{46}$$

Introduce the transform matrix (the unspecified elements of the block matrix T are zero matrices)

$$T = \begin{bmatrix} I_n & 0 & & & & \\ & 0 & I_n & & & \\ & 0 & 0 & I_p & & \\ & I_n & 0 & 0 & & \\ & 0 & 0 & 0 & I_s & \\ & 0 & 0 & 0 & 0 & I_s \end{bmatrix}. \tag{47}$$

Pre-multiplying the left-hand side of (46) by T^{T} and post-multiplying the result by T then implies

$$\begin{bmatrix} -P & F_e^{\mathrm{T}} P & 0 & 0 & C^{\mathrm{T}} R^{\mathrm{T}} & C^{\mathrm{T}} R^{\mathrm{T}} \\ P F_e & -P & P F_\Delta^\circ & P E & 0 & 0 \\ 0 & F_\Delta^{\circ\mathrm{T}} P & -\delta I_n & 0 & 0 & 0 \\ 0 & E^{\mathrm{T}} P & 0 & -\gamma I_p & 0 & 0 \\ R C & 0 & 0 & 0 & -\gamma I_s & 0 \\ R C & 0 & 0 & 0 & 0 & -\delta I_s \end{bmatrix} \prec 0\,. \tag{48}$$

To get (39), we can write

$$P F_e = P F - P J C \tag{49}$$

and with the new matrix variable

$$Y = S J. \tag{50}$$

Note that (48) implies (34). This concludes the proof. ∎

Remark 4. The transformation defined by the matrix T is introduced to obtain more comparable LMI structures, related to different approaches applied in residual filter parameter design. Note that a key property of linear observers is that the map from a steady-state observer error in the faulty free system to the solution $e(i) \in \mathbb{R}^n$ related to the parametric fault on a given time interval is always linear.

In particular, if no other special constraints are considered, the following enhanced result is directly implied.

Theorem 2. (Enhanced design condition) *The Luenberger observer (24), (25) is quadratically stable if there exist positive definite symmetric matrices $P, S \in \mathbb{R}^{n\times n}$, matrices $R \in \mathbb{R}^{s\times s}$, $Y \in \mathbb{R}^{n\times s}$ and positive scalars $\gamma, \delta \in \mathbb{R}$ such that*

$$P = P^T > 0, \quad S = S^T > 0, \quad \gamma > 0, \quad \delta > 0, \tag{51}$$

$$\begin{bmatrix} -\boldsymbol{P} & * & * & * & * & * \\ \boldsymbol{SF}-\boldsymbol{ZC} & \boldsymbol{P}-2\boldsymbol{S} & * & * & * & * \\ \boldsymbol{0} & \boldsymbol{F}_{\Delta}^{\circ\mathrm{T}}\boldsymbol{S} & -\delta\boldsymbol{I}_n & * & * & * \\ \boldsymbol{0} & \boldsymbol{E}^{\mathrm{T}}\boldsymbol{S} & \boldsymbol{0} & -\gamma\boldsymbol{I}_p & * & * \\ \boldsymbol{RC} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & -\gamma\boldsymbol{I}_s & * \\ \boldsymbol{RC} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & -\delta\boldsymbol{I}_s \end{bmatrix} \prec 0 . \quad (52)$$

When the above conditions are satisfied, compute

$$\boldsymbol{J} = \boldsymbol{S}^{-1}\boldsymbol{Z} \quad (53)$$

and the residual generator gain matrix is defined directly by the matrix variable $\boldsymbol{R}$.

Proof. Transform (38) into a singular form such that

$$\boldsymbol{F}_e\boldsymbol{e}(i) + \boldsymbol{F}_{\Delta}^{\circ}\boldsymbol{h}^{\bullet}(i) + \boldsymbol{E}\boldsymbol{d}(i) - \boldsymbol{e}(i+1) = \boldsymbol{0}. \quad (54)$$

Then, with an arbitrary symmetric positive definite square matrix $\boldsymbol{S} \in \mathbb{R}^{n\times n}$, we get

$$\boldsymbol{e}^{\mathrm{T}}(i+1)\boldsymbol{S}(\boldsymbol{F}_e\boldsymbol{e}(i)+\boldsymbol{F}_{\Delta}^{\circ}\boldsymbol{h}^{\bullet}(i)+\boldsymbol{E}\boldsymbol{d}(i)-\boldsymbol{e}(i+1)) = 0 . \quad (55)$$

Adding (55) and its transposition to (37), we obtain the condition

$$\begin{aligned} &\boldsymbol{e}^{\mathrm{T}}(i+1)\boldsymbol{P}\boldsymbol{e}(i+1) - \boldsymbol{e}^{\mathrm{T}}(i)\boldsymbol{P}\boldsymbol{e}(i) \\ &+ \boldsymbol{e}^{\mathrm{T}}(i)(\gamma^{-1}\boldsymbol{C}^{\mathrm{T}}\boldsymbol{R}^{\mathrm{T}}\boldsymbol{RC} + \delta^{-1}\boldsymbol{C}^{\mathrm{T}}\boldsymbol{R}^{\mathrm{T}}\boldsymbol{RC})\boldsymbol{e}(i) \\ &- \gamma\boldsymbol{d}^{\mathrm{T}}(i)\boldsymbol{d}(i) - \delta\boldsymbol{h}^{\bullet\mathrm{T}}(i)\boldsymbol{h}^{\bullet}(i) \\ &- 2\boldsymbol{e}_z^{\mathrm{T}}(i+1)\boldsymbol{S}\boldsymbol{e}_z(i+1) \\ &+ \boldsymbol{e}^{\mathrm{T}}(i+1)\boldsymbol{S}(\boldsymbol{F}_e\boldsymbol{e}(i) + \boldsymbol{F}_{\Delta}^{\circ}\boldsymbol{h}^{\bullet}(i) + \boldsymbol{E}\boldsymbol{d}(i)) \\ &+ (\boldsymbol{F}_e\boldsymbol{e}(i) + \boldsymbol{F}_{\Delta}^{\circ}\boldsymbol{h}^{\bullet}(i) + \boldsymbol{E}\boldsymbol{d}(i))^{\mathrm{T}}\boldsymbol{S}\boldsymbol{e}(i+1) \\ &< 0 . \end{aligned} \quad (56)$$

For

$$\boldsymbol{e}_{\Xi}^{\mathrm{T}}(i) = \begin{bmatrix}\boldsymbol{e}^{\mathrm{T}}(i) & \boldsymbol{e}^{\mathrm{T}}(i+1) & \boldsymbol{h}^{\bullet\mathrm{T}}(i) & \boldsymbol{d}^{\mathrm{T}}(i)\end{bmatrix}, \quad (57)$$

the inequality (56) can be rewritten as

$$\boldsymbol{e}_{\Xi}^{\mathrm{T}}(i)\boldsymbol{P}_{\Xi}\boldsymbol{e}_{\Xi}(i) < 0 , \quad (58)$$

where, with (45),

$$\boldsymbol{P}_{\Xi} = \begin{bmatrix} \boldsymbol{\Pi}_{11} & \boldsymbol{F}_e^{\mathrm{T}}\boldsymbol{S} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{SF}_e & \boldsymbol{P}-2\boldsymbol{S} & \boldsymbol{SF}_{\Delta}^{\circ} & \boldsymbol{SE} \\ \boldsymbol{0} & \boldsymbol{F}_{\Delta}^{\circ\mathrm{T}}\boldsymbol{S} & -\delta\boldsymbol{I}_n & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{E}^{\mathrm{T}}\boldsymbol{S} & \boldsymbol{0} & -\gamma\boldsymbol{I}_p \end{bmatrix} \prec 0 . \quad (59)$$

In much the same way as above, (59) can be transformed to a strict LMI of the form

$$\begin{bmatrix} -\boldsymbol{P} & * & * & * & * & * \\ \boldsymbol{SF}_e & \boldsymbol{P}-2\boldsymbol{S} & * & * & * & * \\ \boldsymbol{0} & \boldsymbol{F}_{\Delta}^{\circ\mathrm{T}}\boldsymbol{S} & -\delta\boldsymbol{I}_n & * & * & * \\ \boldsymbol{0} & \boldsymbol{E}^{\mathrm{T}}\boldsymbol{S} & \boldsymbol{0} & -\gamma\boldsymbol{I}_p & * & * \\ \boldsymbol{RC} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & -\gamma\boldsymbol{I}_s & * \\ \boldsymbol{RC} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & -\delta\boldsymbol{I}_s \end{bmatrix} \prec 0 . \quad (60)$$

Specifically, we can set

$$\boldsymbol{SF}_e = \boldsymbol{SF} - \boldsymbol{SJC} \quad (61)$$

and with the matrix variable

$$\boldsymbol{Z} = \boldsymbol{SJ} \quad (62)$$

(60) implies (52). The proof is completed. ■

Remark 5. Comparing (34) and (52), we can deduce that residual filters become quadratically stable under both feasible design conditions, but in the second case the Lyapunov matrix $\boldsymbol{P}$ is decoupled from the system parameters and the observer gain is defined using the slack matrix $\boldsymbol{S}$. Although the LMI structure seems slightly complex at first glance due to the slack matrix existence, the faulty changes in the system parameters reduce the achievable performance in the design to a lesser extent.

There are parameter restrictions in the sense of H_∞ optimization of the residual filter design task and the slack matrix variable provides additional degrees of freedom, which reduces the conservativeness. In general, it is sufficient for the matrix $\boldsymbol{S}$ to be square and regular, but the comparison of (30) and (48) shows that it is advantageous for discrete-time systems if the matrix is symmetric and positive definite (De Oliveira *et al.*, 2002).

4. Application of the quadratic stability principle

The quadratic stability principle can be applied if the fault magnitude bounds are known. The following additional lemma is presented to illustrate this case.

Lemma 2. (Krokavec and Filasová, 2021) *Consider the uncertain dual autonomous system*

$$\boldsymbol{p}(i+1) = (\boldsymbol{F} + \Delta\boldsymbol{F}(i))^{\mathrm{T}}\boldsymbol{p}(i) , \quad (63)$$

$$\Delta\boldsymbol{F}(i) = \boldsymbol{V}\boldsymbol{W}(i)\boldsymbol{U} , \quad \boldsymbol{W}^{\mathrm{T}}(i)\boldsymbol{W}(i) \preceq \boldsymbol{I}_s , \quad (64)$$

where $\boldsymbol{p}(t) \in \mathbb{R}^n$ and the system matrix parameters are $\boldsymbol{F}, \Delta\boldsymbol{F}(i) \in \mathbb{R}^{n\times n}$, $\boldsymbol{V} \in \mathbb{R}^{n\times s}$, $\boldsymbol{U} \in \mathbb{R}^{s\times n}$, $\mathrm{rank}(\boldsymbol{V}) = s$, the elements of $\boldsymbol{W}(i) \in \mathbb{R}^{s\times s}$ are Lebesgue measurable (Khargonekar and Petersen, 1990) and $\boldsymbol{V}, \boldsymbol{U}$ are known. Then (63) is quadratically stable if and only if there exist a symmetric positive definite matrix $\boldsymbol{Q} \in \mathbb{R}^{n\times n}$ and a positive scalar $\lambda \in \mathbb{R}$ such that, conditioned by the inequality

$$-\boldsymbol{Q} + \lambda\boldsymbol{U}^{\mathrm{T}}\boldsymbol{U} \prec 0 , \quad (65)$$

the set of linear matrix inequalities

$$\boldsymbol{Q} = \boldsymbol{Q}^{\mathrm{T}} \succ 0 , \quad \lambda > 0 , \quad (66)$$

$$\begin{bmatrix} -\boldsymbol{Q} & \boldsymbol{QF} & \boldsymbol{QV} \\ * & -\boldsymbol{Q} + \lambda\boldsymbol{U}^{\mathrm{T}}\boldsymbol{U} & \boldsymbol{0} \\ * & * & -\lambda\boldsymbol{I}_s \end{bmatrix} \prec 0 \quad (67)$$

is feasible.

This result implies the following relevant corollary.

Corollary 1. *Solving the task with the zone arranged parametric uncertainties as presented above, for (1) we can set*

$$\Delta F(i) = VW(i)U\,, \tag{68}$$

where

$$V = F_\Delta^\circ = \begin{bmatrix} 0 & F_\Delta & 0 \end{bmatrix}, \tag{69}$$

$$W(i) = \operatorname{diag}\begin{bmatrix} 0 & H(i) & 0 \end{bmatrix}, \tag{70}$$

$$U = \operatorname{diag}\begin{bmatrix} 0 & N & 0 \end{bmatrix}, \tag{71}$$

$$N = \operatorname{diag}\begin{bmatrix} n_1 & n_2 & \cdots & n_s \end{bmatrix}, \tag{72}$$

where V, $W(i)$, $U \in \mathbb{R}^{n\times s}$ and $n_j,\ j = 1,\dots,s$ are known bounds of the faulty parameters in the given faulty strip.

As far as the conditions of Corollary 1 are satisfied with the admissible uncertainties, the following is implied:

Theorem 3. *The Luenberger observer (24), (25) is quadratically stable if there exist a positive definite symmetric matrix $Q \in \mathbb{R}^{n\times n}$, matrices $R \in \mathbb{R}^{s\times s}$, $X \in \mathbb{R}^{n\times s}$ and positive scalars $\lambda, \xi \in \mathbb{R}$ such that*

$$Q = Q^T > 0\,, \quad \lambda > 0\,, \quad \xi > 0\,, \tag{73}$$

$$\begin{bmatrix} -Q & QF - XC & QV & C^{\mathrm T}R^{\mathrm T} \\ * & -Q + \lambda U^{\mathrm T}U & 0 & 0 \\ * & * & -\lambda I_s & 0 \\ * & * & * & -\xi I_s \end{bmatrix} \prec 0\,, \tag{74}$$

When the above conditions are satisfied, compute

$$J = Q^{-1}X \tag{75}$$

and the residual generator gain matrix is defined directly by the matrix variable R.

Proof. Starting with (67), we can write

$$\begin{bmatrix} -Q + \lambda^{-1}QVV^{\mathrm T}Q & QF \\ F^{\mathrm T}Q & -Q + \lambda U^{\mathrm T}U \end{bmatrix} \prec 0\,, \tag{76}$$

$$\begin{aligned} &-Q + \lambda^{-1}QVV^{\mathrm T}Q \\ &\qquad + QF(Q - \lambda U^{\mathrm T}U)^{-1}F^{\mathrm T}Q \prec 0\,. \end{aligned} \tag{77}$$

Pre-multiplying the left-hand side and post-multiplying the result by $P = Q^{-1}$ with $\vartheta = \lambda^{-1}$ then implies

$$-P + \vartheta VV^{\mathrm T} + F(P^{-1} - \vartheta^{-1}U^{\mathrm T}U)^{-1}F^{\mathrm T} \prec 0\,. \tag{78}$$

For the Lyapunov function

$$v(p(i)) = p^{\mathrm T}(i)Pp(i) > 0\,, \tag{79}$$

$$\begin{aligned} &\Delta v(p(i)) \\ &\quad = -\,p^{\mathrm T}(i)Pp(i) \\ &\qquad + p^{\mathrm T}(i)(F + \Delta F(i))P(F + \Delta F(i))^{\mathrm T}p(i) \\ &\quad < 0\,, \end{aligned} \tag{80}$$

it is evident that the approximation

$$\begin{aligned} &p^{\mathrm T}(i)(F + \Delta F(i))P(F + \Delta F(i))^{\mathrm T}p(i) \\ &\quad \le p^{\mathrm T}(i)(F(P^{-1} - \vartheta^{-1}U^{\mathrm T}U)^{-1}F^{\mathrm T} \\ &\qquad + \vartheta VV^{\mathrm T})p(i) \end{aligned} \tag{81}$$

is realized.

Setting

$$r(i) = RCPp(i)\,. \tag{82}$$

and assuming that $\xi \in \mathbb{R}_+$ is a positive scalar, we get

$$\begin{aligned} &p^{\mathrm T}(i)P\Theta Pp(i) \\ &\quad = \xi^{-1}p^{\mathrm T}(i)PC^{\mathrm T}R^{\mathrm T}RCPp(i) \ge 0\,, \end{aligned} \tag{83}$$

where

$$\Theta = \xi^{-1}C^{\mathrm T}R^{\mathrm T}RC \succeq 0. \tag{84}$$

Then can we set

$$\Delta v(p(i)) \le -p^{\mathrm T}(i)P\Theta Pp(i) < 0\,. \tag{85}$$

Applying (84) and (85), (78) is modified as

$$\begin{aligned} &-P + P\Theta P + \vartheta VV^{\mathrm T} \\ &\qquad + F(P^{-1} - \vartheta^{-1}U^{\mathrm T}U)^{-1}F^{\mathrm T} \prec 0\,. \end{aligned} \tag{86}$$

Pre-multiplying the left-hand side and post-multiplying the result by $Q = P^{-1}$, and then substituting $\vartheta^{-1} = \lambda$, yield

$$\begin{aligned} &-Q + \xi^{-1}C^{\mathrm T}R^{\mathrm T}RC + \lambda^{-1}QVV^{\mathrm T}Q \\ &\qquad + QF(Q - \lambda U^{\mathrm T}U)^{-1}F^{\mathrm T}Q \prec 0\,. \end{aligned} \tag{87}$$

This implies

$$\begin{bmatrix} \Theta - Q + \lambda^{-1}QVV^{\mathrm T}Q & QF \\ * & -Q + \lambda U^{\mathrm T}U \end{bmatrix} \prec 0\,, \tag{88}$$

$$\begin{bmatrix} \Theta - Q & QF & QV \\ * & -Q + \lambda U^{\mathrm T}U & 0 \\ * & * & -\lambda I_s \end{bmatrix} \prec 0\,, \tag{89}$$

$$\begin{bmatrix} -Q & QF & QV & C^{\mathrm T}R^{\mathrm T} \\ * & -Q + \lambda U^{\mathrm T}U & 0 & 0 \\ * & * & -\lambda I_s & 0 \\ * & * & * & -\xi I_s \end{bmatrix} \prec 0\,, \tag{90}$$

respectively.

Replacing F by F_e and analogously introducing the new variable $QJ = X$, from (90) we get (74), which concludes the proof. ∎

This technique, as well as the H_∞ norm structures outlined in the preceding section, give some new insights into the single multiplicative fault detection.

5. Illustrative example

To apply the results of Theorems 1–3 to the synthesis of residual filters, consider the system (1)–(3) with the strip column rank $s = 2$, the sampling period $t_s = 0.02$ s, the system disturbance having normal distribution with zero mean and variance $\sigma^2 = 0.05$, and the model parameters

$$\boldsymbol{F} = \begin{bmatrix} 0.9324 & 0 & 0.1109 & 0.0990 \\ 0.0062 & 0.9197 & 0.0226 & 0.0002 \\ 0.0185 & 0 & 0.8744 & 0.0867 \\ 0.0001 & 0.0428 & 0.0001 & 0.9467 \end{bmatrix},$$

$$\boldsymbol{F}^{\circ}_{\Delta} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \boldsymbol{E} = \begin{bmatrix} 0.0547 \\ 0.0621 \\ 0.0725 \\ 0.0267 \end{bmatrix},$$

$$\boldsymbol{G} = \begin{bmatrix} 0.0081 & 0.0043 \\ 0.0110 & 0.0041 \\ 0.0028 & 0.0063 \\ 0.0025 & 0.0034 \end{bmatrix}, \quad \boldsymbol{h}(i) = \begin{bmatrix} \boldsymbol{0} \\ h_1(i) \\ h_2(i) \\ \boldsymbol{0} \end{bmatrix},$$

$$\boldsymbol{C} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad \boldsymbol{C}_y = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

The matrix of multiplicative fault amplitudes and the fault amplitude bounds are

$$\boldsymbol{H}^{\circ}(i) = \mathrm{diag}\begin{bmatrix} 0 & h_1(i) & h_2(i) & 0 \end{bmatrix},$$

$$\overline{h}_j = 0.001, \quad \underline{h}_j = 0$$

for $j = 1, 2$. The system is stable and the system working point in simulations is set up by the forced mode rule

$$\boldsymbol{u}(i) = \boldsymbol{N}_w \boldsymbol{w},$$

which assigns the system output to the desired steady state value $\boldsymbol{y}_o = \boldsymbol{w}$, while

$$\boldsymbol{N}_w = (\boldsymbol{C}_y(\boldsymbol{I}_n - \boldsymbol{F})^{-1}\boldsymbol{G})^{-1} = \begin{bmatrix} -18.7567 & 77.7863 \\ 25.3041 & -101.1829 \end{bmatrix},$$

$$\boldsymbol{w}^{\mathrm{T}} = \begin{bmatrix} 1 & 2 \end{bmatrix}.$$

Programming (73) and (74) in the SeDuMi toolbox decomposes the feasible design problem using the set of LMI variables $\boldsymbol{Q} \in \mathbb{R}^{n \times n}$, $\boldsymbol{R} \in \mathbb{R}^{s \times s}$, $\boldsymbol{X} \in \mathbb{R}^{n \times s}$, $\lambda, \xi \in \mathbb{R}$. The obtained values are

$$\lambda = 1.3005, \quad \xi = 0.8388,$$

$$\boldsymbol{Q} = \begin{bmatrix} 0.8074 & -0.0026 & -0.0442 & 0.1131 \\ -0.0026 & 0.3849 & -0.0169 & 0.0016 \\ -0.0442 & -0.0169 & 0.2585 & -0.0454 \\ 0.1131 & 0.0016 & -0.0454 & 1.2716 \end{bmatrix} \succ 0,$$

$$\boldsymbol{X} = \begin{bmatrix} 0.0051 & 0.0920 \\ 0.3469 & -0.0052 \\ -0.0165 & 0.2175 \\ 0.0538 & 0.0161 \end{bmatrix},$$

$$\boldsymbol{R} = 10^{-9}\begin{bmatrix} -0.3198 & -0.1738 \\ 0.0283 & -0.0024 \end{bmatrix}.$$

It can be easily shown that (75) yields

$$\boldsymbol{J} = \begin{bmatrix} 0.0036 & 0.1578 \\ 0.9012 & 0.0258 \\ 0.0028 & 0.8753 \\ 0.0410 & 0.0299 \end{bmatrix}$$

and using (39), expressing the summarized behaviour of all estimated variables, the observer dynamics is defined by the observer system matrix and its eigenvalues

$$\boldsymbol{F}_e = \begin{bmatrix} 0.9324 & -0.0036 & -0.0489 & 0.0990 \\ 0.0062 & 0.0185 & -0.0032 & 0.0002 \\ 0.0185 & -0.0028 & -0.0009 & 0.0867 \\ 0.0001 & 0.0018 & -0.0298 & 0.9467 \end{bmatrix},$$

$$\rho(\boldsymbol{F}_e) = \{0.0023 \;\; 0.0190 \;\; 0.9327 \pm 0.0024\,\mathrm{i}\}.$$

Following the algorithm defined by Theorem 1, we get the residual filter parameters

$$\boldsymbol{R} = 10^{-3}\begin{bmatrix} -0.0016 & 0.0053 \\ 0.0048 & -0.1002 \end{bmatrix},$$

$$\boldsymbol{J} = \begin{bmatrix} 0.1691 & 0.2395 \\ 0.7886 & 0.0192 \\ 0.0055 & 0.8377 \\ 0.0705 & 0.0774 \end{bmatrix},$$

$$\boldsymbol{F}_e = \begin{bmatrix} 0.9324 & -0.1691 & -0.1296 & 0.0990 \\ 0.0062 & 0.1311 & 0.0034 & 0.0002 \\ 0.0185 & -0.0055 & 0.0367 & 0.0867 \\ 0.0001 & -0.0277 & -0.0773 & 0.9467 \end{bmatrix},$$

$$\rho(\boldsymbol{F}_e) = \{0.0466 \;\; 0.1324 \;\; 0.9339 \pm 0.0109\,\mathrm{i}\}.$$

For the same problem data, but with the solution according to Theorem 2, we obtain

$$\boldsymbol{R} = 10^{-8}\begin{bmatrix} -0.3895 & 0.4208 \\ -0.0327 & 0.2419 \end{bmatrix},$$

$$\boldsymbol{J} = \begin{bmatrix} 0.1514 & 0.2781 \\ 0.8076 & 0.0207 \\ 0.0054 & 0.8284 \\ 0.0672 & 0.1028 \end{bmatrix},$$

$$\boldsymbol{F}_e = \begin{bmatrix} 0.9324 & -0.1514 & -0.1672 & 0.0990 \\ 0.0062 & 0.1121 & 0.0019 & 0.0002 \\ 0.0185 & -0.0054 & 0.0460 & 0.0867 \\ 0.0001 & -0.0244 & -0.1027 & 0.9467 \end{bmatrix},$$

$$\rho(\boldsymbol{F}_e) = \{0.0593 \;\; 0.1133 \;\; 0.9323 \pm 0.0129\,\mathrm{i}\}.$$

From these results it is immediately clear that all the presented algorithms retain the unchanged first and fourth columns of the system dynamics matrix $\boldsymbol{F}$ even in the structures of the observer dynamics matrices $\boldsymbol{F}_e$. On the other hand, they set the robustness of the second and third columns (sector columns) of the observer matrices $\boldsymbol{F}_e$, which in $\boldsymbol{F}$ are exposed to parametric faults. It is also clear that the residual fault filter designed according to Theorem 3 has the fastest dynamics, but the disadvantage is that its application requires measuring all system state variables, as it follows from the relation (82).

To carry out simulations, demonstrating fault residual filter properties, single multiplicative faults with formal description by the step-like activation of $h_1(i) = 0.001$ while $h_2(i) = 0$, as well as when $h_2(i) = 0.001$ while $h_1(i) = 0$, respectively, are considered in their own time scale from the time instant $t = 3$ s. To remove additive dynamics in the responses, all initial states in simulation are set to zero.

The fault residuals time evaluations are depicted in Figs. 1–3, which show that the residual filters achieve very good time responses and stability performances, which clearly confirms the theoretic analysis. In order for the results of all methodologies to be comparable, the output of each residual filter is normalized by its corresponding norm $\|\boldsymbol{R}\|$. The system model parameters are defined to show cases where the parametric fault is acting on the different column element structure in the fault sector.

The residual fault filter designed according to Theorem 3 has the best directional properties, but in addition to the disadvantage mentioned above, the disturbance is not taken into account in its synthesis, so its use is always determined by the specific situation.

The remaining two methods reflect in the design conditions the upper bound of the H_∞ norm of the transfer matrix functions related to the sector faults and the external disturbances. The H_∞ norm adjustment by LMIs is optimal in the terms of the given row sector structure.

The tenable conclusion is that parametric faults of values of the order of 10^{-3} can be detected using the proposed residual structure, even if they do not cause system instability.

6. Concluding remarks

Observer based fault residual structures for discrete-time linear systems with single multiplicative system faults, have been considered. Because the elements of the system matrix change after the occurrence of a fault, the different synthesis starting points allow enough flexibility to guarantee stability and preferred dynamic properties of the observer structure, as well as satisfactory residual signal sensitivity. Three synthesis algorithms have been proposed being feasible in terms of standard numerical operations regarding linear matrix inequalities. They are completely model based and convenient to use.

In the used configurations, the problem is resolved with respect to measurable state variables having input coincidence with the multiplicative fault parameter structure. The number of residual signals may be equal to the number of control related outputs, as defined for control system performance. In addition, due to partial elimination, estimates of the state and the output vector can be preserved from disturbance disruptions.

This concept is also matched by the choice of the system model in the illustrative example. In order to achieve the independence of the fault residual filter time responses from the control structure, a stable system is used for the simulations with the working mode setting by the forced mode principle. The existence of a potential set of residual filters, reflecting the complementary segmentation of the fault matrix structure, forms a basis for future research directions.

The presented results remain valid for parametric fault detection as long as the sector state variable are measurable for FDI constructions. This constitutes an important step in the solution of the problem. The relaxation of the quadratic stability principle remains an open problem if not all sector state variables are measurable and will be a topic of future research. Further research will be carried out towards the effectiveness of the proposed methods in practical applications in the distributed systems diagnosis.

Acknowledgment

The work presented in this paper was supported by VEGA, the Grant Agency of the Ministry of Education and the Academy of Science of the Slovak Republic under the grant no. 1/0483/21. This support is very gratefully acknowledged.

References

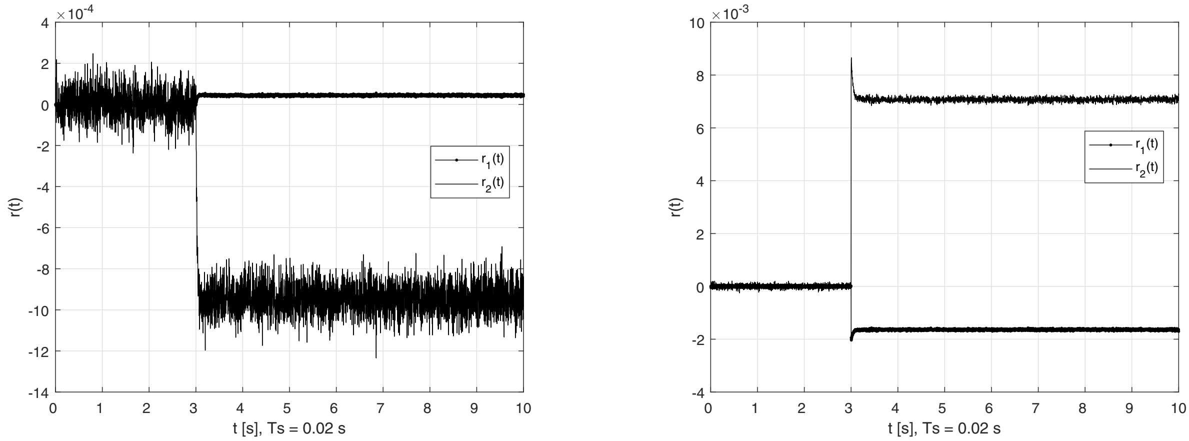
Baïkeche, H., Marx, B., Maquin, D., and Ragot, J. (2006). On parametric and nonparametric fault detection in linear closed-loop systems, *4th Workshop on Advanced Control and Diagnosis, Nancy, France*, pp. 1–6.

Borutzky, W. (2021). *Bond Graph Modelling for Control, Fault Diagnosis and Failure Prognosis*, Springer, Cham.

Chen, W., Khan, A.Q., Abid, M., and Ding, S.X. (2011). Integrated design of observer based fault detection for a class of uncertain nonlinear systems, *International Journal of Applied Mathematics and Computer Science* **21**(3): 423–430, DOI: 10.2478/v10006-011-0031-0.

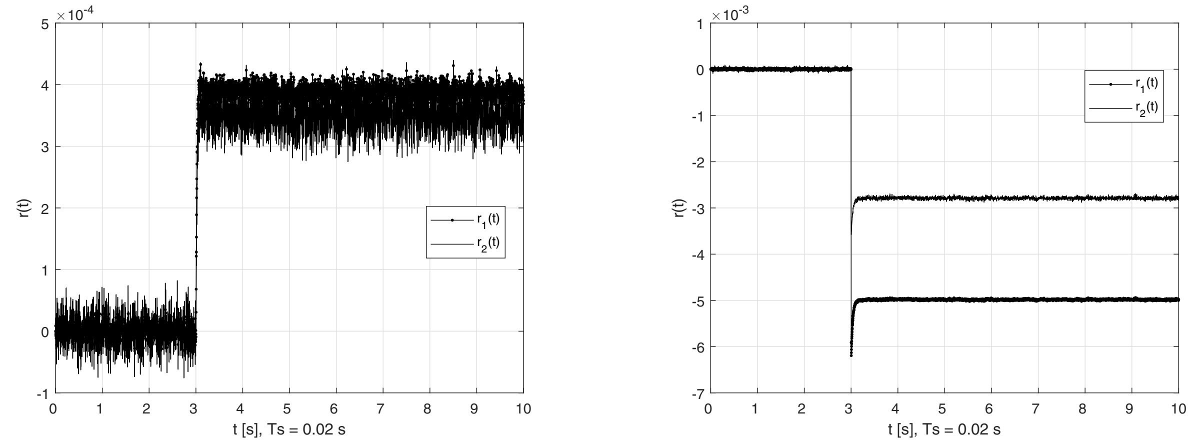
Chiang, L.H., Russell, E.L., and Braatz, R.D. (2001). *Fault Detection and Diagnosis in Industrial Systems*, Springer, London.

De Oliveira, M.C., Geromel, J.C., and Bernussou, J. (2002). Extended H_2 and H_∞ norm characterizations and



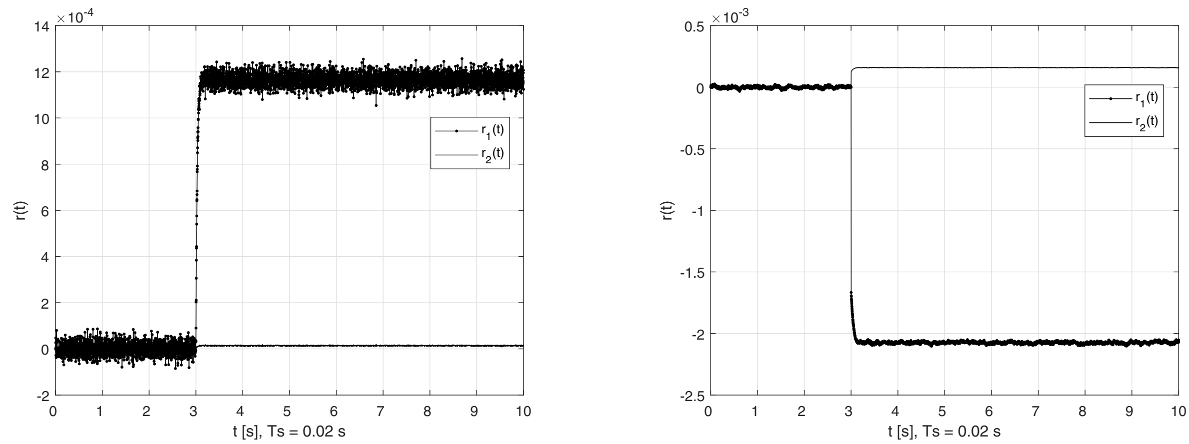
(a) (b)

Fig. 1. Responses of the fault residual filter (Theorem 1): the first sector column fault (a), the second sector column fault (b).



(a) (b)

Fig. 2. Responses of the fault residual filter (Theorem 2): the first sector column fault (a), the second sector column fault (b).



(a) (b)

Fig. 3. Responses of the fault residual filter (Theorem 3): the first sector column fault (a), the second sector column fault (b).

controller parametrizations for discrete-time systems, *International Journal of Control* **75**(9): 666–679.

Ding, S. (2021). *Advanced Methods for Fault Diagnosis and Fault-tolerant Control*, Springer, Cham.

Doraiswami, R. and Cheded, L. (2012). Kalman filter for parametric fault detection. An internal model principle-based approach, *IET Control Theory and Applications* **6**(5): 715–725.

Döhler, M., Mevel, L., and Zhang, Q. (2016) Fault detection, isolation and quantification from Gaussian residuals with application to structural damage diagnosis, *Annual Reviews in Control* **42**(C): 244–256.

Ellis, G. (2002). *Observers in Control Systems: A Practical Guide*, Academic Press, San Diego.

Felicio, P.A.S.A. and Lourtie, P.M.G. (2006). Applicability of standard formulation parametric fault detection methods, *14th Mediterranean Conference on Control and Automation, Ancona, Italy*, pp. 1–6.

Ferdowsi, H., Jagannathan, S. (2011). Unified model-based fault diagnosis scheme for nonlinear discrete-time systems with additive and multiplicative faults, *50th IEEE Conference on Decision and Control and European Control Conference, Orlando, USA*, pp. 1570–1575.

Filasová, A., Krokavec, D., and Liššinský, P. (2016). Conditions with D-stability circle area in design of observer-based fault estimation, *Applied Mathematical Sciences* **10**(35): 1705–1717.

Gao, Z. (2015). Fault estimation and fault-tolerant control for discrete-time dynamic systems, *IEEE Transactions on Industrial Electronics* **62**(4): 3874–3884.

Gao, C. and Duan, G. (2012). Robust adaptive fault estimation for a class of nonlinear systems subject to multiplicative faults, *Circuits, Systems, and Signal Processing* **31**(6): 2035–2046.

Gershon, E., Shaked, U., and Yaesh, I. (2005). *H_∞ Control and Estimation of State-multiplicative Linear Systems*, Springer, London.

Gertler, J. (1998). *Fault Detection and Diagnosis in Engineering Systems*, CRC Press, Boca Raton.

Gil, P., Santos, F., Palma, L., and Cardoso, A. (2015). Recursive subspace system identification for parametric fault detection in nonlinear systems, *Applied Soft Computing* **37**(C): 444–455.

Hamdi, H., Rodrigues, M., Rabaoui, B., and Benhadj Braiek, N. (2021). A fault estimation and fault–tolerant control based sliding mode observer for LPV descriptor systems with time delay, *International Journal of Applied Mathematics and Computer Science* **31**(2): 247–258, DOI: 10.34768/amcs-2021-0017.

Hogben, L. (2007), *Handbook of Linear Algebra*, Chapman & Hall/CRC, Boca Raton.

Huang, L., Ren, H., Chai, Y., and Qu, J. (2021). A fault detection method based on stacking the SAE-SRBM for nonstationary and stationary hybrid processes, *International Journal of Applied Mathematics and Computer Science* **31**(1): 29–43, DOI: 10.34768/amcs-2021-0003.

Khargonekar, P.P. and Petersen, I.R. (1990). Robust stabilization of uncertain linear systems: Quadratic stabilizability and H_∞ control theory, *IEEE Transactions on Automatic Control* **35**(3): 356–361.

Kim, K.S. and Rew, K.H. (2013). Reduced order disturbance observer for discrete-time linear systems, *Automatica* **49**(4): 968–975.

Korovin, S.K. and Fomichev, V.V. (2009). *State Observers for Linear Systems with Uncertainty*, De Gruyter, Berlin.

Krokavec, D. and Filasová, A. (2019a). H_∞ norm principle in residual filter design for discrete-time linear positive systems, *European Journal of Control* **45**: 17–29.

Krokavec, D. and Filasová, A. (2019b). On fault detection for discrete-time linear state-multiplicative systems with uncorrelated multiplicative faults, *6th International Conference on Control, Decision and Information Technologies, Paris, France*, pp. 894-899.

Krokavec, D. and Filasová, A. (2020). Detection of state-multiplicative faults in discrete-time linear systems, *in* A. Bartoszewicz *et al.* (Eds), *Advances in Intelligent Systems and Computing*, Springer, Cham, pp. 1424–1433.

Krokavec, D. and Filasová, A. (2021). Quadratic stabilization of linear uncertain positive discrete-time systems, *Symmetry* **13**(9): 1–20.

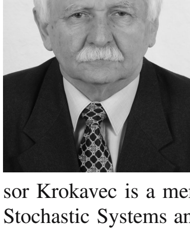
Luenberger, D.G. (1979). *Introduction to Dynamic Systems: Theory, Models and Applications*, Wiley, New York.

Park, Y.J., Fan, S.K.S., and Hsu, C.Y. (2020). A review on fault detection and process diagnostics in industrial processes, *Processes* **8**(9): 1–26.

Sun, Z. and Yang, Z. (2014). Joint parametric fault diagnosis and state estimation using KF-ML method, *IFAC Proceedings Volumes* **47**(3): 8293–8298.

Wu, C., Qi, J., Song, D., Qi, X., and Han, J. (2015). Simultaneous state and parameter estimation based actuator fault detection and diagnosis for an unmanned helicopter, *International Journal of Applied Mathematics and Computer Science* **25**(1): 175–187, DOI: 10.1515/amcs-2015-0013.

Zhong, M., Ma, C., and Ding, S.X. (2005). Design of parametric fault detection systems: An H-infinity optimization approach, *Journal of Control Theory and Applications* **2**(1): 35–41.



Dušan Krokavec received his MSc and PhD degrees in 1967 (automatic control) and 1982 (technical cybernetics), respectively, both from the Faculty of Electrical Engineering, Slovak University of Technology in Bratislava. In 1999 he was appointed as a full professor of automation and control at the Technical University in Košice. He specializes in stochastic processes in dynamic systems, digital control systems, Kalman filtering and dynamic system fault diagnosis. Professor Krokavec is a member of the IFAC Technical Committees TC 1.4. Stochastic Systems and TC 6.4. Fault Detection, Supervision & Safety of Technical Processes (SAFEPROCESS).

Anna Filasová graduated in technical cybernetics and received her MSc and PhD degrees in 1975 and 1993, respectively, both from the Faculty of Electrical Engineering and Informatics, Technical University of Košice, Slovakia. In 1999 she was appointed as an associated professor of technical cybernetics at the Technical University in Košice. Her main research interests are in robust control, decentralized control, fault tolerant control and linear matrix inequalities.

Received: 19 December 2021
Revised: 9 April 2022
Accepted: 20 April 2022