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A NEW METHOD FOR DECISION MAKING PROBLEMS WITH REDUNDANT AND INCOMPLETE INFORMATION BASED ON INCOMPLETE SOFT SETS: FROM CRISP TO FUZZY

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This research is focused on decision-making problems with redundant and incomplete information under a fuzzy environment. Firstly, we present the definition of incomplete fuzzy soft sets and analyze their data structures. Based on that, binary relationships between each pair of objects and the "restricted/relaxed AND" operations in the incomplete fuzzy soft set are discussed. After that, the definition of incomplete fuzzy soft decision systems is proposed. To reduce the inconsistency caused by the redundant information in decision making, the significance of the attribute subset, the reduct attribute set, the optimal reduct attribute set and the core attribute in incomplete fuzzy soft decision systems is also discussed. These definitions can be applied in an incomplete fuzzy soft set directly, so there is no need to convert incomplete data into complete one in the process of reduction. Then a new decision-making algorithm based on the above definitions can be developed, which can deal with redundant information and incomplete information simultaneously, and is independent of some unreliable assumptions about the data generating mechanism to forecast the incomplete information. Lastly, the algorithm is applied in the problem of regional food safety evaluation in Chongqing, China, and the corresponding comparison analysis demonstrates the effectiveness of the proposed method.

Keywords: decision-making, soft set, incomplete fuzzy soft set, incomplete information, redundant information.

1. Introduction

Uncertainties are involved in most real-life problems in engineering, economics, medical science, and so on. Many researchers have proposed some mathematical theories to deal with the uncertainties (Zadeh, 1965; Pawlak, 1984; 1985; Gau and Buehrer, 1993; Liu, 2007). However, Molodtsov (1999) pointed out that the parametrization tools of these theories are inadequate, and instead he presented a new mathematical theory, named soft set theory, which is free from the limitation of inadequacy of the parameterization tools, and can be used to deal with uncertain problems. Potential applications of the soft set theory include function smoothing, game theory, operational research, integration, probability theory, and measurement theory (Molodtsov, 1999). It was also applied in many other domains which contain uncertainties such as forecasting (de Andres *et al.*, 2012; Xu *et al.*, 2014; 2019), decision making (Maji and Roy, 2002; Garg and Arora, 2018; Yang and Yao, 2020), evaluation (Li *et al.*, 2018), and so on.

One of the most common applications of soft sets is in decision-making problems. Based on soft set theory, researchers have proposed many different kinds of hybrid soft sets to deal with decision-making problems in different information environments, such as fuzzy environment (Yang *et al.*, 2013; Li *et al.*, 2015; Liu *et al.*, 2017; Hussain *et al.*, 2020; Qayyum and Shaheen, 2020), intuitionistic fuzzy environment (Zhang, 2012; Cagman

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and Karatas, 2013; Feng *et al.*, 2020), interval-valued fuzzy environment (Xiao *et al.*, 2013; Peng and Yang, 2017; Ali *et al.*, 2020), trapezoidal fuzzy environment (Xiao *et al.*, 2012; Zhang and Zhang, 2013), semantics environment (Yang and Yao, 2020), rough environment (Alcantud *et al.*, 2020), and so on. These studies have widened the scope of application of soft set theory in decision making.

However, the decision-making methods based on the soft sets mentioned above can only deal with decision-making problems with complete and independent information. Typically, decision making is a ranking process about several alternatives with respect to certain criteria. In order to get the exact order and an optimal decision, we may tend to collect information from as many as possible similar attributes. As a result, the corresponding information is redundant and may be incomplete.

Xia *et al.* (2021) proposed a method based on incomplete soft sets to deal with decision-making problems with redundant and incomplete information. However, this method cannot be used to process incomplete soft sets with fuzzy information. There are plenty of cases where evaluation of each alternative just using dichotomous variables is inappropriate. In general, variables in the decision-making problems with uncertainties can be more appropriately described by fuzzy numbers. Accordingly, some effective incomplete information processing methods for fuzzy soft sets were developed.

Zou and Xiao (2008) initiated data analysis approaches for both crisp soft sets and fuzzy soft sets under incomplete information. Especially for fuzzy soft sets, unknown values are predicted based on an average-probability method. However, the prediction is not accurate because all unknown values for a variable are replaced by the same predictive value, in spite of potential differences between objects on the same variable. Deng and Wang (2013) proposed a new prediction method of the incomplete information in fuzzy soft sets, which is based on the notions of the “complete distance” between two objects and the “average dominant degree” between two parameters. Nevertheless, the final prediction results of it may be not between 0 and 1, which is against the property of fuzzy soft sets. Hence, Liu *et al.* (2017) redefined the notion of the dominant degree and provided an improvement of the method (Deng and Wang, 2013). It predicted the unknown values through an adjustable object-parameter approach based on the similarity measures and standardized the predicted values to the interval of 0 and 1 in order to satisfy the property of fuzzy soft sets.

Except for Xia *et al.* (2021), all the above-mentioned methods can be used to deal with incomplete information in decision-making problems in a fuzzy informational environment by filling the unknown or missing data points with predicted values based on a certain algorithm. However, these methods are based on some strict and unnecessary assumptions, and more importantly, they cannot handle redundant information which may disturb our decision making. Therefore, this paper intends to propose a new decision-making method which is capable of processing both incomplete and redundant information based on soft set in a fuzzy informational environment.

This paper is an improvement of the decision-making method of Xia *et al.* (2021) from a crisp informational environment to a fuzzy informational environment. The core of the new method proposed by this paper is the binary relationships between objects in an incomplete fuzzy soft set. In order to make the binary relationships between objects in an incomplete soft set proposed by Xia *et al.* (2021) suitable for a fuzzy environment, this paper improves it and redefines the significance of an attribute set in an incomplete fuzzy soft set. Then the reduct attribute set and the corresponding decision rules can be generated to facilitate decision-making.

In our method, there are no needs to make strict assumptions about the data generating process and to transform the incomplete information into complete one. On the contrary, our method can be applied to the original incomplete data set and essential information can be extracted from redundant information by using the parameter reduction tools. Therefore, information distortion in the process of data transformation can be avoided, and efficiency and precision of decision-making can be improved.

The rest of this paper is organized as follows: Section 2 introduces some preliminary definitions and notions of soft sets and fuzzy soft sets. Based on the definition of incomplete soft sets, Section 3 presents the concept of the incomplete fuzzy soft sets, and analyzes some characteristics of them. In addition, the operations on two incomplete fuzzy soft sets are defined in this section. In Section 4, the incomplete fuzzy soft decision system is introduced, including the concept of the incomplete fuzzy soft decision system, the significance of an attribute subset, reduct attribute set, core attribute and decision rules in incomplete fuzzy soft decision systems. In Section 5, an approach to decision making with incomplete information based on incomplete fuzzy soft sets is proposed, and it is demonstrated by an example. Moreover, some features of the method we proposed are highlighted by a comparative analysis with another relevant method. To illustrate the effectiveness of the method proposed by this paper in practice, Section 6 applies it in the problem of regional food safety evaluation in Chongqing, China. Section 7 concludes the paper.

2. Preliminaries

For the sake of clarity, we briefly introduce basic concepts of soft sets and fuzzy soft sets in the first place.

Suppose that $U = \{h_1, h_2, \dots, h_n\}$ is a common universe set and $A = \{e_1, e_2, \dots, e_m\}$ is a set of parameters.

Definition 1. (*Soft set (cf. Molodtsov, 1999)*) A pair (F, A) is called a soft set over U, where F is a mapping given by $F : A \to P(U)$, and $P(U)$ is the set of all subsets of U.

In other words, the soft set is a parameterized family of subsets of the set U. Every set $F(e)$ ($e \in A$), from this family can be considered as a set of e-approximate elements of soft set (F, A), and it is a subset of U.

Example 1. (*Soft set (adapted from Example 1 of Xia* et al. *(2021))*) Let $U = \{h_1, h_2, \dots, h_6\}$ be a set of houses and suppose that $A = \{e_1, e_2, \dots, e_5\}$ is a set of parameters, which stand for cheap, beautiful, area, location, and in the green surroundings, respectively, each parameter being a word or a sentence. Then

$$U = \{h_1, h_2, \dots, h_6\}$$

and

$$\begin{aligned} A &= \{e_1, e_2, \dots, e_5\} \\ &= \{\text{cheap, beautiful, area, location,} \\ &\quad \text{in the green surroundings}\} \end{aligned}$$

In this case, the soft set (F, A) describes the "attractiveness of the houses" which Mr. X is going to buy and consists of the following five subsets of U:

$$\begin{aligned} (F, A) = \big\{ F(e_1) &= \{\text{cheap houses}\} \\ &= \{h_3, h_5\}, \\ F(e_2) &= \{\text{beautiful houses}\} \\ &= \{h_1, h_2, h_4, h_6\}, \\ F(e_3) &= \{\text{big houses}\} \\ &= \{h_4, h_6\}, \\ F(e_4) &= \{\text{good location houses}\} \\ &= \{h_4, h_5, h_6\}, \\ F(e_5) &= \{\text{in the green suroundings houses}\} \\ &= \{h_1, h_4, h_6\} \big\}. \end{aligned}$$

For ease of data storage and calculation, we can represent (F, A) in the form of Table 1, in which "1" signifies $h_i \in F(e_j)$ ($i = 1, 2, \dots, 6$ and $j = 1, 2, \dots, 5$) and "0" otherwise. ♦

Definition 2. (*Fuzzy soft set (cf. Molodtsov, 1999)*) A pair $(\tilde{F}, A)$ is called a fuzzy soft set over U, where $\tilde{F}$ is a mapping given by $\tilde{F} : A \to \tilde{P}(U)$, and $\tilde{P}(U)$ denotes the set of all fuzzy subsets of U.

Example 2. (*Fuzzy soft set*) Consider Example 1, 'Attractiveness of the houses', under fuzzy information can be described by fuzzy soft set $(\tilde{F}, A)$. It is a set of five fuzzy subsets of houses on U and

$$\begin{aligned} (\tilde{F}, A) = \big\{ \tilde{F}(e_1) &= \{h_1/0.2, h_2/0.3, h_3/0.8, \\ &\quad h_4/0.1, h_5/0.8, h_6/0.2\}, \\ \tilde{F}(e_2) &= \{h_1/0.8, h_2/0.8, h_3/0.2, \\ &\quad h_4/0.7, h_5/0.2, h_6/0.7\}, \\ \tilde{F}(e_3) &= \{h_1/0.4, h_2/0.4, h_3/0.4, \\ &\quad h_4/0.5, h_5/0.2, h_6/0.7\}, \\ \tilde{F}(e_4) &= \{h_1/0.2, h_2/0.3, h_3/0.1, \\ &\quad h_4/0.6, h_5/0.8, h_6/0.8\}, \\ \tilde{F}(e_5) &= \{h_1/0.9, h_2/0.4, h_3/0.4, \\ &\quad h_4/1.0, h_5/0.1, h_6/0.8\} \big\}. \end{aligned}$$

In the same way, a fuzzy soft set can be represented in the form of Table 2. The crisp number 0 or 1 is replaced by the value of a membership function $\mu_{\tilde{A}}(x)$ which associates each element with a real number in the interval [0, 1]. ♦

3. Incomplete soft set

3.1. Concept of incomplete soft sets. Incomplete information and uncertainties in decision-making

Table 1. Tabular representation of (F, A).

U	e_1	e_2	e_3	e_4	e_5
h_1	0	1	0	0	1
h_2	0	1	0	0	0
h_3	1	0	0	0	0
h_4	0	1	1	1	1
h_5	1	0	0	1	0
h_6	0	1	1	1	1

Table 2. Tabular representation of $(\tilde{F}, A)$.

U	e_1	e_2	e_3	e_4	e_5
h_1	0.2	0.8	0.4	0.2	0.9
h_2	0.3	0.8	0.4	0.3	0.4
h_3	0.8	0.2	0.4	0.1	0.4
h_4	0.1	0.7	0.5	0.6	1.0
h_5	0.8	0.2	0.2	0.8	0.1
h_6	0.2	0.7	0.7	0.8	0.8

problems can be described effectively by the frame of incomplete soft sets. The following is the definition of incomplete soft sets and incomplete fuzzy soft sets.

Definition 3. (*Incomplete soft set (cf. Xia* et al.*, 2021)*) A soft set (F, A) is called a complete soft set if and only if $F(e)\,(e \in A)$ does not contain objects with unknown or missing values; otherwise, it is called an incomplete soft set

$$(F', A) = \big\{F'(e) = \{h_i\} \cup \{h_j\}\big\}, \qquad (1)$$

where $e \in A$, $h_i, h_j \in U$, $\{h_i\}$ denotes a set of objects with known information on attribute e, which belong to $F'(e)$ explicitly; and $\{h_j\}$ denotes a set of objects with incomplete information on attribute e, which may or may not belong to $F'(e)$.

Example 3. (*Incomplete soft set (adapted from Example 2 of Xia* et al. *(2021))*) Reconsider Example 1 for demonstration. Suppose that information is lost for object h_2 on attribute e_2, h_3 on e_1 and e_4, h_5 on e_4, and h_6 on e_3. Then the incomplete soft set (F', A) can be defined by

$$\begin{aligned}(F', A) = \{F'(e_1) &= \{\text{cheap houses}\}\\ &= \{h_5\} \cup \{h_3\},\\ F'(e_2) &= \{\text{beautiful houses}\}\\ &= \{h_1, h_4, h_6\} \cup \{h_2\},\\ F'(e_3) &= \{\text{big houses}\}\\ &= \{h_4\} \cup \{h_6\},\\ F'(e_4) &= \{\text{good location houses}\}\\ &= \{h_4, h_6\} \cup \{h_3, h_5\},\\ F'(e_5) &= \{\text{in the green suroundings houses}\}\\ &= \{h_1, h_4, h_6\} \cup \emptyset\}.\end{aligned}$$

Its tabular representation is shown in Table 3, in which "*" is used to indicate that information is incomplete for specific attributes. It should be noted that, in an incomplete soft set, an unknown value does not mean it is useless; instead, it has increased the uncertainty and difficulty in decision making. ♦

According to the definition of incomplete soft sets, we can get the definition of incomplete fuzzy soft sets in the same way.

Table 3. Tabular representation of (F', A).

U	e_1	e_2	e_3	e_4	e_5
h_1	0	1	0	0	1
h_2	0	*	0	0	0
h_3	*	0	0	*	0
h_4	0	1	1	1	1
h_5	1	0	0	*	0
h_6	0	1	*	1	1

Note: * means incomplete information.

Definition 4. (*Incomplete fuzzy soft set*) A fuzzy soft set $(\tilde{F}, A)$ is called a complete fuzzy soft set if, and only if, $\tilde{F}(e)(e \in A)$ does not contain objects with uncertain or unknown features; otherwise, it is an incomplete fuzzy soft set denoted by $(\tilde{F'}, A)$.

Similar to (F', A), each set $\tilde{F'}(e)$ in $(\tilde{F'}, A)$ can also be considered as e-approximate elements of the incomplete fuzzy soft set. It consists of a known-information part and an unknown-information part, and is given by

$$(\tilde{F'}, A) = \big\{\tilde{F'}(e) = \{h_i/\mu_i\} \cup \{h_j\}\big\}, \qquad (2)$$

where $e \in A$, $h_i, h_j \in U$, $\{h_i/\mu_i\}$ is a fuzzy subset of U, μ_i is the value of the membership function of object h_i on attribute e; $\{h_j\}$ denotes the set of objects whose membership function values on attribute e are unknown.

Example 4. (*Incomplete fuzzy soft set*) After applying the same data missing structure of Example 3 to $(\tilde{F'}, A)$ in Example 2, information of houses can be denoted by $(\tilde{F'}, A)$, and

$$\begin{aligned}(\tilde{F'}, A) = \big\{\tilde{F'}(e_1) &= \{h_1/0.2, h_2/0.3, h_4/0.1,\\ &\quad h_5/0.8, h_6/0.2\} \cup \{h_3\},\\ \tilde{F'}(e_2) &= \{h_1/0.8, h_3/0.2, h_4/0.7,\\ &\quad h_5/0.2, h_6/0.7\} \cup \{h_2\},\\ \tilde{F'}(e_3) &= \{h_1/0.4, h_2/0.4, h_3/0.4,\\ &\quad h_4/0.5, h_5/0.2\} \cup \{h_6\},\\ \tilde{F'}(e_4) &= \{h_1/0.2, h_2/0.3,\\ &\quad h_4/0.6, h_6/0.8\} \cup \{h_3, h_5\},\\ \tilde{F'}(e_5) &= \{h_1/0.9, h_2/0.4, h_3/0.4,\\ &\quad h_4/1.0, h_5/0.1, h_6/0.8\} \cup \emptyset\big\}.\end{aligned}$$

Likewise, in $(\tilde{F'}, A)$, the subset $\tilde{F'}(e_1) = \{h_1/0.2, h_2/0.3, h_4/0.1, h_5/0.8, h_6/0.2\} \cup \{h_3\}$, for example, has identified the ranking of price of the five houses h_1, h_2, h_4, h_5, h_6 for certainty according to their membership degrees on attribute e_1, while it is unable to rank house h_3 properly due to its incomplete information on attribute e_1. Its tabular representation is shown in Table 4. ♦

Definition 5. (*Incomplete fuzzy soft subset*) Let $(\tilde{F'}, A)$ and $(\tilde{G'}, B)$ be two incomplete fuzzy soft sets. $(\tilde{F'}, A)$ is said to be an incomplete fuzzy soft subset of $(\tilde{G'}, B)$ and denoted by $(\tilde{F'}, A)\tilde{\subseteq}(\tilde{G'}, B)$ if and only if $A \subseteq B$ and $\forall e \in A, F'(e)\tilde{\subseteq}G'(e)$.

Correspondingly, $(\tilde{G'}, B)$ is said to be an incomplete fuzzy soft superset of $(\tilde{F'}, A)$ if $(\tilde{F'}, A)$ is an incomplete fuzzy soft subset of $(\tilde{G'}, B)$ and denoted by $(G', B)\tilde{\supseteq}(F', A)$.

Example 5. (*Incomplete fuzzy soft subset*) Given two incomplete fuzzy soft sets $(\tilde{F}', A)$ and $(\tilde{G}', B)$, suppose that

$$
\begin{aligned}
U &= \{h_1, h_2, h_3, h_4, h_5, h_6\} \text{ is a set of houses,}\\
A &= \{e_1, e_2\} = \{\text{cheap, beautiful}\},\\
B &= \{e_1, e_2, e_3\} = \{\text{cheap, beautiful, size}\}.
\end{aligned}
$$

Here, A and B are two sets of parameters, and the incomplete fuzzy soft sets $(\tilde{F}', A)$ and $(\tilde{G}', B)$ can be defined respectively by

$$
\begin{aligned}
(\tilde{F}', A) = \big\{\tilde{F}'(e_1) &= \{h_1/0.2, h_2/0.3, h_4/0.1,\\
&\quad h_5/0.8, h_6/0.2\} \cup \{h_3\},\\
\tilde{F}'(e_2) &= \{h_1/0.8, h_3/0.2, h_4/0.7,\\
&\quad h_5/0.2, h_6/0.7\} \cup \{h_2\}\big\}\\
(\tilde{G}', B) &= \big\{\tilde{G}'(e_1)\\
&= \{h_1/0.2, h_2/0.3, h_4/0.1,\\
&\quad h_5/0.8, h_6/0.2\} \cup \{h_3\},\\
\tilde{G}'(e_2) &= \{h_1/0.8, h_3/0.2, h_4/0.7,\\
&\quad h_5/0.2, h_6/0.7\} \cup \{h_2\},\\
\tilde{G}'(e_3) &= \{h_1/0.9, h_2/0.4, h_3/0.4, h_4/1.0,\\
&\quad h_5/0.1, h_6/0.8\} \cup \emptyset\big\}
\end{aligned}
$$

Therefore, we have $(\tilde{F}', A)\tilde{\subseteq}(\tilde{G}', B)$. ♦

Definition 6. (*Equality of incomplete fuzzy soft sets*) $(\tilde{F}', A)$ and $(\tilde{G}', B)$ are two equal incomplete fuzzy soft sets, denoted by $(\tilde{F}', A)\tilde{=}(\tilde{G}', B)$, if and only if $(\tilde{F}', A)\tilde{\subseteq}(\tilde{G}', B)$ and $(\tilde{F}', A)\tilde{\supseteq}(\tilde{G}', B)$.

3.2. Operations on incomplete soft sets.

3.2.1. Binary relationships. In a fuzzy soft set $(\tilde{F}, A)$, the value domain of the mapping function $\tilde{F}$ is a set of all fuzzy subsets of U, which is a class of objects with a continuous membership grade between 0 and 1. Therefore, the binary relationships of incomplete soft sets defined by Xia *et al.* (2021) cannot be used in incomplete fuzzy soft sets, which can only process the incomplete information in a crisp soft set with an extreme membership value of 0 or 1. This section will discuss the binary relationships between two objects in both fuzzy soft sets and incomplete fuzzy soft sets.

The binary relationship between two objects in a fuzzy soft set is discussed first.

Table 4. Tabular representation of $(\tilde{F}', A)$.

U	e_1	e_2	e_3	e_4	e_5
h_1	0.2	0.8	0.4	0.2	0.9
h_2	0.3	*	0.4	0.3	0.4
h_3	*	0.2	0.4	*	0.4
h_4	0.1	0.7	0.5	0.6	1.0
h_5	0.8	0.2	0.2	*	0.1
h_6	0.2	0.7	*	0.8	0.8

Note: * means incomplete information.

Definition 7. (*Indiscernibility relationship*) Let $(\tilde{F}, A)$ be a fuzzy soft set on universe U, and $B \subseteq A$. A binary indiscernibility relationship $\widetilde{\text{IND}}(B)$ on U can be defined as follows:

$$
\begin{aligned}
\widetilde{\text{IND}}(B) = \big\{&(h_i, h_j) \in U \times U\\
&|\tilde{F}(e)/h_i - \tilde{F}(e)/h_j| \le \delta, \forall e \in B\big\}, \quad (3)
\end{aligned}
$$

where δ is an arbitrary small nonnegative number and $|\tilde{F}(e)/h_i - \tilde{F}(e)/h_j|$ is the absolute difference between the membership values of x_i and x_j on e in $(\tilde{F}, A)$; thus

(i) $|\tilde{F}(e)/h_i - \tilde{F}(e)/h_j| \ge 0$;

(ii) $|\tilde{F}(e)/h_i - \tilde{F}(e)/h_j| = |\tilde{F}(e)/h_j - \tilde{F}(e)/h_i|$;

(iii) $|\tilde{F}(e)/h_i - \tilde{F}(e)/h_j| = 0$ if and only if x_i and x_j are the same on the attribute e;

(iv) $|\tilde{F}(e)/h_i - \tilde{F}(e)/h_j| \le |\tilde{F}(e)/h_i - \tilde{F}(e)/h_k| + |\tilde{F}(e)/h_k - \tilde{F}(e)/h_j|$.

From Definition 7, it is clear that a pair (h_i, h_j) of objects from $U \times U$ is indiscernible if and only if the difference between their membership values on each attribute $e(e \in B)$ is less than the nonnegative number δ. That is to say, two objects can be considered as having the same properties with respect to B in reality, if the difference between their membership values on each attribute $e(e \in B)$ is small enough. Thus, the binary indiscernibility relationship $\widetilde{\text{IND}}(B)$ can be used to identify the objects which are in the same class with h_i. Then we can define an indiscernibility class $\tilde{I}_B(h_i)$ from $\widetilde{\text{IND}}(B)$ of the fuzzy soft set $(\tilde{F}, A)$ to describe those objects on U which might be indiscernible to h_i:

$$
\tilde{I}_B(h_i) = \{h_j \in U | (h_i, h_j) \in \widetilde{\text{IND}}(B)\}. \quad (4)
$$

Example 6. (*Indiscernibility relation*) In Example 2, conditions of houses are described by the fuzzy soft set $(\tilde{F}, A)$.

For illustration purposes only, we arbitrarily set $\delta =$

0.2. Then

$$\widetilde{\text{IND}}(A) = \{(h_1,h_1),(h_2,h_2),(h_3,h_3),(h_4,h_4),(h_5,h_5),(h_6,h_6),(h_4,h_6),(h_6,h_4)\}$$

$$\tilde{I}_A(h_1) = \{h_1\},$$
$$\tilde{I}_A(h_2) = \{h_2\},$$
$$\tilde{I}_A(h_3) = \{h_3\},$$
$$\tilde{I}_A(h_4) = \{h_4,h_6\},$$
$$\tilde{I}_A(h_5) = \{h_5\},$$
$$\tilde{I}_A(h_6) = \{h_4,h_6\}.$$

Because of incomplete information, however, the distance between two objects according to Eqn. (3) cannot be evaluated precisely if any of the two contains incomplete information. As a consequence, the indiscernibility relations $\widetilde{\text{IND}}(B)$ of Definition 7 cannot be applied to incomplete fuzzy soft sets. Instead, a similarity relation $\widetilde{\text{SIM}}(B), B \subseteq A$ on U can be defined to describe the objects which may have similar properties with respect to the parameters in B in the incomplete fuzzy soft set $(\tilde{F}', A)$. ♦

Definition 8. (*Similarity relationship*) Let $(\tilde{F}', A)$ be an incomplete fuzzy soft set over a common universe U and $B \subseteq A$. A binary similarity relation $\widetilde{\text{SIM}}(B)$ on U can be defined as follows:

$$\widetilde{\text{SIM}}(B) = \{(h_i,h_j) \in U \times U : |\tilde{F}'(e)/h_i - \tilde{F}'(e)/h_j| \le \delta, \text{ or } \tilde{F}'(e)/h_i = * \text{ or } \tilde{F}'(e)/h_j = *, \forall e \in B\},$$

where * denotes unknown values in incomplete fuzzy soft sets.

Then the similarity class $\tilde{S}_B(h_i)$ of incomplete fuzzy soft set $(\tilde{F}', A)$ defined from $\widetilde{\text{SIM}}(B)$ is given by

$$\tilde{S}_B(h_i) = \{h_j \in U \mid (h_i,h_j) \in \widetilde{\text{SIM}}(B)\}. \quad (5)$$

Example 7. (*Similarity relationship*) Reconsider the incomplete fuzzy soft set $(\tilde{F}', A)$ in Example 4. Set the arbitrarily small nonnegative number $\delta = 0.2$ again. According to Definition 8, we have the following similar relationship with respect to the parameter set A in $(\tilde{F}', A)$:

$$\widetilde{\text{SIM}}(A) = \{(h_1,h_1),(h_2,h_2),(h_3,h_3),(h_4,h_4),(h_5,h_5),(h_6,h_6),(h_2,h_3),(h_3,h_2),(h_4,h_6),(h_6,h_4)\},$$

$$\tilde{S}_A(h_1) = \{h_1\},$$
$$\tilde{S}_A(h_2) = \{h_2,h_3\},$$
$$\tilde{S}_A(h_3) = \{h_2,h_3\},$$
$$\tilde{S}_A(h_4) = \{h_4,h_6\},$$
$$\tilde{S}_A(h_5) = \{h_5\},$$
$$\tilde{S}_A(h_6) = \{h_4,h_6\}.$$

♦

3.2.2. Restricted/relaxed AND operation.

Based on the definitions of incomplete fuzzy soft set and its binary relationships, the operations of the incomplete fuzzy soft sets can be discussed.

Definition 9. (*Restricted AND operation*) Assume that $(\tilde{F}', A)$ is an incomplete fuzzy soft set and $X \subseteq U$. The operation of "$(\tilde{F}', A)$ restricted AND X", denoted by $(\tilde{F}', A)\underset{\sim}{\wedge}X$, is defined as

$$(\tilde{F}', A)\underset{\sim}{\wedge}X = \{h_i \in U \mid \tilde{S}_A(h_i) \subseteq X\}. \quad (6)$$

Definition 10. (*Relaxed AND operation*) Assume that $(\tilde{F}', A)$ is an incomplete fuzzy soft set and $X \subseteq U$. The operation of "$(\tilde{F}', A)$ relaxed AND X", denoted by $(\tilde{F}', A)\tilde{\wedge}X$, is defined as

$$(\tilde{F}', A)\tilde{\wedge}X = \{h_i \in U \mid \tilde{S}_A(h_i) \cap X \ne \emptyset\}. \quad (7)$$

Example 8. (*Restricted/relaxed AND operation*) Reconsider Example 4 and suppose $X = \{h_3,h_5\}$ is a subset of the universe U. Then according to Definition 9 and Definition 10, we can have

$$(\tilde{F}', A)\underset{\sim}{\wedge}X = \{h_5\}, \quad (\tilde{F}', A)\tilde{\wedge}X = \{h_2,h_3,h_5\}.$$

♦

It can be concluded that the result of the restricted AND operation is a set of objects whose similarity class belongs to X with certainty, while the result of the relaxed AND operation is a set of objects whose similarity class possibly belongs to X.

Theorem 1. *Let U be a common universe set and $(\tilde{F}', A)$ be an incomplete soft set. $(\tilde{F}', B_1)$ and $(\tilde{F}', B_2)$ are two incomplete soft subsets of (F', A), and $(\tilde{F}', B_1) \subseteq (\tilde{F}', B_2) \subseteq (F', A)$. For $X \subseteq U$, we have*

$$(\tilde{F}', B_1)\underset{\sim}{\wedge}X \subseteq (\tilde{F}', B_2)\underset{\sim}{\wedge}X, \quad (8)$$

$$(\tilde{F}', B_1)\tilde{\wedge}X \supseteq (\tilde{F}', B_2)\tilde{\wedge}X. \quad (9)$$

Proof. For any $h_i \in U$, if $(\tilde{F}', B_1) \subseteq (\tilde{F}', B_2)$, then $\tilde{S}_{B_1}(h_i) \supseteq \tilde{S}_{B_2}(h_i)$. Assume that $\tilde{S}_{B_1}(h_i) \subseteq X$, where $X \subseteq U$, and then $\tilde{S}_{B_2}(h_i) \subseteq X$. At the same time, there may be $h_j \in U$, $\tilde{S}_{B_1}(h_j) \nsubseteq X$ and $\tilde{S}_{B_2}(h_j) \subseteq X$. Therefore, $(\tilde{F}', B_1)\underset{\sim}{\wedge}X \subseteq (\tilde{F}', B_2)\underset{\sim}{\wedge}X$.

Similarly, $\forall h_i \in U$, if $\tilde{S}_{B_2}(h_i) \cap X \ne \emptyset$, then $\tilde{S}_{B_1}(h_i) \cap X \ne \emptyset$. At the same time, there may be $h_j \in U$, $\tilde{S}_{B_2}(h_j) \cap X = \emptyset$ and $\tilde{S}_{B_1}(h_j) \cap X \ne \emptyset$. Therefore, $(\tilde{F}', B_1)\tilde{\wedge}X \supseteq (\tilde{F}', B_2)\tilde{\wedge}X$. ■

4. Incomplete soft decision system

To develop a decision-making method based on incomplete fuzzy soft sets, it is necessary to introduce incomplete fuzzy soft decision systems and analyze some important characters of them.

4.1. Concept of an incomplete fuzzy soft decision system. Based on the definition of soft decision systems, the definition of incomplete fuzzy soft decision systems can be deduced.

Definition 11. (*Soft decision system*) Suppose that (F, A) and (G, B) are two soft sets over a common universe U and $A \cap B = \emptyset$. Then the triple $((F, A), (G, B), U)$ is defined as a soft decision system over the common universe U, where (F, A) is the condition soft set and (G, B) is the decision soft set.

Definition 12. (*Incomplete fuzzy soft decision system*) A system $((F, A), (G, B), U)$, in which both condition and decision soft sets are crisp soft sets, is called an incomplete crisp soft decision system. If the condition soft set is an incomplete fuzzy soft set, then it is called an incomplete fuzzy soft decision system and denoted by $((\tilde{F'}, A), (G, B), U)$.

Example 9. (*Incomplete fuzzy soft decision system*) Consider again Example 4. Let $(\tilde{F'}, A)$ be the condition soft set, a complete soft set (G, B) be the decision soft set, and

$$(G, B) = \{G(\varepsilon_1) = \{h_3, h_5\},\\ G(\varepsilon_2) = \{h_1, h_2, h_4, h_6\}\},$$

where ε_1 and ε_2 are two attributes in the attribute set B, which denote unattractive house and attractive house respectively. Then the triple $((\tilde{F'}, A), (G, B), U)$ is an incomplete fuzzy soft decision system. ♦

4.2. Significance of an attribute subset. Owing to redundant information, parameter reduction is an important step in decision making. This section presents the definition of the significance of an attribute subset in a fuzzy incomplete soft set, which is an important indicator for parameter reduction.

Definition 13. (*Significance of an attribute set*) Let $((\tilde{F'}, A), (G, B), U)$ be an incomplete fuzzy soft decision system and $C \subseteq A$ be an attribute subset. The significance of C can be defined as

$$\mathrm{SIG}(C) = \Big| \bigcup_{\varepsilon_i \in B} (\tilde{F'}, C) \underset{\sim}{\wedge} G(\varepsilon_i) \Big|, \tag{10}$$

where $|\cdot|$ means the cardinal number of a set.

Example 10. (*Significance of an attribute set*) In Example 9, according to Definition 13, the significance of A in $(\tilde{F'}, A)$ can be computed by

$$\begin{aligned}\mathrm{SIG}(C) &= \Big| \bigcup_{\varepsilon_i \in B} (\tilde{F'}, A) \underset{\sim}{\wedge} G(\varepsilon_i) \Big| \\ &= |(\tilde{F'}, A) \underset{\sim}{\wedge} G(\varepsilon_1) \cup (\tilde{F'}, A) \underset{\sim}{\wedge} G(\varepsilon_2)| \\ &= |\{h_5\} \cup \{h_1, h_4, h_6\}| \\ &= 4.\end{aligned}$$

♦

Theorem 2. *Let $((\tilde{F'}, A), (G, B), U)$ be an incomplete soft decision system and $C_1 \subseteq C_2 \subseteq A$. Then we have*

$$SIG(C_1) \leq SIG(C_2). \tag{11}$$

Proof. We have

$$\mathrm{SIG}(C_1) = \Big| \bigcup_{\varepsilon_i \in B} (\tilde{F'}, C_1) \underset{\sim}{\wedge} G(\varepsilon_i) \Big|$$

$$\mathrm{SIG}(C_2) = \Big| \bigcup_{\varepsilon_i \in B} (\tilde{F'}, C_2) \underset{\sim}{\wedge} G(\varepsilon_i) \Big|.$$

From Theorem 1, for each $\varepsilon_i \in B$,

$$(\tilde{F'}, C_1) \underset{\sim}{\wedge} G(\varepsilon_i) \subseteq (\tilde{F'}, C_2) \underset{\sim}{\wedge} G(\varepsilon_i),$$

$$\Big| \bigcup_{\varepsilon_i \in B} (\tilde{F'}, C_1) \underset{\sim}{\wedge} G(\varepsilon_i) \Big| \leq \Big| \bigcup_{\varepsilon_i \in B} (\tilde{F'}, C_2) \underset{\sim}{\wedge} G(\varepsilon_i) \Big|,$$

i.e., $\mathrm{SIG}(C_1) \leq \mathrm{SIG}(C_2)$. ∎

Theorem 2 shows that the significance of an attribute subset monotonically increases with the number of attributes, which means that adding a new attribute in an attribute subset at least does not decrease the significance of the attribute subset in any incomplete fuzzy soft decision system. This property is very important for parameter reduction.

4.3. Parameter reduction. Based on the definition of the significance of the attribute subset, the reduct attribute set, optimal reduct attribute set, core attribute, and core attribute set can be defined.

Definition 14. (*Reduct attribute set*) Assume that $((\tilde{F'}, A), (G, B), U)$ is a fuzzy incomplete soft decision system and $C \subseteq A$. Then C is a reduct attribute set of $((\tilde{F'}, A), (G, B), U)$ if

$$\mathrm{SIG}(C) = \mathrm{SIG}(A). \tag{12}$$

Definition 15. (*Optimal reduct attribute set*) If for any subset of C, $D \subset C \subseteq A$, $\mathrm{SIG}(D) < \mathrm{SIG}(C) = \mathrm{SIG}(A)$, then C is an optimal reduct attribute set of $((\tilde{F'}, A), (G, B), U)$.

Definitions 14 and 15 offer a good tool to find an (optimal) reduct attribute set. It is based on the definition of the significance of an attribute set, which can help us identify the necessary attributes and the unnecessary attributes through comparing similarities between the soft subset $(\tilde{F'}, C)$ and the decision soft set (G, B). This process can be conducted in an incomplete fuzzy soft set directly according to the definitions of the similarity relationship between objects in incomplete fuzzy soft sets and the restricted AND operation. This does not need to transfer incomplete information into complete one. Therefore, parameter reduction can be realized in decision-making problems with incomplete information under a fuzzy environment directly without any information loss or distortion.

Example 11. (*Reduct attribute set*) Reconsider Example 9. For $((\tilde{F'}, A), (G, B), U)$, according to Definition 13, the significance of attribute subset $C = \{e_2, e_3, e_4, e_5\} \subset A$ in $((\tilde{F'}, A), (G, B), U)$ can be computed as follows:

$$\begin{aligned}(\tilde{F'}, C) = \{\tilde{F'}(e_2) &= \{h_1/0.8, h_3/0.2, h_4/0.7,\\ &\quad h_5/0.2, h_6/0.7\} \cup \{h_2\},\\ \tilde{F'}(e_3) &= \{h_1/0.4, h_2/0.4, h_3/0.4,\\ &\quad h_4/0.5, h_5/0.2\} \cup \{h_6\},\\ \tilde{F'}(e_4) &= \{h_1/0.2, h_2/0.3, h_4/0.6,\\ &\quad h_6/0.8\} \cup \{h_3, h_5\},\\ \tilde{F'}(e_5) &= \{h_1/0.9, h_2/0.4, h_3/0.4,\\ &\quad h_4/1, h_5/0.1, h_6/0.8\} \cup \emptyset\}.\end{aligned}$$

Then

$$\begin{aligned}\widetilde{\mathrm{SIM}}(C) &= \{(h_1, h_1), (h_2, h_2), (h_3, h_3),\\ &\quad (h_4, h_4), (h_5, h_5), (h_6, h_6),\\ &\quad (h_2, h_3), (h_3, h_2), (h_4, h_6), (h_6, h_4)\},\\ \tilde{S}_C(h_1) &= \{h_1\},\\ \tilde{S}_C(h_2) &= \{h_2, h_3\},\\ \tilde{S}_C(h_3) &= \{h_2, h_3\},\\ \tilde{S}_C(h_4) &= \{h_4, h_6\},\\ \tilde{S}_C(h_5) &= \{h_5\},\\ \tilde{S}_C(h_6) &= \{h_4, h_6\}.\\ \widetilde{\mathrm{SIG}}(C) &= \Big| \bigcup_{\varepsilon_i \in B} (\tilde{F'}, C) \underset{\sim}{\wedge} G(\varepsilon_i) \Big|\\ &= |(\tilde{F'}, C) \underset{\sim}{\wedge} G(\varepsilon_1) \cup (\tilde{F'}, C) \underset{\sim}{\wedge} G(\varepsilon_2)|\\ &= |\{h_5\} \cup \{h_1, h_4, h_6\}|\\ &= 4 = \widetilde{\mathrm{SIG}}(A).\end{aligned}$$

According to Definition 14, C is a reduct attribute subset of$((\tilde{F'}, A), (G, B), U)$, but it may be not an optimal reduct attribute set. This is because there may be an attribute subset $D \subset C$ and $\widetilde{\mathrm{SIG}}(D) = \widetilde{\mathrm{SIG}}(A)$. ♦

Based on the definition of the reduction in the incomplete soft decision system, we give the definitions of a core attribute and a core attribute set.

Definition 16. (*Core attribute*) Attribute e is a core attribute of an incomplete fuzzy soft decision system if it belongs to every reduct attribute set of $((\tilde{F'}, A), (G, B), U)$.

Definition 17. (*Core attribute set*) An attribute set $C\,(C \subseteq A)$ is a core attribute set of $((\tilde{F'}, A), (G, B), U)$ if all the elements in C are core attributes of $((\tilde{F'}, A), (G, B), U)$.

4.4. Decision rules. Decision rules of the soft decision system $((F, A), (G, B), U)$ can be established according to the attributes set A as follows:

$$\wedge(e_i, v) \rightarrow \vee(\varepsilon_i, w), \tag{13}$$

where $e_i \in A, \varepsilon_i \in B, v = F(e_i)/h_i, w = G(\varepsilon_i)/h_i$; $\wedge$ means "and"; $\vee$ means "or"; $\wedge(e_i, v)$ signifies the condition part of the rule and $\vee(\varepsilon_i, w)$ stands for the decision part of the rule.

Accordingly, we can get the decision rules from the attributes set on an incomplete fuzzy soft decision system.

Example 12. (*Decision rules of incomplete fuzzy soft decision systems*) In Example 9 , the decision rules of $((\tilde{F'}, A), (G, B), U)$ from A are as follows:

$r_1 : (e_1, 0.2) \wedge (e_2, 0.8) \wedge (e_3, 0.4) \wedge (e_4, 0.2) \wedge (e_5, 0.9)$
$\rightarrow (\varepsilon_2, 1)$ (attractive house);
$r_2 : (e_1, 0.3) \wedge (e_2, *) \wedge (e_3, 0.4) \wedge (e_4, 0.3) \wedge (e_5, 0.4)$
$\rightarrow (\varepsilon_2, 1)$ (attractive house);
$r_3 : (e_1, *) \wedge (e_2, 0.2) \wedge (e_3, 0.4) \wedge (e_4, *) \wedge (e_5, 0.4)$
$\rightarrow (\varepsilon_1, 1)$ (unattractive house);
$r_4 : (e_1, 0.1) \wedge (e_2, 0.7) \wedge (e_3, 0.5) \wedge (e_4, 0.6) \wedge (e_5, 1)$
$\rightarrow (\varepsilon_2, 1)$ (attractive house);
$r_5 : (e_1, 0.8) \wedge (e_2, 0.2) \wedge (e_3, 0.2) \wedge (e_4, *) \wedge (e_5, 0.1)$
$\rightarrow (\varepsilon_1, 1)$ (unattractive house);
$r_6 : (e_1, 0.2) \wedge (e_2, 0.7) \wedge (e_3, *) \wedge (e_4, 0.8) \wedge (e_5, 0.8)$
$\rightarrow (\varepsilon_2, 1)$ (attractive house),

where * denotes incomplete information in $((\tilde{F'}, A), (G, B), U)$.

Obviously, the above rules generated by the attributes set A contain too much redundant information. Decisions cannot be made effectively based on them, and these are not optimal decision rules. But on the upside, optimal decision rules can be derived by an optimal reduct attribute set of a soft information system, because it contains less redundant information. ♦

5. Method based on incomplete soft sets

5.1. Algorithm. This section attempts to apply the incomplete fuzzy soft set developed above to decision-making problems with redundant information and incomplete information, and illustrate it with an example of the house choice. Firstly, we formulate Algorithm 1 for dealing with decision-making problems based on incomplete fuzzy soft sets.

Algorithm 1. Algorithm to solve decision-making problems with redundant and incomplete information under fuzzy environment.

Step 1. Construct an incomplete fuzzy soft decision system $((\tilde{F}', A), (G, B), U)$.

Step 2. Calculate $\widetilde{\mathrm{SIG}}(A)$ according to Definition 13.

Step 3. Calculate $\widetilde{\mathrm{SIG}}(A_i)$, where $A_i \subseteq A$.

Step 4. Find an optimal reduct attribute set of $((\tilde{F}', A), (G, B), U))$ according to Definition 15.

Step 5. Obtain optimal decision rules and make a decision.

5.2. Numerical example. Then we can apply Algorithm 1 to complete the demonstration of Example 4.

In Step 1, an incomplete fuzzy soft decision system $((\tilde{F}', A), (G, B), U))$ is established on the initial data set.

In Step 2, the significance of the attribute set A in $((\tilde{F}', A), (G, B), U))$ is given by

$$\begin{aligned}\mathrm{SIG}(A) &= \Big| \bigcup_{\varepsilon_i \in B} (\tilde{F}', A) \underset{\sim}{\wedge} G(\varepsilon_i) \Big| \\ &= |(\tilde{F}', A) \underset{\sim}{\wedge} G(\varepsilon_1) \cup (\tilde{F}', A) \underset{\sim}{\wedge} G(\varepsilon_2)| \\ &= |\{h_5\} \cup \{h_1, h_4, h_6\}| \\ &= 4.\end{aligned}$$

In Step 3, the significance of each attribute subset in $((\tilde{F}', A), (G, B), U))$ can be calculated by the same way. We get

$$\begin{aligned}C &= A = \{e_1, e_2, e_3, e_4, e_5\}, \\ C_1 &= C - e_1 = \{e_2, e_3, e_4, e_5\}, \\ \widetilde{\mathrm{SIG}}(C_1) &= |\cup_{\varepsilon \in B} (\tilde{F}', C_1) \underset{\sim}{\wedge} G(\varepsilon)| \\ &= |\{h_5\} \cup \{h_1, h_4, h_6\}| \\ &= 4, \\ \widetilde{\mathrm{SIG}}(C_1) &= \widetilde{\mathrm{SIG}}(A), \\ C &= C_1 = \{e_2, e_3, e_4, e_5\}, \\ C_2 &= C - e_2 = \{e_3, e_4, e_5\}, \\ \widetilde{\mathrm{SIG}}(C_2) &= |\cup_{\varepsilon \in B} (\tilde{F}', C_2) \underset{\sim}{\wedge} G(\varepsilon)| \\ &= |\{h_5\} \cup \{h_1, h_4, h_6\}| \\ &= 4, \\ \widetilde{\mathrm{SIG}}(C_2) &= \widetilde{\mathrm{SIG}}(A), \\ C &= C_2 = \{e_3, e_4, e_5\}, \\ C_3 &= C - e_3 = \{e_4, e_5\}, \\ \widetilde{\mathrm{SIG}}(C_3) &= |\cup_{\varepsilon \in B} (\tilde{F}', C_3) \underset{\sim}{\wedge} G(\varepsilon)| \\ &= |\{h_5\} \cup \{h_1, h_4, h_6\}| \\ &= 4, \\ \widetilde{\mathrm{SIG}}(C_3) &= \widetilde{\mathrm{SIG}}(A), \\ C &= C_3 = \{e_4, e_5\}, \\ C_4 &= C - e_4 = \{e_5\}, \\ \widetilde{\mathrm{SIG}}(C_4) &= |\cup_{\varepsilon \in B} (\tilde{F}', C_4) \underset{\sim}{\wedge} G(\varepsilon)| \\ &= |\{h_5\} \cup \{h_1, h_4, h_6\}| \\ &= 4, \\ \widetilde{\mathrm{SIG}}(C_4) &= \widetilde{\mathrm{SIG}}(A), \\ C &= C_4 = \{e_5\}, \\ C_5 &= C - e_5 = \emptyset.\end{aligned}$$

In Step 4, according to Definitions 14 and 15, we can conclude that the attribute subset $C = C_4 = \{e_5\}$ is the optimal reduct attribute set of $((\tilde{F}', A), (G, B), U))$, because $\widetilde{\mathrm{SIG}}(C_4) = \widetilde{\mathrm{SIG}}(A)$, and C_4 is the minimum subset of A.

In Step 5, we can derive the optimal decision rules as follows:

$$\begin{aligned}
&r_1: (e_5, 0.9) \to (\varepsilon_2, 1) \quad \text{(attractive house);} \\
&r_2: (e_5, 0.4) \to (\varepsilon_2, 1) \quad \text{(attractive house);} \\
&r_3: (e_5, 0.4) \to (\varepsilon_1, 1) \quad \text{(unattractive house);} \\
&r_4: e_5, 1.0) \to (\varepsilon_2, 1) \quad \text{(attractive house);} \\
&r_5: (e_5, 0.1) \to (\varepsilon_1, 1) \quad \text{(unattractive house);} \\
&r_6: (e_5, 0.8) \to (\varepsilon_2, 1) \quad \text{(attractive house).}
\end{aligned}$$

In other words, if the house is in a good green surrounding (the membership value at e_5 is close to 1), then the house is attractive, like h_1, h_4 and h_6. If the house is not in a good green surrounding (the membership value at e_5 is close to 0), then the house is unattractive, like h_5. If the house is in a normal green surrounding (the membership value at e_5 is close to 0.5), then the house may be attractive or not, like h_2 and h_3.

5.3. Comparative analysis. Zou and Xiao (2008) developed the average-probability approach to process incomplete information by using fuzzy soft sets (FSSs). This section compares the FSS method with our method based on incomplete fuzzy soft sets (IFSSs) to demonstrate the advantages of the IFSS method to solve decision-making problems with redundant and incomplete information under a fuzzy environment.

5.3.1. Results from FSSs. Let p_e denote the average-probability that an object belongs to $F'(e)$, and

$$\tilde{p}_e = \frac{1}{b} \sum_{h_{ie} \neq * \wedge 1 \leq i \leq m} h_{ie}, \tag{14}$$

where the h_{ie}'s are the entries in the fuzzy soft set, m is the number of objects in universe U and b is the number of objects that belong to $F(e)$ with complete information. Then the cells with incomplete data in the fuzzy soft set can be replaced with p_e.

In Example 4, according to Eqn. (14), it is easy to see that $p_{e_1} = 0.3, p_{e_2} = 0.5, p_{e_3} = 0.4, p_{e_4} = 0.5$, and we can get the choice value of each object. Let $c_{i(\text{avg})}$ be the choice value of an object generated by the method of average probability. For comparison, let $c_{i(0.5)}$, $c_{i(0)}$ and $c_{i(1)}$ stand for the choice values of an object by setting all cells with incomplete data to 0.5, 0 and 1, respectively. As is shown in Table 5, the outcomes based on $c_{i(0)}$ and $c_{i(1)}$ are considerably different from $c_{i(\text{avg})}$, but the results based on $c_{i(0.5)}$ are almost the same as those for $c_{i(\text{avg})}$, except for h_6. That is because all membership values of known objects under e_3 are less than 0.5 and close to 0, which can result in an error if the incomplete data of h_6 on e_3 are replaced by 0.5. From Table 5, we can conclude that h_4 and h_6 are two optimal choices because their choice values are the highest based on the average-probability approach. Here h_1 is an ordinary house, and h_2, h_3, h_5 might be poor choices. These results are different from the classification results of this paper. The method proposed by this paper can classify objects into three classes according to the optimal decision rules: h_1, h_4 and h_6 are attractive houses, h_5 is an unattractive house, and h_2 and h_3 are two uncertain houses, based on their performance in the green surrounding.

5.3.2. Comparison. The differences between the FSS method and the IFSS method can be summarized as follows. First, the approaches to handle incomplete information are different. In the FSS method, the missing membership value of an object is replaced by the average probabilities of all objects with complete information on the attribute, the replacement is made possible by assuming that the membership values of each object follow a normal distribution. On the contrary, in the IFSS method of this paper, the decisions rules are derived based on binary indiscernibility and similarity relations which can be directly applied to attributes with incomplete information. Thus, no unreliable assumptions about the distribution of the membership values to fill the missing data points need to be made. Second, the FSS method does not make full use of the decision values in the process of decision making, and there is no connection between the conditional attributes and the decision attributes. The IFSS proposed by this paper can generate decision rules by making a connection between the conditional parameters and decision values straightforwardly. Third, in the FSS method, the objects can be ranked according to their decision values but cannot be grouped as several classes, because no grouping rules were provided, while this is not a problem in the IFSS method in our study as the grouping rules can be clearly defined.

Table 5. Choice values of the incomplete fuzzy soft set $(\tilde{F'}, A)$.

U	$c_{i(\text{avg})}$	$c_{i(0.5)}$	$c_{i(0)}$	$c_{i(1)}$
h_1	2.5	2.5	2.5	2.5
h_2	1.9	1.9	1.4	2.4
h_3	1.8	2.0	1.0	3.0
h_4	2.9	2.9	2.9	2.9
h_5	1.8	1.8	1.3	2.3
h_6	2.9	3.0	2.5	3.5

6. Application to evaluation of regional food safety

This section describes the application of the proposed decision-making method to evaluate the regional food safety situation of Chongqing, China. We obtained the inspection results of 40 districts (see Section A1 in Appendix for a list of these 40 districts) of Chongqing on 12 attributes (a description of these 12 attributes can be found in Section A2 of Appendix) regarding food safety in 2018. Our research question was how to predict the regional satisfaction level (SL) of local people about food safety issues when we had data about food safety inspection.

All the twelve attributes had values in the interval from 0 to 1. Because there were no missing values in the original data, we randomly set 30% of the observations with one or at most four missing variables to create an incomplete fuzzy data set based on the raw data. 30 districts were randomly selected as the training set and the remaining 10 districts as the testing set. Both sets had 30% observations with missing values. With the corresponding satisfaction of food safety, we built an incomplete fuzzy soft decision system. Then, by using the DM algorithm proposed in this paper, the significance of each attribute subset and the optimal parameter reduction

Table 6. Prediction accuracy of FSS and IFSS.

	FSS	IFSS
Prediction accuracy (%)	60	100

could be obtained.

Our algorithm was programmed and implemented in R, and the FSS method was also applied in the same problem. Table 6 shows the forecasting accuracy of the two methods (an explanation of the forecasting process can be found in Section A3 of Appendix). Although the small testing sample leads to a significant difference between the accuracies of the two methods, and the 100% predictive accuracy of the IFSS method is not convincible, the results can also show that our method outperformed the FSS.

7. Conclusion

Based on the research of Molodtsov, this paper proposed a method to solve decision-making problems which contain incomplete and redundant information based on incomplete fuzzy soft sets. Before investigating the decision method, we presented the concepts of incomplete fuzzy soft sets and incomplete fuzzy soft decision systems. Then incomplete fuzzy soft subsets, incomplete fuzzy soft supersets, and the equality of incomplete fuzzy soft sets were defined. Based on these basic definitions about the incomplete fuzzy soft set, binary relationships (a binary indiscernibility relation and a binary similarity relation) of incomplete fuzzy soft sets were discussed, and some operations such as the restricted/relaxed AND operation on an incomplete fuzzy soft set and a subset of the universe were defined. After that, the definition of the significance of an attribute subset in an incomplete fuzzy soft decision system was proposed. Following this definition, we got the definitions of a reduct attribute set, an optimal reduct attribute set and core attributes of an incomplete fuzzy soft decision system $\big((\tilde{F'}, A), (G, B), U)\big)$. According to the optimal reduct attribute set, an optimal decision rules can be derived. Finally, the incomplete fuzzy soft set based method of MCDM with incomplete information was proposed and illustrated with an example.

The results have demonstrated the capability of the incomplete fuzzy soft set to integrate data, and avoid an information loss or a distortion caused by incomplete and redundant information. A corresponding comparative analysis with Zou and Xiao's research about a data analysis approach to fuzzy soft sets under incomplete information was performed, and the effectiveness of the method proposed by this study was emphasized. The approach proposed by this paper can be applied to a wide range of areas such as feature selection, decision making and forecasting problems.

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Appendix

A1. Districts of Chongqing

There were a total of 40 districts in Chongqing as of 2018 which include Ba'nan, Beibei, Changshou, Dadukou, Dazu, Dianjiang, Fengdu, Fengjie, Fuling, Hechuan, Jiangbei, Jiangjin, Jiulongpo, Kaizhou, Liangping, Liangjiang, Nan'an, Nanchuan, Pengshui, Qianjiang, Rongchang, Shapingba, Shizhu, Tongliang, Wansheng, Wanzhou, Wushan, Wuxi, Wulong, Xiushan, Yongchuan, Youyang, Yubei, Yuzhong, Yunyang, Zhongxian, Tongnan, Bishan, and Qijiang.

A2. Attributes to assess food safety

It should be noted that we did not access to the raw data of food safety inspection at the firm levels in each district of Chongqing, but only a summary of levels of consumer satisfaction on food safety and the inspection results on the twelve attributes at a district level. All these attributes were used in the application analysis in Section 6. A full list of these twelve attributes can be found in Table A1.

The data were provided by the Chongqing Administration for Market Regulation according to the data disclosure agreement between the Administration and the authors and could not be revealed to the public. Therefore, we did not show the data in Appendix.

Table A1. Description of food safety attributes

Attribute name	Description
e_1	Residues of agricultural chemicals and veterinary drugs
e_2	Non-edible substance
e_3	Misuse or overuse of food additives
e_4	Contamination of foods by heavy metals
e_5	Other pollutants
e_6	Microbial contamination
e_7	Biotoxins in food
e_8	Other biological substance
e_9	Food quality index
e_{10}	Functional component in health care products
e_{11}	Labeling
e_{12}	Others

A3. Forecasting satisfactory level of food safety

First, according to the incomplete fuzzy soft decision system, we can derive an optimal reduct attribute. In the case of food safety inspection, the optimal reduct attribute is attribute 6, microbial contamination.

Second, according to the incomplete fuzzy soft decision system, we can use the optimal reduct attribute to classify the training observations into 8 groups. We name these 8 groups as condition groups.

Third, we use the satisfactory level of consumers to local food safety (with values from 0 to 1) as the decision attribute. We set to 0.8 as the threshold. If the consumer's satisfactory level is larger than or equal to 0.8, then the district is classified as satisfied, otherwise, it is classified as unsatisfied. We name these two groups as decision groups.

Fourth, according to the similarity between the condition group and the decision group, we can form 8 decision rules. For example, suppose that condition group A and the satisfied group contain common districts. Furthermore, suppose that district D from the testing set belongs to the domain defined by condition group A. Then according to the decision rules, we can predict that district D would be in the satisfied group.

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