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ALGEBRAIC ACTIVE DISTURBANCE REJECTION TO CONTROL A GENERALIZED UNCERTAIN SECOND–ORDER FLAT SYSTEM

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We introduce an algebraically active disturbance rejection-based control solution for the trajectory tracking problem of an uncertain second-order flat system with unknown external disturbances. To this end, we first algebraically identify the system's unknown dynamics and the external disturbances with a linear set of time-varying integral expressions for the output and the control signal. We use the identified dynamics on an online feedback cancellation scheme to linearize the second-order system and cancel the uncertainties. With a proportional-integral controller we stabilize the linearized system without the need to estimate the velocity and have feedback from it. We carry out the stability analysis using linear systems theory. Finally, we evaluate the effectiveness of the proposed controller in a partially known 2-DOF manipulator.

Keywords: active disturbance rejection control, uncertain flat system, numerical methods, algebraic estimation, 2-DOF manipulator.

1. Introduction

It is well known that the presence of uncertainties and perturbations is one of the main problems when designing a control strategy. The complexity of the technological systems developed in recent years has given rise to the emergence of the robust control area, which consists of the different techniques and principles that guarantee stability and satisfactory performance of dynamic systems despite their parameter uncertainties and the presence of disturbances (de Jesús Rubio, 2016; de Jesús Rubio *et al.*, 2015; Dullerud and Paganini, 2013; Liu and Yao,

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2016; Rubio *et al*., 2019). Robust controllers have been developed in different engineering fields, such as robotics, power systems, automotive, and aerospace, to ensure safety and reliability.

Among the most popular robust control techniques, one can find sliding mode control (Bartolini *et al*., 2008; Edwards *et al*., 2006; Shtessel *et al*., 2014; Utkin *et al*., 2017), the attractive (invariant) ellipsoid method (Azhmyakov *et al*., 2013; Davila and Poznyak, 2011; Poznyak *et al*., 2014), multi-model control (Boltyansky, 1999; Poznyak *et al*., 2002) and H^∞ control (Petersen and Tempo, 2014). Other control fields that handle uncertain systems are adaptive control (Astolfi *et al*., 2007; Krstic *et al*., 1995), identification schemes (Åström and Eykhoff, 1971; Åström and Wittenmark, 1971; Fliess and Sira-Ramirez, 2008; 2003; Romero *et al*., 2014; de Jesús Rubio, 2016; de Jesús Rubio *et al*., 2015; Dullerud and Paganini, 2013; Liu and Yao, 2016; Runio *et al*., 2019) and high-gain observers in nonlinear feedback control (Khalil, 2017; Khalil and Praly, 2014).

In the work of Freidovich and Khalil (2008), a robust output feedback controller for feedback linearizable systems is introduced, where the model uncertainties and perturbations are estimated using an extended high-gain observer. Another disturbance observer-based control is introduced by Ferreira *et al*. (2010), who estimate the states and the disturbance in finite time using a sliding-mode-based observer. Finally, we mention the work of Sanchez and Moreno (2021), where the authors also propose a disturbance observer-based control scheme for a class of nonlinear systems. Therein, an extended-order higher-order sliding-mode observer is used to estimate the states of the system and the matched external disturbances exactly and in finite time. Active disturbance rejection control (ADRC) (Gao *et al*., 2001; Han, 2009) and the model-free control approach (MFCA) (Fliess and Join, 2009; 2013) have been demonstrated to be practical and efficient options to address the trajectory tracking control problem for uncertain flat systems. We refer the reader interested in these two topics to the work of Lafont *et al*. (2015) and Sira-Ramírez *et al*. (2018). Finally, we invited the reader to see the related and interesting recently published works by Ordaz *et al*. (2023) and Ding *et al*. (2020) and the papers therein.

ADRC is a suitable tool to solve the output-feedback control problem for uncertain plants, where the input-output behavior of the plant is assumed to be well approximated within its operating range by an ordinary differential equation. Roughly speaking, ADRC algebraically estimates the unmodeled dynamics and unknown external perturbations. A feedback controller uses these estimates to cancel the undesired effects they produce online. Following the ideas behind ADRC, this study proposes an explicit control solution for the trajectory tracking control problem for a second-order uncertain flat system without requiring velocity feedback, which is our study's main contribution. The proposed approach consists of an algebraic version of ADRC for the online estimation and cancellation of the system uncertainties, avoiding the use of extended state observers. Inspired by the results presented by Aguilar-Ibanez *et al*. (2021) and Cortés-Romero *et al*. (2017), the use of an algebraic estimator consisting of iterated integrals of both the system's output and the control signal is proposed.

The suggested solution to the trajectory tracking problem for an uncertain second-order flat system is presented in two stages. Firstly, it is derived for a scalar system; then, it is extended to a generalized multivariable system. The main contribution of our algebraic control approach is that it allows us to solve the output-feedback control problem for uncertain flat second-order systems without needing to estimate the first-time derivative of the system position. Our proposal avoids using extended observers and exhibits fast convergence, performing similarly to the controllers that use them to identify uncertainties and velocities. As far as we know, our algebraic approach has not yet been extensively studied at this point. To illustrate the effectiveness of the proposed control method, numerical simulations with satisfactory outcomes are presented.

We organize the rest of this work as follows. Section 2 presents the rationale of algebraic ADRC for a second-order system and introduces the control problem statement. Section 3 discusses the proposed control strategy. Section 4 presents numerical simulations that assess the control strategy's effectiveness. Finally, some concluding remarks are provided in Section 5.

2. Motivation

Consider the uncertain second-order flat plant

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= f_0(x, d, t) + bu, \\ y &= x_1, \end{aligned} \tag{1}$$

where x_1 and x_2 are the states, f_0 represents the system's unknown dynamics, u and y are respectively the input and output of the plant, $b > 0$ is a constant input gain; d is the unknown bounded disturbance. Please note that function f_0 lumps the endogenous and exogenous perturbations, which, additionally, can contain noisy signals that may affect any system state variables. Regarding the plant input-output behavior and the spirit of the MFCA, we can rewrite the nonlinear model (1) as the following linear perturbed system:

$$\ddot{y} = F(t) + bu, \tag{2}$$

where F is defined by

$$F(t) = f_0(y, \dot{y}, d, t),$$

which is regarded as a completely unknown function of time. For simplicity, we assume that the input gain b is known and the exogenous disturbance $F(t)$ is uniformly and absolutely bounded in some compact set in the region of interest. It is necessary to notice that the perturbations and the unknown endogenous injections must be uniformly and absolutely bounded to guarantee that the closed-loop differential equation (2) exhibits a solution and avoids the corresponding finite-scape time problem (Gliklikh, 2006; Cortés–Romero *et al.*, 2017; Guo and Zhao, 2011; 2013).

The main idea of our control solution consists in algebraically estimating online the uncertainty $F(t)$ using the ADRC approach while simultaneously canceling it (Fliess and Join, 2009). Summarizing, we apply to both sides of Eqn. (2) the estimator operator $[\cdot]_E$, which acts during a very short sliding time window of width δ, that is

$$[\ddot{y}]_E = [F]_E + b\,[u]_E\,. \tag{3}$$

Now, under some suitable assumptions, the uncertainty estimate $[F]_E$ can be approximated as

$$[F]_E \approx \kappa \widehat{F}_t, \tag{4}$$

where $\kappa > 0$ is a convenient constant that depends on the selection of $[\cdot]_E$, and, as mentioned above, $\widehat{F}_t$ is an estimate of F computed inside the sliding time window of width δ. Then, from Eqns. (3) and (4), we have that

$$\widehat{F}_t = \frac{1}{\kappa}\left([\ddot{y}]_E - b\,[u]_E\right),$$

and, in general, $\widehat{F}_t$ satisfies the following inequality:

$$\left|F(t) - \widehat{F}_t\right| \le \epsilon(\delta), \quad \forall t \ge T_0 > 0, \tag{5}$$

where $\delta > 0$ and $\epsilon(\delta)$ are small constants, and the time T_0 is sufficiently small. Specifically, the operator $[\cdot]_E$ acts during a very short sliding time window of width δ. Therefore, if $\widehat{F}_t$ is very close to the current values of $F(t)$, then the control action can be proposed as

$$u = -\frac{1}{b}(\widehat{F}_t - u_0), \tag{6}$$

where u_0 is the new system input. Evidently, if the relation (5) is fulfilled, the system (2), in closed-loop with (6), reads as

$$\ddot{y} \approx u_0.$$

Control problem. Based on the differential-algebraic approach, we develop a control scheme that solves the output trajectory tracking control problem for the uncertain flat system (2). That is, we propose the following piece-wise continuous controller:

$$u = C(y - y_r),$$

where $C(\cdot)$ is a smooth function and y_r is a continuous and bounded time-varying reference, with its first and second-time derivatives being bounded. Then, the above-introduced controller u ensures that

$$|y^{(i-1)}(t) - y_r^{(i-1)}(t)| \le \epsilon_i, \qquad t \ge T_0 > 0, \quad i = \{1, 2\},$$

where $T_0 > 0$ is the time required for the states to enter a region of confinement and the constants $\epsilon_i > 0$ are as small as needed.

To solve the problem under consideration, we must take into account that $f_0(x, d, t)$ is a continuous and locally Lipschitz function with respect to x, in some region of interest $x \in D \subset \mathbb{R}^2$. Besides, $F(t) = F_0(x, d, t)$ and $\dot{F}(t) = \dot{F}_0(x, d, t)$ are continuous and uniformly bounded in D. That is, F is uniformly continuous.

Consequently, F could be approximated by a piece-wise continuous function in the sliding time window $[T - \delta, T]$.

Novelty. This study proposes a suitable scheme for simultaneous online algebraic estimation and cancellation of the function $F(t)$, corresponding to the unknown dynamics. Our approach differs from previously presented solutions by extending the method to apply it to more complex unknown flat systems. We accomplish this generalization based on the following estimation operator:

$$[x(T)]_\delta^m = \int_{T-\delta}^{T}\int_{T_m-\delta}^{T_m}\cdots\int_{\tau_2-\delta}^{\tau_2} x(\tau_1)\,\mathrm{d}\tau_1\,\mathrm{d}\tau_2\ldots\mathrm{d}\tau_m, \tag{7}$$

where the window of width δ suggests evaluating the integral during the interval $(T - \delta, T)$, where T stands for the running time and $T > \delta$, and m is the number of iterated integrals. This operator was introduced by Aguilar-Ibanez *et al.* (2021) and Cortés-Romero *et al.* (2017).

Remark 1. The idea behind the estimation operator consists in obtaining an algebraic expression of the estimated function in a sufficiently short sliding time window $[T - \delta, T]$, where $\delta < T$ is the size of this window and T is the evolution time during which we want to obtain the estimate. In other words, we want to recover information by using the integral operator several times to estimate the function.

Because our approach algebraically recovers the unknown dynamics, it has the advantage of doing it in finite time, contrarily to what has been done when applying the ADRC approach, where the solution is accomplished using asymptotic extended-order state observers (Chen *et al.*, 2007; Han, 2009; Tian and Gao, 2007; Zhao and Gao, 2013; Zhou *et al.*, 2009).

Figure 1 shows an illustrative scheme that summarizes the particularities of the problem under consideration and the proposed solution.

3. Control strategy

3.1. Design of algebraically active disturbance rejection-based control for a scalar system. To solve the trajectory tracking control problem for the uncertain flat system (2), we first algebraically estimate the uncertain disturbance F, using measurements of y and u. Afterward, we propose the corresponding stabilizing controller. Then, to obtain the estimate $\widehat{F}_t$, we consider the short sliding time window $[T-\delta, T]$, where $\delta > 0$ is the window size and T is the evolution time during which we want to obtain the approximated value of $F(T)$. Evidently, the current sliding time window starts after $T - \delta > 0$. The tracking error is defined as

$$\varepsilon = y - y_r, \tag{8}$$

where y_r is the desired reference trajectory, with its first and second-time derivatives bounded. Hence, according to (2), we have

$$\ddot{\varepsilon} = F(T) + bu - \ddot{y}_r. \tag{9}$$

To stabilize (9), we need to feed back the damping term defined by $k_d\dot{\varepsilon}$, with $k_d > 0$. To accomplish that, this term is added to both the sides of Eqn. (9), which yields

$$\ddot{\varepsilon} + k_d\dot{\varepsilon} = F_n(T) + bu, \tag{10}$$

where $F_n(t)$ is the new uncertainty of the system (10), defined as

$$F_n(T) = k_d\dot{\varepsilon} + F(T) - \ddot{y}_r.$$

To estimate the new uncertain term $F_n(T)$, we use the estimation operator $[\cdot]_\delta^{m=2}$ on both the sides of Eqn. (10) to obtain

$$\begin{aligned}\int_{\tau_1=T-\delta}^{T}\int_{\tau_2=\tau_1-\delta}^{\tau_1} (\ddot{\varepsilon} + k_d\dot{\varepsilon})\,\mathrm{d}\tau_2\,\mathrm{d}\tau_1 \\ = \int_{\tau_1=T-\delta}^{T}\int_{\tau_2=\tau_1-\delta}^{\tau_1} (F_n(\cdot) + bu(\cdot))\,\mathrm{d}\tau_2\,\mathrm{d}\tau_1.\end{aligned}$$

Assuming that F_n could be approximated by a piece-wise continuous function in the sliding time window $[T-\delta, T]$, after some simple calculus it is easy to see that

$$\begin{aligned}&\varepsilon(T) - 2\varepsilon(T-\delta) + \varepsilon(T-2\delta) \\ &\quad + k_d\left([\varepsilon(T)]_\delta^1 - [\varepsilon(T-\delta)]_\delta^1\right) \\ &\qquad = \delta^2\widehat{F}_n(T) + b\,[u(T)]_\delta^2.\end{aligned} \tag{11}$$

Algebraic manipulations of Eqn. (11) lead us to the following estimate of F_n:

$$\begin{aligned}\widehat{F}_n(T) = \frac{1}{\delta^2}\Big(&\varepsilon(T) - 2\varepsilon(T-\delta) + \varepsilon(T-2\delta) \\ &+ k_d\left([\varepsilon(T)]_\delta^1 - [\varepsilon(T-\delta)]_\delta^1\right) \\ &- b\,[u(T)]_\delta^2\Big).\end{aligned} \tag{12}$$

Notice that, using (11), we actually approximate the double integral (12), employing only the error position (8); therefore, there is no need to measure the acceleration. We underscore that the estimations of quantities $[u(T)]_\delta^2$, $[\varepsilon(T)]_\delta^1$ and $[\varepsilon(T-\delta)]_\delta^1$ may be implemented as a discrete linear filter. According to (6) and (10), we propose the current controller u as

$$u = -\frac{1}{b}\left(\widehat{F}_n - u_0\right), \tag{13}$$

where we compute $\widehat{F}_n$ using Eqn. (12), and u_0 can be fixed as

$$u_0 = -k_p\varepsilon(T), \tag{14}$$

where $k_p > 0$. Therefore, by substituting (13) and (14) into (10), we have that

$$\ddot{\varepsilon} + k_d\dot{\varepsilon} + k_p\varepsilon = F_n - \widehat{F}_n, \tag{15}$$

where $k_d > 0$ and $k_p > 0$. Notice that, by construction, $\widehat{F}_n$ is a very good numerical approximation. Hence, if the roots of the characteristic polynomial $p(s) = s^2 + k_d s + k_p$ are far enough from the imaginary axis in the open left half complex s-plane, we can ensure that the error ε tends to be very close to zero. Alternatively, we can define u_0 as a PI controller as follows:

$$u_0 = -k_p\varepsilon(T) - k_I\int_{\tau=0}^{T}\varepsilon(\tau)\,\mathrm{d}\tau, \tag{16}$$

where $k_I > 0$. Now, introducing (14) and (16) into (10), we obtain the following dynamic error equation

$$\ddot{\varepsilon} + k_d\dot{\varepsilon} + k_p\varepsilon + k_I\int_{\tau=0}^{T}\varepsilon(s)\,\mathrm{d}s = F_n - \widehat{F}_n.$$

Once again, choosing k_d, k_p, and k_I such that the roots of the characteristic polynomial $p(s) = s^3 + k_d s^2 + k_p s + k_I$ $p(s)$ are far enough from the imaginary axis in the open left half complex s-plane, we can ensure again that the error ε tends to be very close to zero. Finally, according to (13), the control action u can be taken as

$$u = \begin{cases} \dfrac{1}{b}u_0 & \text{for } t \in [0, 2\delta], \\[2ex] \dfrac{1}{b}(-\widehat{F}_n + u_0) & \text{for } t > 2\delta, \end{cases} \tag{17}$$

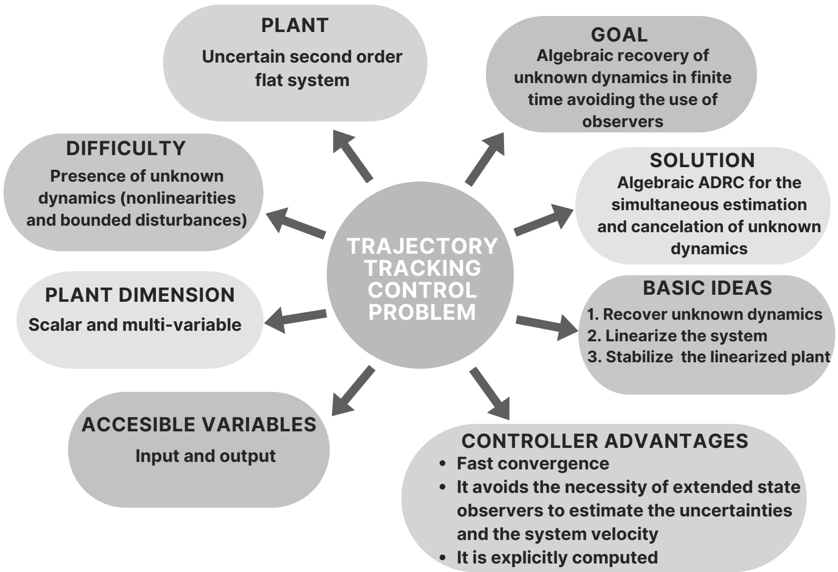

Fig. 1. Block diagram of the control problem in question.

where u_0 can be fixed as (14) or (16), and $\widehat{F}_n$ is computed as (12).

Notice that, if we introduce the control law (17) in combination with (14), we will only be introducing the controller proportional action into the system (10), leading to the following closed-loop equation:

$$\ddot{\varepsilon} + k_d\dot{\varepsilon} + k_p\varepsilon = F_n(t) - \widehat{F}_n(t). \tag{18}$$

When we conveniently place the dominant linear characteristic polynomial roots in the open left half of the complex s-plane, the tracking error ε converges to a small neighborhood of zero. We must note that the estimation error $F_n(t) - \widehat{F}_n(t)$ is very small because the algebraic estimation method is highly accurate; consequently, the trajectory tracking error is linearly dominated by the characteristic polynomial $p(s)$. Furthermore, suppose we conveniently place the dominating linear characteristic polynomial roots in the open left half of the complex s-plane. In that case, the tracking error converges to a small neighborhood of zero of the output's associated phase space once the disturbance cancelation is accomplished. As the estimates rely on the sliding time window size and the integration step size, the error ε is confined inside a small ε-vicinity. Similarly, we can alternatively substitute the controller (16) into (17), also introducing the integral action of the controller in the system (10), obtaining the following optional closed-loop equation:

$$\ddot{\varepsilon} + k_d\dot{\varepsilon} + k_{pI}\varepsilon + k_I\int_0^T \varepsilon = F_n(t) - \widehat{F}_n(t), \tag{19}$$

where constants $\{k_d, k_p, k_I\}$ have to be chosen in such a way that the characteristic polynomial $p(s) = s^3 + k_d s^2 + k_p s + k_I$ is Hurtwitz, and its roots are far enough from the imaginary axis in the open left half complex s-plane. Once again, because the tracking error $F_n(t) - \widehat{F}_n(t)$ is very small, it is linearly dominated by the characteristic polynomial $p(s)$.

To implement the proposed algebraic approach, the following assumption is required:

A1 The input gain $b \neq 0$ is known, and the uncertainty $F_n(t)$ is uniformly continuous and absolutely bounded.

Additionally, to guarantee that Eqn. (9), in closed-loop with (17), possesses a solution, the unknown uncertainty has to be uniformly and absolutely bounded and avoid the finite time of escape (Gliklikh, 2006; Guo and Zhao, 2011; 2013). As for the time window of width δ that moves along time T, we must note that it allows us to evaluate the iterated integrals of both the output and control, letting us estimate the uncertainties F_n in time T. When time T increases, that is, the new interval changes to $[T-\delta, T]$, the integration interval also changes, we need to estimate once again the uncertainties F_n through $\widehat{F}_n$. In short, the change in T implies a continuous estimate of the uncertainties for a valid $T > \delta > 0$.

Comments regarding the above discussion.
(i) The string of obtained outputs, before $T > 2\delta$, of the states $\varepsilon(T - k\delta)$, for $k = \{0, 1, 2\}$, $[\varepsilon(T)]_\delta^1$, and $[\varepsilon(T-\delta)]_\delta^1$ are available or computable. Also, the quantities $[\varepsilon(T)]_\delta^1$, $[\varepsilon(T-\delta)]_\delta^1$ and $[u(T)]_\delta^2$ can be

numerically estimated, with high precision, using either a digital filter or an implementation of any numerical integration algorithm.

(ii) According to the integral mean value theorem, it can be proven that $\widehat{F}_n$ is well defined, that is

$$\widehat{F}_n = F_n(\zeta) = \frac{1}{\delta^2}\left[F_n(T)\right]_\delta^2, \quad \zeta \in [T-2\delta, T].$$

For a detailed explanation of how the above expression was obtained, please refer to Cheney and Kincaid (2012). On the other hand, using the Taylor series expansion is evident that:

$$\left|\widehat{F}_n - F_n(T)\right| = O(\delta). \tag{20}$$

In fact, the numerical error between $F_n(T)$ and its estimate $\widehat{F}_n$ is equal to $O(\delta)$ times two. Recall that $g(h) = O(h)$ means that there exists a finite constant k, such that $|g(t)| \le kh$.

(iii) Finally, we underscore that $\dot{\varepsilon}(t)$ is not fed back or implemented in the current controller (17). However, we must use integrators of the outputs $\varepsilon(T)$ and $\varepsilon(T-\delta)$, and we do not need to use observers.

We summarize the above discussion in the form of the following proposition.

Proposition 1. *Under Assumption A1, consider the uncertain system (2), in closed loop with (17), where $\widehat{F}_n$ is computed via (12). Then, the tracking error $\varepsilon = y - y_r$ of the closed-loop system uniformly and asymptotically converges to a δ-vicinity.*

Proof. First of all, the control u is active after $t > 2\delta$ (cf. (17)). From Assumption A1, the state ε remains bounded for $t \in [0, 2\delta]$. For simplicity, we only consider the proportional control action (14) and the controller (17) in closed loop with (10), and for the case where $k_d = 2w$ and $k_p = w^2$, with $w > 0$; the latter leads us to the following state-space realization:

$$\dot{\xi} = \begin{bmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -w^2 & -2w \end{bmatrix}\xi + \begin{bmatrix} 0 \\ \widetilde{F}_n(t) \end{bmatrix},$$

where $\xi_1 = \varepsilon$, $\xi_2 = \dot{\varepsilon}$, and $\widetilde{F}_n(t) = \widehat{F}_n(t) - F_n(t)$. Therefore, the error ξ evolution is given by

$$\xi(t) = e^{A(t-T_0)}\xi(T_0) + \int_{\zeta=T_0}^{t} e^{A(t-\zeta)}\begin{bmatrix} 0 \\ \widetilde{F}_n(\zeta) \end{bmatrix} \mathrm{d}\zeta, \tag{21}$$

with $T_0 \ge 2\delta$, and

$$A = \begin{bmatrix} 0 & 1 \\ -w^2 & -2w \end{bmatrix}.$$

Notice that, according to the inequalities (20), we have that $\left|\widetilde{F}_n(\zeta)\right| = O(\delta)$, for all $t > T_0 \ge 2\delta > 0$.

Because the exponential matrix A is Hurwitz, it can be upper-bounded as

$$\|e^{At}\| \le \rho e^{-wt},$$

where $\rho > 0$ is a suitable constant and $w = \lambda_{\max}(A)$. Therefore, from (21) we can upper bound $\xi(t)$ as

$$\|\xi(t)\| \le \rho e^{-w(t-T_0)}\|\xi_0\| + \rho \max_{\zeta\in[T_0,t]}\left|\widetilde{F}_n(\zeta)\right| \int_{\zeta=T_0}^{t} e^{-w(t-\zeta)}\,\mathrm{d}\zeta.$$

Therefore, as $t \to \infty$, we have that

$$\|\xi(t)\| \le \frac{\rho}{w}O(\delta).$$

Now, selecting w sufficiently large, we can claim that $\|\xi(t)\| = O(\delta/w)$, for t large enough, holds. That is, the tracking trajectory error ε is bounded. Following verbatim the above arguments, we can probe the boundedness of the proportional-integral action of the closed loop (19). ■

To have an intuitive and practical idea of how the proposed control approach works for the case when we use PD-control, we sketched an algorithm in pseudo-code-like notation (Algorithm 1).

Remark 2. Notice that the estimation error ξ relies on the ratio between the size δ of the sliding time window and the control gain w. Finally, our proposed solution does not use an observer or differentiator to achieve the control objective. This advantage has been previously explored by Ramírez-Neria *et al.* (2020) and Moreno-Valenzuela (2019; 2007). Additionally, because our methodology uses iterative integrals, the disturbance estimation in terms of noise with zero mean is improved, and the convergence time of the uncertainties estimator is generally small. Conversely, if a biased noise exists in the measurements of the system's output, the error can be incremented. In this scenario, our approach is not the most suitable option.

Remark 3. When fixing the sliding window size, we must consider the signal-to-noise ratio and the perturbation frequency. Because it is hard to know the noise level involved in the signal, a trial-and-error process helps to find a suitable window size. Additionally, if the perturbation frequency is high, the sliding window size must also be small, with a smaller integration step size, and the resulting increase in the computing time. On the other hand, a large window size yields a lower computation time but a longer estimation time.

Algorithm 1. PD-control usage case.

Consider the plant $\ddot{y} = F(T) + bu$ defined in (2). Set $k_p, k_d, \delta > 0$.

1. Define the tracking error as in (8) $\varepsilon = y - y_r$, which leads to the dynamic error equation
$$\ddot{\varepsilon} + k_d\dot{\varepsilon} = F_n + bu$$
with the new uncertain system
$$F_n(T) = k_d\dot{\varepsilon} + F(T) - \ddot{y}.$$

2. Define the sliding time window $w_T = [T - \delta, T]$ with $T > \delta$. Store $y(t)$ and $u(t)\,\forall t \in [T - 2\delta, T]$ for future computations.

3. If $T > 2\delta$, compute the estimate of $F_n(T)$ using (12) and the string of past values saved in Step 2. Otherwise,
$$\widehat{F}_n(T) = 0.$$

4. Control action: If $T > 2\delta$ compute $u = -\frac{1}{b}(\widehat{F}_n + k_p\varepsilon)$ defined in (13). Otherwise,
$$u = -\frac{1}{b}k_p\varepsilon.$$

3.2. Generalization: A multi-variable second-order system. We extend the proposed scheme for the vectorial case. To this end, we consider the following generalized second-order system:

$$\begin{aligned}\dot{\mathbf{x}}_1 &= \mathbf{x}_2,\\ \dot{\mathbf{x}}_2 &= \mathbf{f}_0(\mathbf{x}, \mathbf{d},\, t) + \mathbf{b}(\mathbf{x},t)\mathbf{u},\\ \mathbf{y} &= \mathbf{x}_1,\end{aligned} \tag{22}$$

where $\mathbf{x}_1 \in \mathbb{R}^n$ and $\mathbf{x}_2 \in \mathbb{R}^n$ are the states, $\mathbf{f}_0$ is the unknown dynamics, $\mathbf{u} \in \mathbb{R}^n$ and $\mathbf{y} \in \mathbb{R}^n$ are, respectively, the input and output of the system, $\mathbf{b}(\mathbf{x}, t) \in \mathbb{R}^{n\times n}$ is known and fulfills $\mathbf{b}(\mathbf{x}, t) \geq \beta I_n$ with $\beta > 0$. Clearly, the system (22) can be expressed as

$$\ddot{\mathbf{y}} = \mathbf{F}(t) + \mathbf{b}(\mathbf{x}, t)\mathbf{u}, \tag{23}$$

where $\mathbf{F}(t)$ is defined as

$$\mathbf{F}(t) = \mathbf{f}_0(\mathbf{x}, \mathbf{d}, t)$$

and lumps the whole unknown system dynamics, once again, under the following assumptions:

B1 $\mathbf{f}_0(\mathbf{x}, \mathbf{d}, t)$ is a continuous and locally Lipschitz function with respect to $\mathbf{x}$, in some region of interest $\mathbf{x} \in D \subset \mathbb{R}^{2n}$. Besides, $\mathbf{F}$ and $\dot{\mathbf{F}}$ are uniformly bounded in D.

B2 $\mathbf{F}$ can be approximated by a piece-wise continuous function in the sliding time window $[T - \delta, T]$.

As before, the objective consists in finding a control law $\mathbf{u}$ for the system (23), such that the output $\mathbf{y}$ tracks a given smooth output reference signal $\mathbf{y}_r(t)$, with its first and second-time derivatives bounded, in spite of the presence of uncertainty $\mathbf{F}$. That is, we propose $\mathbf{u}$ as

$$\mathbf{u} = \begin{cases} \mathbf{b}^{-1}(\mathbf{x}, t)\mathbf{u}_0 & \text{for} \quad t \in [0, 2\delta], \\ \mathbf{b}^{-1}(\mathbf{x}, t)\left(-\widehat{\mathbf{F}}_n + \mathbf{u}_0\right) & \text{for} \;\; t > 2\delta, \end{cases} \tag{24}$$

where

$$\begin{aligned}\widehat{\mathbf{F}}_n(T) = \frac{1}{\delta^2}(&\varepsilon(T) - 2\varepsilon(T-\delta) + \varepsilon(T - 2\delta)\\ &+ K_d\left([\varepsilon(T)]_\delta^1 - [\varepsilon(T-\delta)]_\delta^1\right)\\ &- \mathbf{b}\,[\mathbf{u}(T)]_\delta^2),\end{aligned} \tag{25}$$

and $K_d = \operatorname{diag}\{k_d, k_d, \dots, k_d\} \in \mathbb{R}^{n\times n}$, where $k_d > 0$, and $\mathbf{u}_0$ can be fixed as

$$\mathbf{u}_0 = -K_p\varepsilon,$$

with $K_p = \operatorname{diag}\{k_p, k_p, \dots, k_p\} \in \mathbb{R}^{n\times n}$, where $k_p > 0$, and the tracking error is defined as $\varepsilon = \mathbf{y} - \mathbf{y}_r$. Alternatively, $\mathbf{u}_0$ can take the following integral form:

$$\mathbf{u}_0 = -K_p\varepsilon - K_I\int_{\tau=0}^{T}\varepsilon(\tau)\,\mathrm{d}\tau,$$

with $K_I = \operatorname{diag}\{k_I, k_I, \dots, k_I\} \in \mathbb{R}^{n\times n}$, where $k_I > 0$. The constants k_d, k_p, k_I are tunable, which implies that the characteristic polynomial

$$p(s) = s^3 + k_d s^2 + k_p s + k_I$$

is Hurwitz.

Now, we are able to summarize the previous discussion related to the multi-variable second-order system (22), in the following proposition.

Proposition 2. *Under Assumptions B1 and B2, consider the uncertain system (22) in closed loop with (24). Then, the tracking error of the closed loop system, $\varepsilon = \mathbf{y} - \mathbf{y}_r$, converges uniformly and asymptotically to a δ-vicinity of the origin. Besides, the trajectory tracking error $\|\varepsilon\| = O(\delta/w)$ is ultimately bounded.*

We can prove Proposition 2 following verbatim the arguments in the proof of Proposition 1.

4. Numerical simulations

We assess the effectiveness of our control approach through three numerical experiments in which the 2-DOF robot, presented by Reyes and Kelly (1997) and shown in Fig. 2, performs three control tasks. We programmed and ran the simulations in the numerical simulator for nonlinear systems Simnon 3.0, instantiated to use the fifth-order Runge–Kutta method. The motion equations of this robot in the joint space are defined as follows:

$$\ddot{\mathbf{q}} = M^{-1}(\mathbf{q})\left(-C(\mathbf{q},\dot{\mathbf{q}})\dot{\mathbf{q}} - G(\mathbf{q}) + \tau + \mathbf{d}\right), \qquad (26)$$

where $\mathbf{q} = (q_1, q_2)$ is the measurable state vector, $\tau = (\tau_1, \tau_2)$ is the input vector, and $\mathbf{d}$ is a bounded and low-frequency perturbation.

Figure 2 shows that q_1 and q_2 are the angular positions of Joints 1 and 2, respectively. The inertia matrix of this system, which is positive definite, is defined as[1]

$$M(\mathbf{q}) = \begin{bmatrix} \theta_1 + 2\theta_2 c_2 & \theta_3 + \theta_2 c_2 \\ \theta_3 + \theta_2 c_2 & \theta_3 \end{bmatrix}, \qquad (27)$$

while the centripetal and Coriolis forces matrix is defined as

$$C(\mathbf{q},\dot{\mathbf{q}}) = \begin{bmatrix} -2\theta_2 s_2 \dot{q}_2 & -\theta_2 s_2 \dot{q}_2, \\ -\theta_2 s_2 \dot{q}_2 & 0 \end{bmatrix}. \qquad (28)$$

The gravitational torque is given by

$$G(\mathbf{q}) = \begin{bmatrix} \theta_4 s_1 + \theta_5 s_{12} \\ \theta_5 s_{12} \end{bmatrix}, \qquad (29)$$

with the parameters

$$\begin{aligned} &\theta_1 = 2.35\,\text{Nm-s}^2, \quad \theta_2 = 0.1\,\text{Nm}, \\ &\theta_3 = .12\,\text{Nm-s}^2, \quad \theta_4 = 38.4\,\text{Nm}, \\ &\theta_5 = 1.82\,\text{Nm}. \end{aligned} \qquad (30)$$

The previous setup was taken from the above-mentioned references (Moreno-Valenzuela *et al.*, 2008; Reyes and Kelly, 1997).

First control maneuver task. The control task consists of a rest-to-rest maneuver tracking a hyperbolic tangent reference signal, that is, from the initial position $\mathbf{q}_i = (0, 0)$ to the final rest position $\mathbf{q}_f = (q_{1f} = 1.5\,[\text{rad}], q_{2f} = 2\,[\text{rad}])$. To this end, we choose a smooth step-like trajectory defined as

$$\mathbf{q}_r = \mathbf{q}_i + \frac{\mathbf{q}_f - \mathbf{q}_i}{2}\left(1 + \tanh(t - T)\right), \qquad (31)$$

where we fix the settling time as $T = 10\,\text{s}$.

The closed-loop system with its corresponding estimators was simulated in the Matlab environment, using a Runge-Kutta integration option with an integration step of 10^{-3}, and the size of the sliding time window was fixed at $\delta = 0.05$. We set the initial conditions as follows:

[1]For simplicity, the following notation is used: $s_1 = \sin q_2$ $s_2 = \sin q_2$, $c_2 = \cos q_2$ $s_{12} = \sin(q_1 + q_2)$.

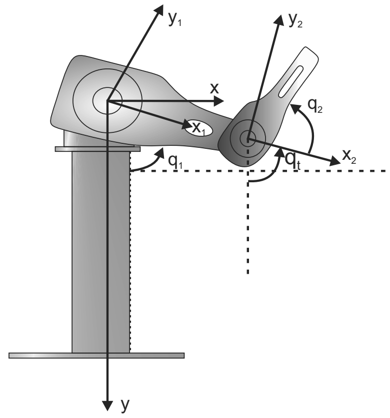

Fig. 2. 2-DOF manipulator.

$$q_1(0) = 0.1\text{ rad}, \quad q_2(0) = 0.05\,\text{rad}, \qquad (32)$$

and we define the system perturbation as

$$\mathbf{d}^T = [\sin(t) + 1, \cos(t) - 1]. \qquad (33)$$

To accomplish this control maneuver task, we first define the tracking error as $\varepsilon = \mathbf{q} - \mathbf{q}_r$. According to (26), the dynamic equation is given by

$$\ddot{\varepsilon} = \mathbf{F}_0(t) + M^{-1}\tau - \ddot{\mathbf{q}}_r, \qquad (34)$$

where

$$\mathbf{F}_0(t) = M^{-1}(\mathbf{q})\left(-C(\mathbf{q},\dot{\mathbf{q}})\dot{\mathbf{q}} - G(\mathbf{q}) + \mathbf{d}\right). \qquad (35)$$

Note that the system position and velocity are bounded in actual applications. For instance, a robot manipulator has a fixed maximal velocity, and its range of movement is restricted. Therefore, the system can be approximately expressed as a perturbed flat one. Consequently, our approach can be used in reality. Therefore, after using Proposition 2, we have that the stabilizing controller can be written as

$$\tau = \begin{cases} M(\mathbf{q})\mathbf{u}_0 & \text{for } t \in [0, 2\delta], \\ M(\mathbf{q})\big(-\widehat{\mathbf{F}}_n + \mathbf{u}_0\big) & \text{for } t > 2\delta, \end{cases} \qquad (36)$$

where

$$\begin{aligned} \widehat{\mathbf{F}}_n(T) = &\frac{1}{\delta^2}\varepsilon(T) - 2\varepsilon(T-\delta) + \varepsilon(T - 2\delta) \\ &+ K_d\left([\varepsilon(T)]_\delta^1 - [\varepsilon(T-\delta)]_\delta^1\right) \\ &- M^{-1}\left[\mathbf{u}(T)\right]_\delta^2 \end{aligned} \qquad (37)$$

and

$$\mathbf{u}_0 = -K_p\varepsilon - K_I \int_{\tau=0}^{T} \varepsilon(\tau)\,\mathrm{d}\tau, \tag{38}$$

with the controller gains chosen as

$$K_d = 3wI_2, \quad K_p = 3w^2I_2, \quad K_I = w^3I_2, \tag{39}$$

and $w = 10$.

We present the outcome of this simulation in Fig. 3, where we show the system trajectories $\mathbf{q} = [q_1, q_2]$, $\dot{\mathbf{q}} = [\dot{q}_1, \dot{q}_2]$, and $\tau = [\tau_1, \tau_2]$. We can see that the position states q_1 and q_2 reach their corresponding reference signals, respectively after 2 s and 1 s, while both velocity states $\dot{q}_1$ and $\dot{q}_2$ do it after 1 s. Finally, we can see that the corresponding control actions never reach their steady state. This is because both controllers constantly counteract the undesirable effect of perturbation $\mathbf{d}$. Figure 3 shows that our control approach performance is satisfactory, even when we never feed back the velocity $\dot{\mathbf{q}}$, and the first and second-time derivatives of $\mathbf{q}_r$. To complement Fig. 3, we show in Fig. 4 the position and velocity tracking errors. It can be observed that the position and velocity errors are small enough; they are of orders of 10^{-3} and 10^{-2}, respectively. From Figs. 3 and 4, we must note that, after around 10 s, there is an inflection point because the reference signal reaches its maximum velocity. Consequently, the tracking error in both coordinates, q_1 and q_2, exhibits an abrupt increment. However, the controller can almost immediately render the system very close to a steady state.

Second control maneuver task. The control task consists in tracking a smooth trajectory defined by

$$q_{r_1} = \sin(0.8t), \qquad q_{r_2} = 2\cos(0.8t). \tag{40}$$

That is, we force the system to track an elliptically shaped trajectory in the phase space portrait of q_1 and q_2. The system's initial condition was set at $\mathbf{q}(0) = (0.25, 1)$. To make the task more challenging, we assume that the inertial matrix M is partially known and given by

$$\widehat{M} = \begin{bmatrix} 2.5 & 0.2 \\ 0.2 & 0.2 \end{bmatrix}, \tag{41}$$

and the system is perturbed by

$$\mathbf{d} = \left(2 + \frac{1}{2}\sin(t), 2 + \frac{1}{2}\sin\left(\frac{t}{4}\right)\cos\left(\frac{t}{4}\right)\right). \tag{42}$$

To run the experiment, we used the same setup as before, except for matrix $M = \widehat{M}$. In other words, we consider M to be a constant matrix. We show the outcome of this numerical experiment in Fig. 5. In Fig. 5(a), we can see that the 2-DOF robot effectively tracks the smooth elliptic-shaped trajectory. In Figs. 5(b)–(e), we can see that the position and velocity errors converge close to zero in less than 2 s, and they are of orders of 10^{-2} and 10^{-3}, respectively. Finally, Fig. 6 shows the corresponding control responses, τ_1 and τ_2. We can claim that our control approach's performance is satisfactory.

Remark 4. To avoid the significant initial peaking phenomena found in the response corresponding to the estimation of F_n, we use a clutch function to smooth these transient peaking responses in the controller. We defined the clutch as a time function smoothly increasing from 0 to 1 during a short time interval $[0, \alpha)$, where we fixed $\alpha = 0.25$ for this particular case. That is, we introduce the function (Sira-Ramírez *et al.*, 2014)

$$s(t) = \begin{cases} 1, & t > \alpha, \\ \sin^{2k}\left(\dfrac{t\pi}{2\alpha}\right), & t \le \alpha, \end{cases} \tag{43}$$

where k is a sufficiently large integer. Thus, the observer variables may be adjusted as

$$\widehat{\mathbf{F}}_a = s(t)\widehat{\mathbf{F}}_n. \tag{44}$$

Third control maneuver task and comparison with the classic ADRC approach. We propose a third experiment to draw a comparison between our control approach and the classical ADRC. In this experiment, we consider the following smooth circular reference trajectory:

$$\mathbf{q}_r = [2\sin(t), 2\cos(t)]^T. \tag{45}$$

We use the same initial conditions, $\mathbf{q}(0) = (0.25, 1)$, and the same partially known constant inertial matrix, $\widehat{M}$ (41). To make the experiment more challenging and evaluate our controller's performance, we compare it with the classical ADRC approach (see, e.g., Gao *et al.*, 2001), and as the disturbance generator we use the chaotic Duffing mechanical oscillator defined by (Kapitaniak, 2000):

$$\dot{x} = v, \quad \dot{v} = -p_1v - p_3x^3 + p_2x + A\cos(1.8t). \tag{46}$$

It is known that this system exhibits a chaotic behavior (Parker and Chua, 2012) for the fixed values of parameters in a neighborhood of $p_1 = 0.4$, $p_2 = -1.1$, $p_3 = 1$, and $A = 2.1$. The perturbation signal is defined as

$$\mathbf{d} = (1 + x, -1 + v)^T, \tag{47}$$

where x and v are respectively, the Duffing oscillator position and velocity. Finally, we use the traditional ADRC proposed as

$$\tau_n = \widehat{M}\left(-\widehat{\mathbf{z}} + \mathbf{u}_0\right), \tag{48}$$

where $\mathbf{u}_0$ is fixed as

$$\mathbf{u}_0 = -K_d(\widehat{\mathbf{v}} - \dot{\mathbf{q}}_r) - K_p\varepsilon - K_I \int_{\tau=0}^{T} \varepsilon(\tau)\,\mathrm{d}\tau, \tag{49}$$

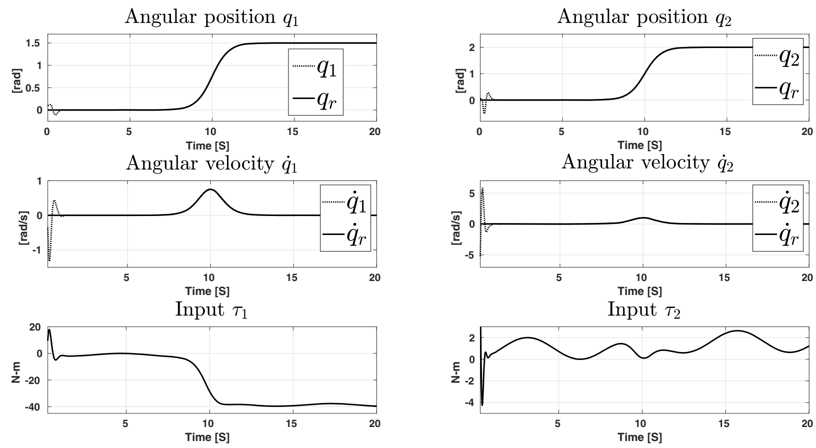

Fig. 3. Closed-loop responses of the proposed algebraic ADRC control for the uncertain 2-DOF manipulator when carrying out a rest-to-rest maneuver tracking a hyperbolic tangent reference signal.

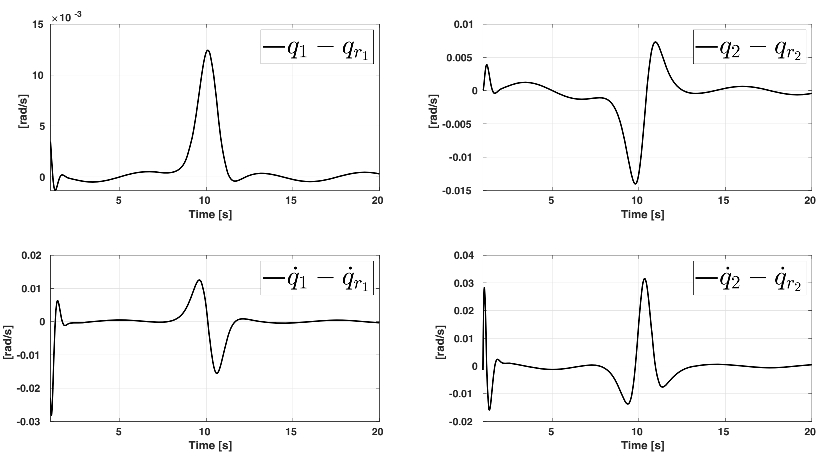

Fig. 4. Position and velocity tracking errors when the 2-DOF manipulator is carrying out a rest-to-rest maneuver tracking a hyperbolic tangent reference signal.

and $\widehat{\mathbf{q}}$ and $\widehat{\mathbf{z}}$ evolve according to the following equations:

$$\begin{aligned} \dot{\widehat{\mathbf{q}}} &= \widehat{\mathbf{v}} - K_2\left(\widehat{\mathbf{q}} - \mathbf{q}\right), \\ \dot{\widehat{\mathbf{v}}} &= \widehat{M}^{-1}\mathbf{u}_0 + \widehat{\mathbf{z}} - K_1\left(\widehat{\mathbf{q}} - \mathbf{q}\right), \\ \dot{\widehat{\mathbf{z}}} &= -K_0\left(\widehat{\mathbf{q}} - \mathbf{q}\right), \end{aligned} \tag{50}$$

where

$$K_2 = 3w_i I_2, \quad K_1 = 3w_i^2 I_2, \quad K_0 = w_i^3 I_2 \tag{51}$$

for some $w_i > 1$. To counteract the peaking phenomena exhibited by the traditional ADRC, we use the same clutch function as in the second experiment, that is,

$$\tau = s(t)\tau_n. \tag{52}$$

To make a fair comparison, we used, for both controllers, the same control gains and the same characteristic polynomial for the ideal closed loops. For fast and efficient recovery of variables $\widehat{\mathbf{v}}$ and $\widehat{\mathbf{z}}$, we fixed the observer's constants as $w_i = 20$. We show the obtained results in Fig. 7, where we can see how our controller and the ADRC one accomplish the trajectory tracking task. This figure shows the corresponding tracking errors and the control effort of both the approaches. Related to the tracking errors, it is evident that they remain bounded and very close to zero, with our controller having a slightly better performance. However, if we use a cascaded GPI observer with the ADRC proposed by Ramírez-Neria *et al*. (2014), the performance may improve considerably, having the inconvenience of finding empirically the number of integrators added to the observer. Finally, notice that both the control signals have a similar magnitude.

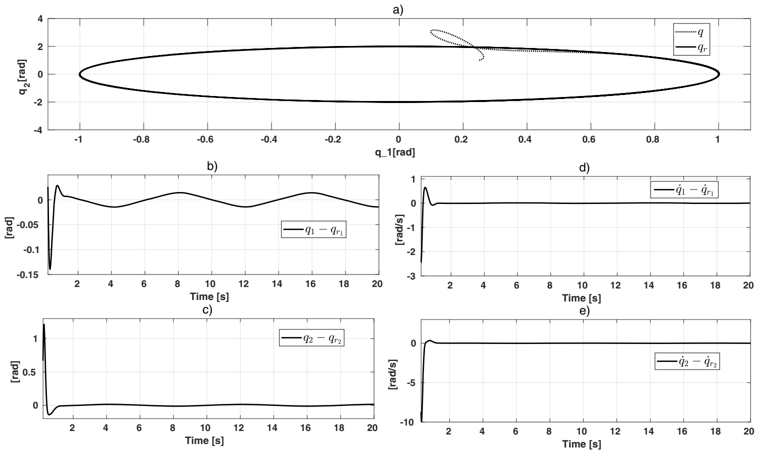

Fig. 5. Closed-loop responses of the proposed algebraic ADRC control for the uncertain 2-DOF manipulator when tracking a smooth elliptic-shaped trajectory in the q_1-q_2-phase space; the inertia matrix M is partially known.

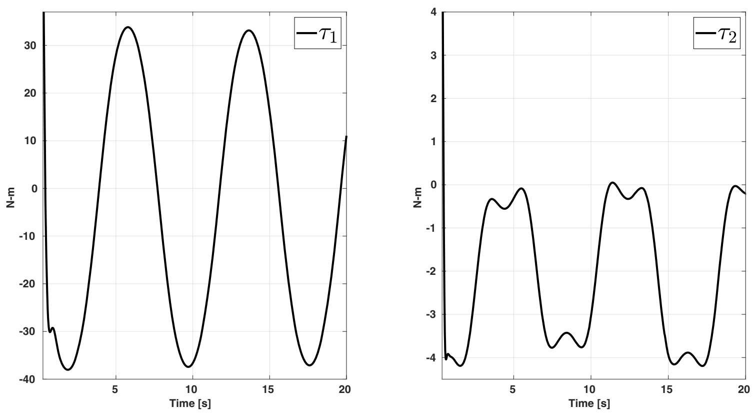

Fig. 6. Control responses τ_1 and τ_2 when tracking a smooth elliptic-shaped trajectory in the q_1-q_2-phase space; the inertia matrix M is partially known.

5. Conclusions

This study proposes an algebraic ADRC-based controller for solving the trajectory tracking problem of an uncertain flat second-order system. The idea behind the method was inspired by the works of Fliess, Sira-Ramirez, and their colleagues (Fliess and Join, 2009; 2013; Fliess and Sira-Ramirez, 2008; Cortés-Romero *et al.*, 2017; Aguilar-Ibanez *et al.*, 2019).

Our controller works as follows: first, the system's unknown dynamics, together with external disturbances, are algebraically identified using a linear set of time-varying integral expressions for the output and the control signal. Then, this information is used in a control law to linearize the perturbed system. We designed the controller to efficiently achieve the disturbance's cancellation by using the algebraically estimated perturbation value and simultaneously solving the tracking control problem for a flat system. We first solve the uncertain flat second-order system case to obtain the above-mentioned solution. Then, we extended it to the case of a second-order multivariable system.

In both the cases, we avoid the necessity of computing the time derivative of the system position. Instead, we compute time-delayed functions. Another essential characteristic of our controller is that it is explicitly computed. That is, we expressed it as an algebraic formula. We use the linear systems theory to carry out the proof of stability. To assess our control solution's effectiveness, we tested it with three numerical experiments, where a 2-DOF manipulator accomplished a number of control tasks. The first one involved

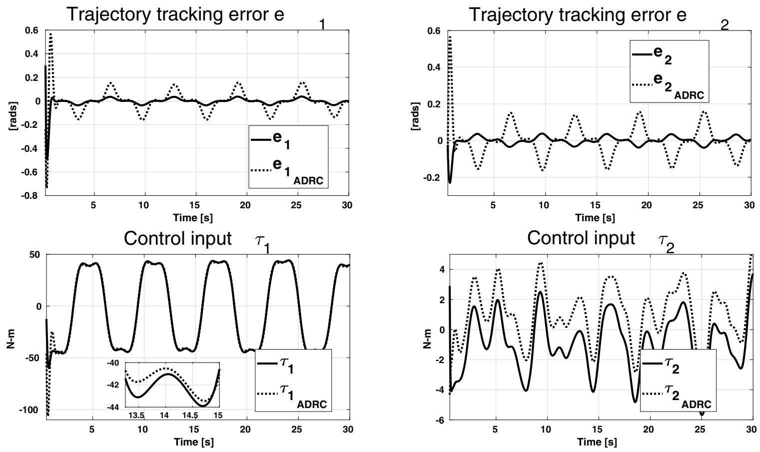

Fig. 7. Comparative results between our control strategy and the traditional ADRC approach; when tracking a circular-shaped trajectory, the inertia matrix M is partially known.

a rest-to-rest maneuver tracking a hyperbolic tangent reference signal while the second involved tracking a smooth, elliptically shaped trajectory. We compared our control solution and the classical ADRC approach in the third experiment. In the three experiments, we obtained satisfactory results.

The paper's main contribution is using the algebraic estimator and avoiding the necessity of using extended state observers to estimate the uncertainties and the system velocity. Additionally, not using extended state observer helps us circumvent the presence of undesired "peaking" and noise amplification associated with the observer needing high gains. Finally, our approach can be further improved for future developments to deal with noise-corrupted measurable output, even considering that the noise is not zero mean.

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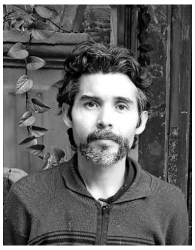

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