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APPLICATIONS OF THE FRACTIONAL STURM–LIOUVILLE DIFFERENCE PROBLEM TO THE FRACTIONAL DIFFUSION DIFFERENCE EQUATION

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This paper deals with homogeneous and non-homogeneous fractional diffusion difference equations. The fractional operators in space and time are defined in the sense of Grünwald and Letnikov. Applying results on the existence of eigenvalues and corresponding eigenfunctions of the Sturm–Liouville problem, we show that solutions of fractional diffusion difference equations exist and are given by a finite series.

Keywords: anomalous diffusion, fractional diffusion equations, fractional calculus, difference equations.

1. Introduction

The diffusion process is in general understood as spontaneous spreading and permeation of particles from a region of higher concentration to a region of lower concentration. It is an irreversible phenomenon, as a result of which the initially inhomogeneous distribution of matter is evened out. Diffusion is a considerable part of many biological processes, e.g., it can be observed when matter, such as water, ions, and molecules needed for cellular processes, enters and leaves cells. In fact, it was discovered, that the diffusive motion of substances occurring in cell biology is anomalous (Woringer *et al.*, 2020).

Mathematical models appropriate to analyze the complexity of anomalous diffusion assume a non-linear connection between the mean square displacement (MSD) and time (given by a power-law relation), and employ diffusion equations involving fractional (real or complex order) differential operators (Metzler and Klafter, 2000). This is in contrast to the normal diffusion systems, which are characterized by the MSD linear in time and are successfully modeled by integer-order diffusion equations. In this context, fractional diffusion equations, are found to be particularly interesting for researchers and one can notice an increasing number of papers in this subject (see, e.g., Płociniczak and Świtała, 2022; 2018; Płociniczak, 2019; D'Ovidio, 2012; Meerschaert, 2011; Wu *et al.*, 2015; Cresson *et al.*, 2021; Ciesielski *et al.*, 2017). Because of their non-local character, however, the problem of finding exact solutions of most fractional diffusion equations still remains unsolved.

As a consequence, there has been a growing interest in developing numerical schemes for such equations (Hanert and Piret, 2012; Meerschaert and Tadjeran, 2004; Bayrak *et al.*, 2020; Ciesielski *et al.*, 2017). These numerical methods are often based on the Grünwald–Letnikov approximations of the Riemann–Liouville or the Caputo operators. One can substitute the discrete unknowns for the continuous ones and replace fractional derivatives by the discrete Grünwald–Letnikov operators (Podlubny, 1999; Meerschaert and Tadjeran, 2004). Consequently, in this work, we study fractional diffusion difference equations, where the fractional operators in space and time are defined in the sense of Grünwald and Letnikov. Precisely, we apply results on the

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discrete fractional Sturm–Liouville problem in order to prove the existence of solutions of homogeneous and non-homogeneous fractional diffusion difference equations. A similar method was used by Klimek *et al.* (2016), but they considered the fractional diffusion equation that involved fractional derivatives in space and time.

The article is organized as follows. In Section 2 we recall basic definitions of the Grünwald–Letnikov fractional differences and bring back results concerning the discrete fractional Sturm–Liouville problem and the discrete fractional initial value problem. Section 3 is devoted to the homogeneous and non-homogeneous fractional diffusion difference equations; all results presented in this section are original. Finally, in Section 4, we illustrate our results through an example.

2. Preliminaries

In this section, we gather all preliminary definitions and theorems that are needed in the sequel. We present definitions of the Grünwald–Letnikov fractional differences and recall results regarding the discrete fractional Sturm–Liouville problem and the discrete fractional initial value problem (Miller and Ross, 1989; Kaczorek, 2011; Mozyrska *et al.*, 2013; Mozyrska and Girejko, 2013; Almeida *et al.*, 2017; Abdeljawad, 2011; Atici and Eloe, 2009). For a comprehensive treatment of discrete fractional calculus, we refer the reader to the book by Goodrich and Peterson (2015).

2.1. Grünwald–Letnikov fractional differences. Let $\mathbb{T} = \{x_k\}_{k=0,\dots,M} = \{a + kh\}_{k=0,\dots,M}$ be the usual regular partition of the interval $[a, b]$ with $M \geq 2$ and $h = (b - a)/M$. Having in mind that all functions acting from $\mathbb{T}$ to $\mathbb{R}^n$ are continuous, we denote by $C(\mathbb{T}, \mathbb{R}^n)$ the set of all those functions. Moreover, for $\alpha > 0$, we set

$$(w_i^\alpha) := \begin{cases} 1 & \text{if } i = 0 \\ (-1)^i \dfrac{\alpha(\alpha-1)\cdots(\alpha-i+1)}{i!} & \text{if } i = 1, 2, \dots . \end{cases} \tag{1}$$

Observe that if $0 < \alpha < 1$, then $(w_i^\alpha) < 0$ and $\lim_{i\to\infty}(w_i^\alpha) = 0$ for $i = 1, 2, \dots$.

Definition 1. Let $y \in C(\mathbb{T}, \mathbb{R}^n)$. The left Grünwald–Letnikov fractional difference of order $\alpha > 0$ of function y is defined by

$$\Delta_{a+}^\alpha\, y(x_k) := \frac{1}{h^\alpha} \sum_{r=0}^{k} (w_r^\alpha) y(x_{k-r}) \tag{2}$$

for $k = 1, \dots, M$, while the right Grünwald–Letnikov fractional difference of order $\alpha > 0$ of function y is defined by

$$\Delta_{b-}^\alpha y(x_k) := \frac{1}{h^\alpha} \sum_{r=0}^{M-k} (w_r^\alpha) y(x_{k+r}) \tag{3}$$

for $k = 0, \dots, M - 1$.

Remark 1. Note that

$$\Delta_{a+}^\alpha : C(\mathbb{T}; \mathbb{R}^n) \to C(\mathbb{T} \setminus \{a\}; \mathbb{R}^n)$$

(resp. $\Delta_{b-}^\alpha : C(\mathbb{T}; \mathbb{R}^n) \to C(\mathbb{T} \setminus \{b\}; \mathbb{R}^n)$).

Remark 2. For $\alpha = 1$, Δ_{a+}^α and Δ_{b-}^α recover the backward and forward differences, i.e.,

$$\Delta_{a+}^1 y(x_k) = \frac{y(x_k) - y(x_{k-1})}{h} = \nabla_h y(x_k),$$

for $k = 1, \dots, M$, and

$$\Delta_{b-}^1 y(x_k) = \frac{y(x_k) - y(x_{k+1})}{h} = -\Delta_h y(x_k).$$

for $k = 0, \dots, M - 1$.

Remark 3. In Definition 1 we consider the Grünwald–Letnikov fractional differences on the finite set $\mathbb{T}$. However, analogous definitions can be formulated for functions acting on $h\mathbb{N} := \{hn : n \in \mathbb{N}\} = \{0, h, 2h, \dots\}$ for $h > 0$ (Kaczorek, 2011; Mozyrska and Wyrwas, 2015). In this case, we have

$$\begin{aligned} \Delta_{a+}^\alpha y(x_k) &= \frac{1}{h^\alpha} \sum_{r=0}^{k} (w_r^\alpha) y(x_{k-r}) \\ &= \frac{1}{h^\alpha} \Big(y(x_k) + \sum_{r=1}^{k} (w_r^\alpha) y(x_{k-r}) \Big). \end{aligned}$$

Note that as $\alpha \to 1$, the first two coefficients in the above sum have the highest absolute values and the remaining coefficients rapidly converge to 0. As $\alpha \to 0$, the first coefficient (being 1) in the above sum has the highest absolute value and the remaining coefficients rapidly converge to 0 (Ostalczyk, 2015). Obviously, the Grünwald–Letnikov fractional difference of order 0 of y is a simply input function y.

We also define the left and right Grünwald–Letnikov fractional sums of order $\alpha > 0$ by replacing α by $-\alpha$ in (2) and (3).

Definition 2. For $y \in C(\mathbb{T}, \mathbb{R}^n)$ the left Grünwald–Letnikov fractional sum of order $\alpha > 0$ of function y is given by

$$\Delta_{a+}^{-\alpha} y(x_k) := \frac{1}{h^\alpha} \sum_{r=0}^{k} (w_r^{-\alpha}) y(x_{k-r})$$

for $k = 1, \dots, M$, while the right Grünwald–Letnikov fractional sum of order $\alpha > 0$ of function y by

$$\Delta_{b-}^{-\alpha} y(x_k) := \frac{1}{h^\alpha} \sum_{r=0}^{M-k} (w_r^{-\alpha}) y(x_{k+r})$$

for $k = 0, \dots, M-1$.

Now, following Ostalczyk (2008), we present definitions of the partial Grünwald–Letnikov fractional differences in a two-dimensional finite domain. Let $\mathbb{T}_1 = \{x_{k_1}\}_{k_1=0,\dots,N} = \{a_1 + k_1 h_1\}_{k_1=0,\dots,N}$ and $\mathbb{T}_2 = \{x_{k_2}\}_{k_2=0,\dots,M} = \{a_2 + k_2 h_2\}_{k_2=0,\dots,M}$ be the usual regular partitions of the intervals $[a_1, b_1]$ and $[a_2, b_2]$, respectively, with $M, N \geq 2$ and $h_1 = (b_1 - a_1)/N$, $h_2 = (b_2 - a_2)/M$. Moreover, we set $\mathbb{D} = \mathbb{T}_1 \times \mathbb{T}_2 = \{(x_{k_1}, x_{k_2}) : x_{k_1} \in \mathbb{T}_1 \wedge x_{k_2} \in \mathbb{T}_2\}$, which is a complete metric space with the metric defined by

$$\begin{aligned} &d((x_{k_1}, x_{k_2}), (x'_{k_1}, x'_{k_2})) \\ &\qquad = \sqrt{(x'_{k_1} - x_{k_1})^2 + (x'_{k_2} - x_{k_2})^2} \end{aligned}$$

for $(x_{k_1}, x_{k_2}), (x'_{k_1}, x'_{k_2}) \in \mathbb{D}$. Given $\varepsilon > 0$, we define the ε-neighborhood of (t'_{k_1}, t'_{k_2}) by

$$\begin{aligned} &U_\varepsilon(x'_{k_1}, x'_{k_2}) \\ &:= \left\{(x_{k_1}, x_{k_2}) \in \mathbb{D} : d((x_{k_1}, x_{k_2}), (x'_{k_1}, x'_{k_2})) < \varepsilon\right\}. \end{aligned}$$

Definition 3. Assume that $y \in C(\mathbb{D}, \mathbb{R}^n)$. The left Grünwald–Letnikov partial fractional difference of order $\alpha > 0$ with respect to x_{k_1} of function y is defined by

$$\Delta_{a_1+,k_1}^{\alpha} y(x_{k_1}, x_{k_2}) := \frac{1}{h_1^\alpha} \sum_{r=0}^{k_1} (w_r^\alpha) y(x_{k_1-r}, x_{k_2}),$$

$k_1 = 1, \dots, N$, while

$$\Delta_{a_2+,k_2}^{\beta} y(x_{k_1}, x_{k_2}) := \frac{1}{h_2^\beta} \sum_{r=0}^{k_2} (w_r^\beta) y(x_{k_1}, x_{k_2-r}),$$

$k_2 = 1, \dots, M$, denotes the left Grünwald–Letnikov partial fractional difference of order $\beta > 0$ with respect to x_{k_2} of function y.

Definition 4. Assume that $y \in C(\mathbb{D}, \mathbb{R}^n)$. The right Grünwald–Letnikov partial fractional difference of order $\alpha > 0$ with respect to x_{k_1} of function y is defined by

$$\Delta_{b_1-,k_1}^{\alpha} y(x_{k_1}, x_{k_2}) := \frac{1}{h_1^\alpha} \sum_{r=0}^{N-k_1} (w_r^\alpha) y(x_{k_1+r}, x_{k_2}),$$

$k_1 = 0, \dots, N-1$, while

$$\Delta_{b_2-,k_2}^{\beta} y(x_{k_1}, x_{k_2}) := \frac{1}{h_2^\beta} \sum_{r=0}^{M-k_2} (w_r^\beta) y(x_{k_1}, x_{k_2+r}),$$

$k_2 = 0, \dots, M-1$ denotes the right Grünwald–Letnikov partial fractional difference of order $\beta > 0$ with respect to x_{k_2} of function y.

Remark 4. Note that in Definitions 3 and 4 the Grünwald–Letnikov partial fractional differences are considered on the finite sets $\mathbb{T}_1$ and $\mathbb{T}_2$. Still such definitions can be formulated for infinite sets $h_1\mathbb{N}$ and $h_2\mathbb{N}$ with $h_1, h_2 > 0$ (see Remark 3).

2.2. Fractional Sturm–Liouville difference equation. Let us recall the important result, proved by Almeida *et al.* (2017), concerning the existence of eigenvalues and eigenfunctions of the following Sturm–Liouville fractional difference equation:

$$\begin{aligned} \Delta_{b-}^{\alpha}(p(x_l)\Delta_{a+}^{\alpha} y(x_l)) + q(x_l) y(x_l) = \lambda y(x_l), \\ l = 1, \dots M-1, \quad (4) \end{aligned}$$

with boundary conditions

$$y(x_0) = y(x_M) = 0, \tag{5}$$

where $p(x_i) > 0$ and $q(x_i)$ are defined and real valued for all x_i, $i = 0, \dots M$, and $\lambda \in \mathbb{R}$ is a parameter. Precisely, the following result holds, see Theorem 2.4 by Almeida *et al.* (2017).

Theorem 1. *The Sturm–Liouville problem* (4)–(5) *has* $M - 1$ *real eigenvalues denoted by*

$$\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_{M-1}.$$

The corresponding eigenfunctions

$$y_1, y_2, \dots, y_{M-1} : \{x_1, \dots, x_{M-1}\} \to \mathbb{R},$$

are mutually orthogonal: if $i \neq j$, *then*

$$\langle y_i, y_j \rangle := \sum_{l=1}^{M-1} y_i(x_l) y_j(x_l) = 0,$$

and they span $\mathbb{R}^{M-1}$*: any vector* $\varphi = (\varphi(x_l))_{l=1}^{M-1} \in \mathbb{R}^{M-1}$ *has a unique expansion*

$$\varphi(x_l) = \sum_{i=1}^{M-1} c_i y_i(x_l), \quad 1 \leq l \leq M-1.$$

The coefficients c_i *are given by*

$$c_i = \frac{\langle \varphi, y_i \rangle}{\langle y_i, y_i \rangle}.$$

2.3. Discrete fractional initial value problem. For the Grünwald–Letnikov fractional difference operator one can formulate the following results (see, e.g., Kaczorek, 2011, Theorem 1.2; Mozyrska and Girejko, 2013, Proposition 5.5).

Theorem 2. *Let $\beta \in (0,1]$. The initial value problem*

$$\Delta^{\beta}_{0+,k} T(t_{k+1}) = -\lambda T(t_k), \tag{6}$$
$$T(0) = c_0, \quad c_0 \in \mathbb{R}, \tag{7}$$

where $k = 0,1,2,\dots$, has the unique solution given by

$$T(t_{k+1}) = c_0 \Phi_{\beta,\lambda}(t_{k+1}) \tag{8}$$

and the transition function $\Phi_{\beta,\lambda}$ is determined by the recurrence formula

$$\Phi_{\beta,\lambda}(t_{k+1}) = (\beta - \lambda h^{\beta})\Phi_{\beta,\lambda}(t_k) - \sum_{s=2}^{k+1} (w^{\beta}_s)\Phi_{\beta,\lambda}(t_{k-s+1}), \tag{9}$$

$k = 0,1,2,\dots$, with $\Phi_{\beta,\lambda}(0) = 1$.

The next theorem states that a unique solution to the non-homogeneous initial value problem exists and can be determined by a recurrence formula.

Theorem 3. *Let $\beta \in (0,1]$. Then a unique solution to the non-homogeneous initial value problem*

$$\Delta^{\beta}_{0+,k} d(t_{k+1}) = -\lambda d(t_k) + A(t_k), \tag{10}$$
$$d(0) = c_0, \quad c_0 \in \mathbb{R}, \tag{11}$$

where $k = 0,1,2,\dots$, exists and is given by

$$d(t_{k+1}) = c_0 \Psi_{\beta,\lambda}(t_{k+1}), \tag{12}$$

where the transition function $\Psi_{\beta,\lambda}$ is determined by the recurrence formula

$$\Psi_{\beta,\lambda}(t_{k+1}) = (\beta - \lambda h^{\beta})\Psi_{\beta,\lambda}(t_k) + h^{\beta}A(t_k) - \sum_{s=2}^{k+1} (w^{\beta}_s)\Psi_{\beta,\lambda}(t_{k-s+1}), \tag{13}$$

$k = 0,1,2,\dots$, with $\Psi_{\beta,\lambda}(0) = 1$.

Proof. By the definition of the operator $\Delta^{\beta}_{0+,k}$ we have

$$\frac{1}{h^{\beta}} \sum_{r=0}^{k+1} (w^{\beta}_r) d(t_{k+1-r}) = -\lambda d(t_k) + A(t_k)$$

and, consequently,

$$d(t_{k+1}) = (\beta - \lambda h^{\beta}) d(t_k) + h^{\beta} A(t_k) - \sum_{r=2}^{k+1} (w^{\beta}_r) d(t_{k+1-r}).$$

∎

For a treatment of more general cases of fractional difference linear systems, we refer the reader to the works by Kaczorek (2011; 2018; 2019) and Ostalczyk (2012).

3. Main results

In this section, we prove that solutions to the homogeneous and non-homogeneous diffusion difference equations exist and are given by finite series. In what follows, suppose that $h_1 > 0$, $a, b \in \mathbb{R}$, $a < b$, and $h_2 = (b-a)/M$, with $M \geq 2$. Moreover, let $\tilde{\mathbb{T}}_1 = \{t_k\}_{k=0,1,\dots} = \{0 + kh_1\}_{k=0,1,\dots}$, $\tilde{\mathbb{T}}_2 = \{x_l\}_{l=0,1,\dots,M} = \{a + lh_2\}_{l=0,1,\dots,M}$ be the usual regular partition of the interval $[a,b]$ and $\tilde{\mathbb{D}} = \tilde{\mathbb{T}}_1 \setminus \{t_0\} \times \tilde{\mathbb{T}}_2$.

3.1. Homogeneous fractional diffusion difference equation.

Consider the following fractional diffusion difference equation:

$$\Delta^{\beta}_{0+,k} u(t_{k+1}, x_l) = -\Delta^{\alpha}_{b-,l}(p(x_l)\Delta^{\alpha}_{a+,l} u(t_k, x_l)) - q(x_l) u(t_k, x_l), \quad k = 0,1,2,\dots, \quad l = 1,\dots M-1, \tag{14}$$

subject to the boundary and initial conditions

$$u(t_k, a) = u(t_k, b) = 0, \qquad k = 1,2,\dots, \tag{15}$$
$$u(0, x_l) = f(x_l), \qquad l = 0,1,\dots,M, \tag{16}$$

where $\alpha, \beta \in (0,1]$ and $f \in C(\tilde{\mathbb{T}}_2, \mathbb{R})$ is such that $f(a) = f(b) = 0$.

Theorem 4. *Fractional diffusion difference equation* (14) *with boundary and initial conditions* (15) *and* (16) *has a solution $u : \tilde{\mathbb{T}}_1 \times \tilde{\mathbb{T}}_2 \to \mathbb{R}$ given by the following sum:*

$$u(t_{k+1}, x_l) = \sum_{n=1}^{M-1} \frac{\langle f, y_n \rangle}{\langle y_n, y_n \rangle} \Phi_{\beta,\lambda_n}(t_{k+1}) y_n(x_l), \tag{17}$$

where $\lambda_1, \lambda_2, \dots, \lambda_{M-1}$ are the eigenvalues, $y_1, y_2, \dots, y_{M-1}$ are the corresponding eigenfunctions of the discrete fractional Sturm–Liouville problem, and Φ_{β,λ_n} are the transition functions corresponding to the discrete initial value problem.

Proof. The proof is based on separation of variables, i.e., we expect the particular solution of (14)–(16) to have the following form:

$$u(t_{k+1}, x_l) = T(t_{k+1}) y(x_l), \quad k = 0,1,2,\dots, \quad l = 1,2,\dots,M-1. \tag{18}$$

Substitute (18) into (14). Then

$$\frac{1}{T(t_k)} \Delta^{\beta}_{0+,k} T(t_{k+1}) = -\frac{1}{y(x_l)} \left(\Delta^{\alpha}_{b-,l}(p(x_l)\Delta^{\alpha}_{a+,l} y(x_l)) + q(x_l) y(x_l) \right), \tag{19}$$

which is satisfied for all t_{k+1} and all x_l. Note that the left-hand side of (19) does not depend on x_l and the right-hand side of (19) does not depend on t_{k+1}. Hence, each side of (19) must be a constant. We will write this separation constant as $-\lambda$, $\lambda \in \mathbb{R}$ (the minus sign is for convenience). Then

$$\begin{aligned}&\frac{1}{T(t_k)}\Delta^{\beta}_{0+,k}T(t_{k+1})\\&= -\frac{1}{y(x_l)}\left(\Delta^{\alpha}_{b-,l}(p(x_l)\Delta^{\alpha}_{a+,l}y(x_l)) + q(x_l)y(x_l)\right)\\&= -\lambda.\end{aligned}$$

We get the following two fractional difference equations depending separately on t_k and x_l:

$$\Delta^{\beta}_{0+,k}T(t_{k+1}) = -\lambda T(t_k), \tag{20}$$

$$\Delta^{\alpha}_{b-,l}(p(x_l)\Delta^{\alpha}_{a+,l}y(x_l)) + q(x_l)y(x_l) = \lambda y(x_l),\qquad y(a) = y(b) = 0. \tag{21}$$

Note that, by Theorem 2, Eqn. (20) has a solution given by

$$T(t_{k+1}) = c_0\Phi_{\beta,\lambda}(t_{k+1}), \quad k = 0, 1, 2, \dots. \tag{22}$$

Furthermore, note that (21) is the Sturm–Liouville problem of the form (4) and (5). Consequently, by Theorem 1, there exists a non-decreasing sequence of $M-1$ eigenvalues $\lambda_1 \leq \cdots \leq \lambda_{M-1}$ and the corresponding sequence of mutually orthogonal eigenfunctions $y_1(x_l), \dots, y_{M-1}(x_l)$ for (4) and (5). Hence

$$u(t_{k+1}, x_l) = c_n\Phi_{\beta,\lambda_n}(t_{k+1})y_n(x_l),\qquad n = 1, \dots, M-1,$$

and plugging to (18), we obtain

$$u(t_{k+1}, x_l) = \sum_{n=1}^{M-1} c_n\Phi_{\beta,\lambda_n}(t_{k+1})y_n(x_l).$$

Coefficients c_n can be determined using the mutual orthogonality of $y_1(x_l), \dots, y_{M-1}(x_l)$ and the initial condition (16). Precisely, we have

$$f(x_l) = u(0, x_l) = \sum_{n=1}^{M-1} c_n y_n(x_l), \quad l = 1, \dots, M-1. \tag{23}$$

Multiplying (23) by $y_m(x_l)$, $m = 1, \dots, M-1$ and summing the results we obtain

$$\sum_{l=1}^{M-1} f(x_l)y_m(x_l) = \sum_{n=1}^{M-1} c_n \sum_{l=1}^{M-1} y_n(x_l)y_m(x_l).$$

Finally, using the orthogonality condition, we get

$$\langle f, y_m\rangle = \sum_{n=1}^{M-1} c_n \langle y_n, y_m\rangle = c_m \langle y_m, y_m\rangle,$$

which means that $c_m = \langle f, y_m\rangle/\langle y_m, y_m\rangle$. ■

Note that, if we choose $\beta = 1$, then the following can be easily deduced from Theorem 4.

Corollary 1. *Consider the fractional diffusion difference equation*

$$\begin{aligned}&\nabla_{h,k}u(t_{k+1}, x_l)\\&= -\Delta^{\alpha}_{b-,l}(p(x_l)\Delta^{\alpha}_{a+,l}u(t_k, x_l)) - q(x_l)u(t_k, x_l),\\&\qquad k = 0, 1, 2, \dots, \quad l = 1, \dots M-1,\end{aligned} \tag{24}$$

subject to boundary and initial conditions (15) *and* (16). *Then Eqn.* (24) *subject to* (15) *and* (16) *has a solution* $u : \tilde{\mathbb{T}}_1 \times \tilde{\mathbb{T}}_2 \to \mathbb{R}$ *given by*

$$u(t_k, x_l) = \sum_{n=1}^{M-1} \frac{\langle f, y_n\rangle}{\langle y_n, y_n\rangle}(1-\lambda_n h)^k y_n(x_l),$$

where $\lambda_1, \dots, \lambda_{M-1}$ *are the eigenvalues and* $y_1, \dots, y_{M-1}$ *are the corresponding eigenfunctions of the discrete fractional Sturm–Liouville problem.*

Proof. Set $\beta = 1$ in Theorem 4. Then the solution to (24) subject to (15) and (16) is given by

$$u(t_{k+1}, x_l) = \sum_{n=1}^{M-1} \frac{\langle f, y_n\rangle}{\langle y_n, y_n\rangle}\Phi_{1,\lambda_n}(t_{k+1})y_n(x_l),$$

where $\Phi_{1,\lambda_n}(t_{k+1})$ is the transition function satisfying the recurrence formula

$$\Phi_{1,\lambda_n}(t_{k+1}) = (1-\lambda_n h)\Phi_{1,\lambda_n}(t_k),$$

with $\Phi_{1,\lambda_n}(0) = 1$. Therefore, $\Phi_{1,\lambda_n}(t_{k+1}) = (1-\lambda_n h)^{k+1}$. ■

3.2. Non-homogeneous fractional diffusion difference equation. In this section, we consider the following non-homogeneous fractional diffusion difference equation:

$$\begin{aligned}\Delta^{\beta}_{0+,k}u(t_{k+1}, x_l) &= -\Delta^{\alpha}_{b-,l}(p(x_l)\Delta^{\alpha}_{a+,l}u(t_k, x_l))\\&\quad - q(x_l)u(t_k, x_l) + g(t_k, x_l),\\&\qquad k = 0, 1, 2, \dots, \quad l = 1, \dots M-1,\end{aligned} \tag{25}$$

subject to boundary and initial conditions (15) and (16). We assume that $\alpha, \beta \in (0, 1]$ and that the function $g \in C(\tilde{\mathbb{T}}_1 \times \tilde{\mathbb{T}}_2, \mathbb{R})$ is given as a series i.e.,

$$g(t_k, x_l) = \sum_{n=1}^{M-1} A_n(t_k)y_n(x_l).$$

We look for a solution of Eqn. (25) subject to (15) and (16) in the form of the series

$$u(t_{k+1}, x_l) = \sum_{n=1}^{M-1} d_n(t_{k+1}) y_n(x_l), \tag{26}$$

where $y_1, \ldots, y_{M-1}$ are the orthogonal eigenfunctions of the discrete fractional Sturm–Liouville problem. Substituting (26) into (25), we arrive at the following system of difference equations:

$$\Delta^{\beta}_{0+,k}(d_n(t_{k+1})) = -\lambda d_n(t_k) + A_n(t_k), \quad n = 1, \ldots, M-1. \tag{27}$$

One can easily check that the solution to (27) is given by

$$d_n(t_{k+1}) = c_0 \Psi_{\beta,\lambda_n}(t_{k+1}), \quad n = 1, \ldots, M-1.$$

Hence

$$u(t_{k+1}, x_l) = \sum_{n=1}^{M-1} c_0 \Psi_{\beta,\lambda_n}(t_{k+1}) y_n(x_l).$$

Now, using similar arguments as in the proof of Theorem 4, we get

$$u(t_{k+1}, x_l) = \sum_{n=1}^{M-1} \frac{\langle f, y_n \rangle}{\langle y_n, y_n \rangle} \Psi_{\beta,\lambda_n}(t_{k+1}) y_n(x_l). \tag{28}$$

Accordingly, we have just proved the following result.

Theorem 5. *Equation* (25) *subject to* (15) *and* (16) *has a solution* $u : \tilde{\mathbb{T}}_1 \times \tilde{\mathbb{T}}_2 \to \mathbb{R}$ *given by the sum* (28) *where* $\lambda_1, \ldots, \lambda_{M-1}$ *are the eigenvalues and* $y_1, \ldots, y_{M-1}$ *are the corresponding eigenfunctions of the discrete fractional Sturm–Liouville problem.*

Corollary 2. *Consider the following non-homogeneous fractional diffusion difference equation:*

$$\begin{aligned} \nabla_{h,k} u(t_{k+1}, x_l) = &-\Delta^{\alpha}_{b-,l}(p(x_l)\Delta^{\alpha}_{a+,l} u(t_k, x_l)) \\ &- q(x_l) u(t_k, x_l) + g(t_k, x_l), \\ &k = 0, 1, 2, \ldots, \quad l = 1, \ldots M-1, \end{aligned} \tag{29}$$

subject to boundary and initial conditions (15) *and* (16). *Then Eqn.* (29) *subject to* (15) *and* (16) *has a solution* $u : \tilde{\mathbb{T}}_1 \times \tilde{\mathbb{T}}_2 \to \mathbb{R}$ *given by the following sum:*

$$\begin{aligned} u(t_{k+1}, x_l) = &\sum_{n=1}^{M-1} \frac{\langle f, y_n \rangle}{\langle y_n, y_n \rangle} \Big((1 - \lambda_n h)^{k+1} \\ &+ h \sum_{r=0}^{k} (1 - \lambda_n h)^{k-r} A_n(t_r) \Big) y_n(x_l). \end{aligned} \tag{30}$$

where $\lambda_1, \ldots, \lambda_{M-1}$ *are the eigenvalues and* $y_1, \ldots, y_{M-1}$ *are the corresponding eigenfunctions of the discrete fractional Sturm–Liouville problem.*

Proof. If we choose $\beta = 1$ in Theorem 5, then the solution to (29) subject to (15) and (16) is given by

$$u(t_{k+1}, x_l) = \sum_{n=1}^{M-1} \frac{\langle f, y_n \rangle}{\langle y_n, y_n \rangle} \Psi_{1,\lambda_n}(t_{k+1}) y_n(x_l),$$

where $\Psi_{1,\lambda_n}(t_{k+1})$ is the transition function defined recursively by

$$\Psi_{1,\lambda_n}(t_{k+1}) = (1 - \lambda_n h)\Psi_{1,\lambda_n}(t_k) + h A_n(t_k), \tag{31}$$

with $\Psi_{1,\lambda_n}(0) = 1$. According to Saber Elaydi (2005, formula (1.2.5)), Eqn. (31) is satisfied by

$$\begin{aligned} \Psi_{1,\lambda_n}(t_{k+1}) = &(1 - \lambda_n h)^{k+1} \\ &+ h \sum_{r=0}^{k} (1 - \lambda_n h)^{k-r} A_n(t_r). \end{aligned}$$

∎

4. Illustrative examples

Consider the fractional difference diffusion equation (14) subject to the boundary and initial conditions (15) and (16), with $\tilde{\mathbb{T}}_1 = \{t_k\}_{k=0,1,\ldots} = \{0 + kh\}_{k=0,1,\ldots}$, $M = 4$, $a = 0$, $b = 4$, $p(x_l) = 1$, $q(x_l) = 0$, $f(x_l) = 4 - (x_l - 2)^2$, and $\alpha = 1$. Specifically, we analyze the following problem:

$$\Delta^{\beta}_{0+,k} u(t_{k+1}, x_l) = \Delta_{1,l}(\nabla_{1,l} u(t_k, x_l)), \quad k = 0, 1, 2, \ldots, \quad l = 1, 2, 3, \tag{32}$$

$$u(t_k, 0) = u(t_k, 4) = 0, \quad k = 1, 2, \ldots, \tag{33}$$

$$u(0, x_l) = 4 - (x_l - 2)^2, \quad l = 0, 1, 2, 3, 4. \tag{34}$$

By Theorem 4, the solution to (32)–(34) is given by the sum

$$u(t_k, x_l) = \sum_{n=1}^{3} \frac{\langle f, y_n \rangle}{\langle y_n, y_n \rangle} \Phi_{\beta,\lambda_n}(t_{k+1}) y_n(x_l), \tag{35}$$

where $\lambda_1, \lambda_2, \lambda_3$ are the eigenvalues and y_1, y_2, y_3 are the corresponding eigenfunctions of the following Sturm–Liouville problem:

$$\Delta_1(\nabla_1 y(x_l)) = -\lambda y(x_l), \quad l = 1, 2, 3, \tag{36}$$

$$y(x_0) = y(x_4) = 0. \tag{37}$$

Using the definitions of forward and backward differences, Eqn. (36) can be rewritten as

$$y(x_{l+1}) + y(x_{l-1}) = (2 - \lambda) y(x_l), \quad l = 1, 2, 3.$$

Hence

- for $l = 1$, we have

$$y(x_2) = (2 - \lambda)y(x_1),$$

- for $l = 2$, we have

$$y(x_3) = ((2 - \lambda)^2 - 1)y(x_1),$$

- for $l = 3$, we have

$$y(x_4) = (2 - \lambda)\left(((2 - \lambda)^2 - 1) - 1\right)y(x_1).$$

Since $y(x_4) = 0$, we have $y(x_1) = 0$ or $(2-\lambda)((2-\lambda)^2 - 2) = 0$. If $y(x_1) = 0$, then the solution to problem (36) and (37) is trivial. Therefore, for $y(x_1) = c \in \mathbb{R}\setminus\{0\}$, the eigenvalues of (36) and (37) are given by $\lambda_1 = -\sqrt{2} + 2$, $\lambda_2 = 2$, $\lambda_3 = \sqrt{2} + 2$ and the corresponding eigenfunctions by $y_1(x_l) = \sin\frac{l\pi}{4}$, $y_2(x_l) = \sin\frac{l\pi}{2}$, $y_3(x_l) = \sin\frac{3l\pi}{4}$, $l = 1, 2, 3$. Determining $\langle y_1, y_1\rangle = 2c^2$, $\langle y_2, y_2\rangle = 2c^2$, $\langle y_3, y_3\rangle = 2c^2$ and $\langle f, y_1\rangle = (4 + 3\sqrt{2})c$, $\langle f, y_2\rangle = 0$, $\langle f, y_3\rangle = (-4 + 3\sqrt{2})c$, we obtain

$$\begin{aligned} u(t_{k+1}, x_l) = \frac{1}{2c}\Big[&(4 + 3\sqrt{2})\Phi_{\beta,-\sqrt{2}+2}(t_{k+1})y_1(x_l) \\ &+(-4 + 3\sqrt{2}))\Phi_{\beta,\sqrt{2}+2}(t_{k+1})y_3(x_l)\Big], \end{aligned}$$

that is,

$$\begin{aligned} u(t_{k+1}, x_1) &= \frac{\sqrt{2}}{4}\Big[(4 + 3\sqrt{2}))\Phi_{\beta,-\sqrt{2}+2}(t_{k+1}) \\ &\qquad +(-4 + 3\sqrt{2}))\Phi_{\beta,\sqrt{2}+2}(t_{k+1})\Big], \\ u(t_{k+1}, x_2) &= \frac{1}{2}\Big[(4 + 3\sqrt{2}))\Phi_{\beta,-\sqrt{2}+2}(t_{k+1}) \\ &\qquad -(-4 + 3\sqrt{2}))\Phi_{\beta,\sqrt{2}+2}(t_{k+1})\Big], \\ u(t_{k+1}, x_3) &= \frac{\sqrt{2}}{4}\Big[(4 + 3\sqrt{2}))\Phi_{\beta,-\sqrt{2}+2}(t_{k+1}) \\ &\qquad +(-4 + 3\sqrt{2}))\Phi_{\beta,\sqrt{2}+2}(t_{k+1})\Big], \end{aligned}$$

for $k = 0, 1, 2, \dots$.

In numerical simulations, we analyze

(i) the diffusion concentration versus the discrete space for different values of the derivative's order β and different values of the sampling time h;

(ii) the diffusion concentration versus the discrete time for a fixed x and different values of the derivative's order β and different values of the sampling time h.

Figures 1 and 2 illustrate Case (i): the left panel for $h = 0.0001$, $k = 20$; the middle panel for $h = 0.01$, $k = 20$; the right panel for $h = 1$, $k = 20$. Consequently, the plots show the diffusion concentration versus the discrete space for $t_k = 0.002, 0.2, 20$ and $\beta = 0.25, 0.5, 0.75, 1$. There is not a significant difference between Figs. 1 and 2. The piecewise linear curve is introduced for the reader's convenience.

Figure 3 illustrates Case (ii): the left panel for $h = 0.0001$, $k = 20$, and $x_1 = 1$; the middle panel for $h = 0.01$, $k = 20$, and $x_1 = 1$; the right panel for $h = 1$, $k = 20$, and $x_1 = 1$. Consequently, the plots show the diffusion concentration versus the discrete time t (with different sampling times) for $x_1 = 1$ and $\beta = 0.25, 0.5, 0.75, 1$.

It is worth noticing that the memory (expressed by the order of the fractional difference β) makes the diffusion process sluggish, tending to remain in the previous state. As β approaches 1, the memory becomes weaker and the viscosity becomes tinier.

Remark 5. Lin and Xu (2007) considered a time-fractional diffusion equation, which was obtained from the standard diffusion equation by replacing the first-order time derivative with the Caputo fractional derivative of order β, with $0 < \beta < 1$. Roughly speaking, they analyzed the continuous counterpart of (32). One may observe that our numerical simulations presented in Fig. 2 for small values of h are consistent with their results (cf. Lin and Xu, 2007, Fig. 1). Clearly, the behavior of the model strongly depends of the magnitude of β. Our numerical simulation are also consistent with those presented by Wu *et al.* (2015, Figs. 1 and 2), who considered a fractional diffusion model of time discretization with the Caputo-like difference.

Now we analyze the following non-homogeneous fractional diffusion difference equation:

$$\begin{aligned} &\Delta^{\beta}_{0+,k}u(t_{k+1}, x_l) \\ &\quad = \Delta_{1,l}(\nabla_{1,l}u(t_k, x_l)) + \sin\frac{\pi x_l}{4} + t_k\sin\frac{3\pi x_l}{4}, \\ &\qquad\qquad k = 0, 1, 2, \dots, \quad l = 1, 2, 3, \end{aligned} \tag{38}$$

subject to the boundary and initial conditions (33) and (34). According to Theorem 5 and since $g(t_k, x_l) = y_1(x_l) + t_k y_3(x_l)$, the solution to (38) is given by

$$\begin{aligned} u(t_{k+1}, x_l) = \frac{1}{2c}\Big[&(4 + 3\sqrt{2})\Psi_{\beta,-\sqrt{2}+2}(t_{k+1})y_1(x_l) \\ &+ (-4 + 3\sqrt{2}))\Psi_{\beta,\sqrt{2}+2}(t_{k+1})y_3(x_l)\Big], \end{aligned}$$

that is,

$$\begin{aligned} u(t_{k+1}, x_1) &= \frac{\sqrt{2}}{4}\Big[(4 + 3\sqrt{2}))\Psi_{\beta,-\sqrt{2}+2}(t_{k+1}) \\ &\qquad +(-4 + 3\sqrt{2}))\Psi_{\beta,\sqrt{2}+2}(t_{k+1})\Big], \\ u(t_{k+1}, x_2) &= \frac{1}{2}\Big[(4 + 3\sqrt{2}))\Psi_{\beta,-\sqrt{2}+2}(t_{k+1}) \\ &\qquad -(-4 + 3\sqrt{2}))\Psi_{\beta,\sqrt{2}+2}(t_{k+1})\Big], \end{aligned}$$

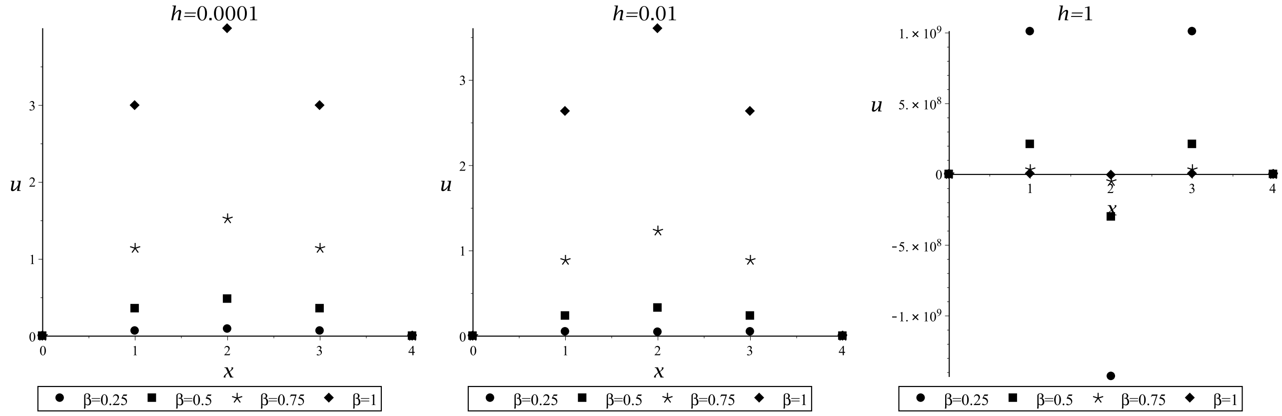

Fig. 1. Diffusion concentration versus the discrete space x for $\beta = 0.25, 0.5, 0.75, 1$ and $h = 0.0001, 0.01, 1$ (in the case of Eqn. (32)).

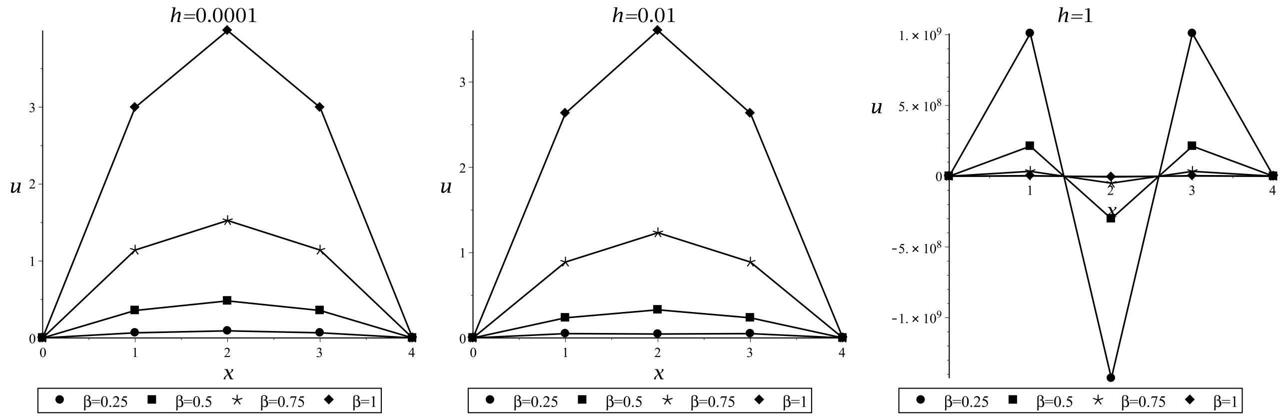

Fig. 2. Diffusion concentration versus the discrete space x for $\beta = 0.25, 0.5, 0.75, 1$ and $h = 0.0001, 0.01, 1$ (in the case of Eqn. (32)).

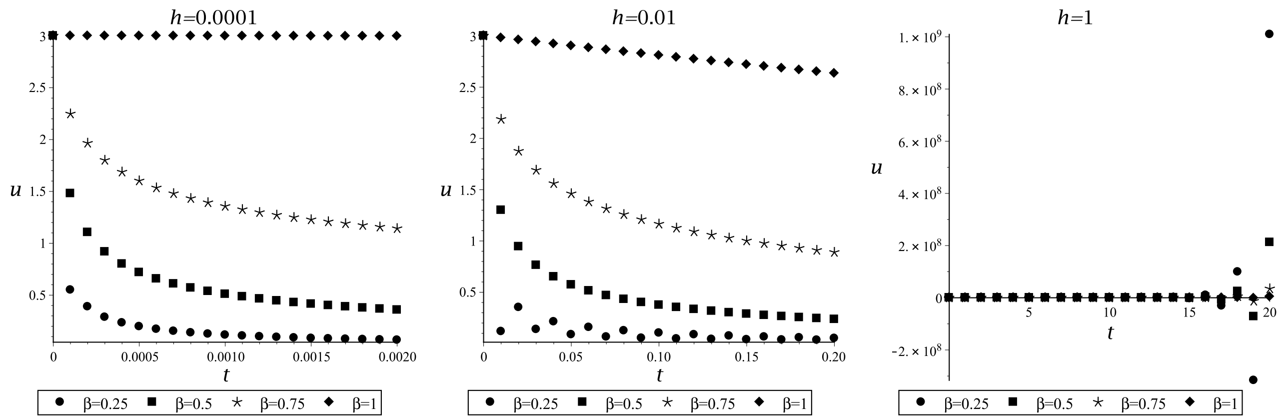

Fig. 3. Diffusion concentration versus the discrete time t for $x_1 = 1$, $\beta = 0.25, 0.5, 0.75, 1$ and $h = 0.0001, 0.01, 1$ (in the case of Eqn. (32)).

$$u(t_{k+1},x_3) = \frac{\sqrt{2}}{4}\Big[(4+3\sqrt{2}))\Psi_{\beta,-\sqrt{2}+2}(t_{k+1}) \\ +(-4+3\sqrt{2}))\Psi_{\beta,\sqrt{2}+2}(t_{k+1})\Big],$$

for $k = 0, 1, 2, \ldots$.

For comparison purposes, we conduct similar numerical simulations as in the case of the solution to Eqn. (32). Namely, Figs. 4 and 5 illustrate Case (i): on the left for $h = 0.0001$, $k = 20$; in the middle for $h = 0.01$, $k = 20$; on the right for $h = 1$, $k = 20$. Hence, the plots show the diffusion concentration versus the discrete space for $t_k = 0.002, 0.2, 20$ and $\beta = 0.25, 0.5, 0.75, 1$. There is not a significant difference between Figs. 4 and 5. The piecewise linear curve is introduced for the reader's convenience.

Figure 6 illustrates Case (ii): on the left for $h = 0.0001$, $k = 20$, and $x_1 = 1$; in the middle for $h = 0.01$, $k = 20$, and $x_1 = 1$; on the right for $h = 1$, $k = 20$, and $x_1 = 1$. Thus, the plots show the diffusion concentration versus the discrete time t (with different sampling time) for $x_1 = 1$ and $\beta = 0.25, 0.5, 0.75, 1$.

Finally, consider the fractional difference diffusion equation of the form

$$\Delta^{\beta}_{0+,k}u(t_{k+1},x_l) = -\Delta^{\alpha}_{4-,l}(\Delta^{\alpha}_{0+,l}u(t_k,x_l)), \\ k = 0,1,2,\ldots, \quad l = 1,2,3, \quad (39)$$

subject to the boundary and initial conditions (33) and (34). In this case, the fractional Sturm–Liouville problem is

$$\Delta^{\alpha}_{4-,l}(\Delta^{\alpha}_{0+,l}y(x_l)) = \lambda y(x_l), \quad l = 1,2,3, \quad (40)$$

$$y(x_0) = y(x_4) = 0. \quad (41)$$

Problem (40) and (41) can be solved using methods presented by Almeida *et al.* (2017). Specifically, by Theorem 2.5 by Almeida *et al.* (2017), Eqn. (40) is the Euler–Lagrange equation for the isoperimetric problem of the form

$$\min(\max) \quad J[y] = \sum_{l=1}^{4} h(\Delta^{\alpha}_{0+,l}y(x_l))^2 \quad (42)$$

subject to $y(x_0) = y(x_4) = 0$ and

$$I[y] = \sum_{l=1}^{4} h(y(x_l))^2 = 1. \quad (43)$$

Problem (42) and (43) can be replaced by the finite-dimensional optimization problem

$$\min(\max) \quad \Phi(y_1,y_2,y_3,\lambda) \\ = \sum_{k=l}^{4} h\left((\Delta^{\alpha}_{0+,l}y_l)^2 - \lambda((y_l)^2-1)\right).$$

subject to $y_0 = y_4 = 0$, where $y_l := y(x_l)$. Using the first-order necessary optimality conditions given by the system of four equations, we obtain the eigenfunctions and the eigenvalues of Eqn. (40). Then, the construction of the solution to (39) goes as in the case of Eqn. (32).

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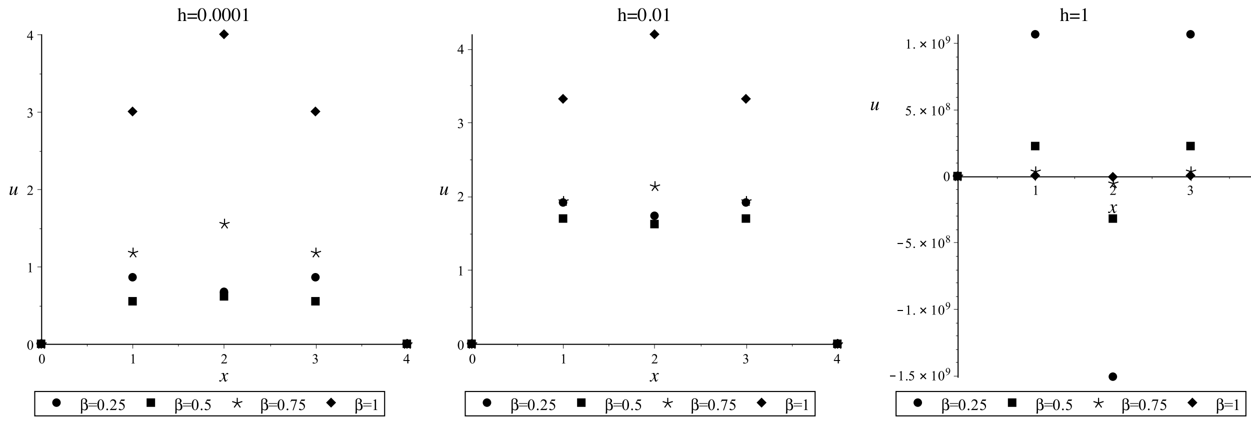

Fig. 4. Diffusion concentration versus the discrete space x for $\beta = 0.25, 0.5, 0.75, 1$ and $h = 0.0001, 0.01, 1$ (in the case of Eqn. (38)).

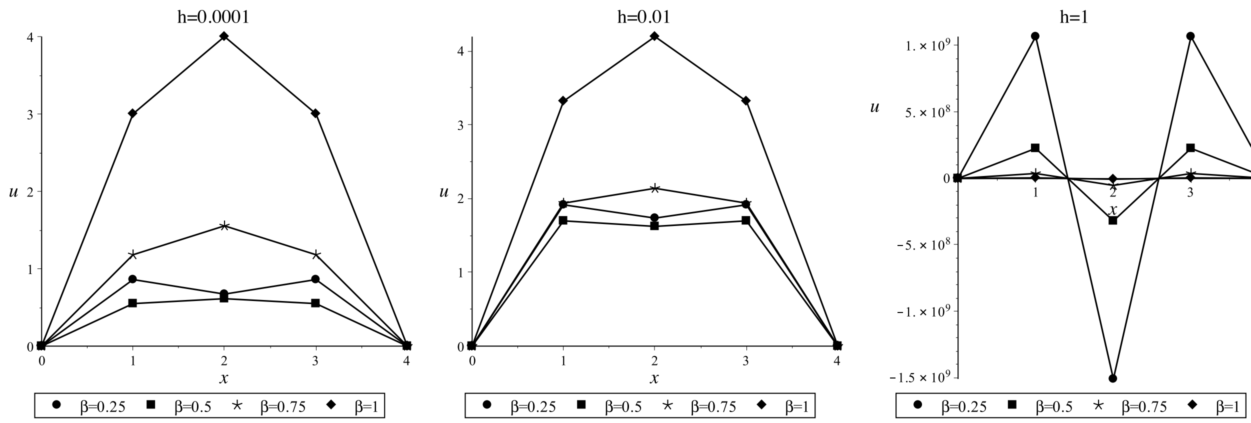

Fig. 5. Diffusion concentration versus the discrete space x for $\beta = 0.25, 0.5, 0.75, 1$ and $h = 0.0001, 0.01, 1$ (in the case of Eqn. (38)).

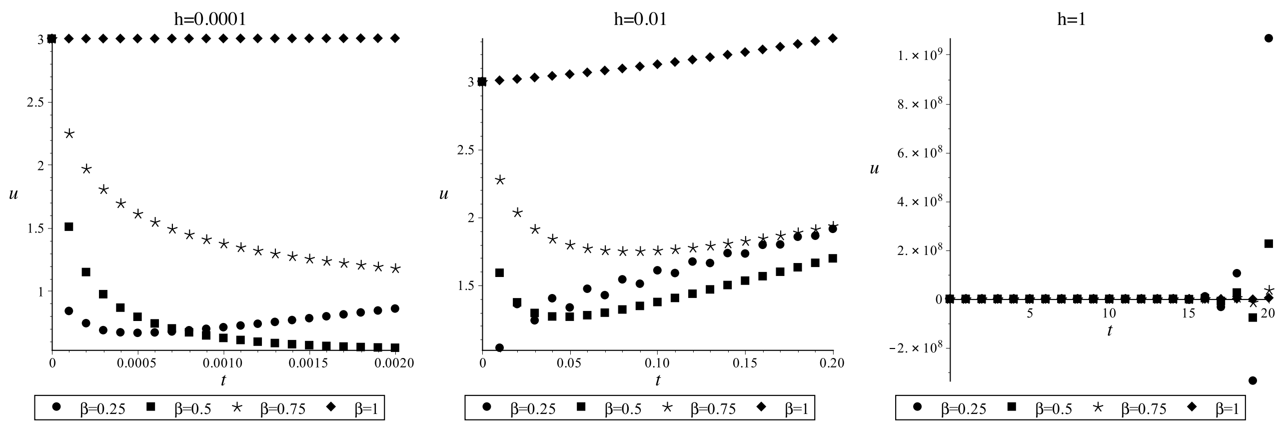

Fig. 6. Diffusion concentration versus the discrete time t for $x_1 = 1$, $\beta = 0.25, 0.5, 0.75, 1$ and $h = 0.0001, 0.01, 1$ (in the case of Eqn. (38)).

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