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TRANSFORMATIONS OF LINEAR STANDARD SYSTEMS TO POSITIVE ASYMPTOTICALLY STABLE LINEAR ONES

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New approaches to transformations of linear continuous-time systems to their positive asymptotically stable canonical controllable (observable) forms are proposed. It is shown that, if the system matrix is nonsingular, then the desired transformation matrix can be chosen in block diagonal form. Procedures for the computation of the transformation matrices are proposed and illustrated with simple numerical examples.

Keywords: asymptotical stability, positive system, continuous-time system, linear system.

1. Introduction

In positive systems inputs, state variables and outputs take only nonnegative values. Examples of positive systems are industrial processes involving chemical reactors, heat exchangers and distillation columns, storage systems, compartmental systems, or water and atmospheric pollution models. A variety of models having positive linear behavior can be found in engineering, management science, economics, social sciences, biology and medicine. An overview of state of art in positive systems theory is given by Kailath (1980), Mitkowski (2019) and Zak (2003).

The concepts of controllability and observability introduced by Kalman (Kaczorek, 2002; Kaczorek and Borawski, 2021) have been the basic notions in modern control theory. It well known that, if the linear system is controllable, then with the use of state feedback it is possible to modify the dynamical properties of closed-loop systems (Antsaklis and Michel, 1997; Hautus and Heymann, 1978; Kaczorek, 1992; 2002; Kaczorek and Borawski, 2021; Kaczorek and Rogowski, 2015; Kailath, 1980; Kalman, 1960; 1963; Klamka, 1991; 2018; Mitkowski, 2019). If the linear system is observable then it is possible to design an observer which reconstructs the state vector of the system (Antsaklis and Michel, 1997; Hautus and Heymann, 1978; Kaczorek, 1992; 2002; Kaczorek and Borawski, 2021; Kaczorek and Rogowski, 2015; Kailath, 1980; Kalman, 1960; 1963; Klamka, 1991; 2018; Mitkowski, 2019). Descriptor systems of integer and fractional order were analyzed by Kaczorek (2002) and Klamka (2018). The stabilization of positive descriptor fractional linear systems with two different fractional orders by a decentralized controller was investigated by Sajewski (2018; 2017). The eigenvalue assignment in uncontrollable linear continuous-time systems was analyzed by Kaczorek (2022).

In this paper, new approaches to the transformations of linear continuous-time systems to their positive asymptotically stable canonical controllable (observable) forms are proposed. In Section 2 some basic definitions and theorems concerning linear standard continuous-time systems are recalled. A new approach to the transformations of linear systems to their asymptotically stable controllable canonical forms is proposed in Section 3 and extended to observable canonical forms in Section 4. Concluding remarks are given in Section 5.

The following notation will be used: $\mathbb{R}$, the set of real numbers; $\mathbb{R}^{n\times m}$, the set of $n \times m$ real matrices; I_n, the $n \times n$ identity matrix; M_n, the set of Metzler matrices (matrices with nonnegative off-diagonal entries); $\mathbb{R}_+^{n\times m}$, the set of $n \times m$ matrices with nonnegative entries.

2. Preliminaries

Consider the linear continuous-time system

$$\dot{x} = Ax + Bu, \tag{1a}$$
$$y = Cx, \tag{1b}$$

where $x = x(t) \in \mathbb{R}^n$, $u = u(t) \in \mathbb{R}^m$, $y = y(t) \in \mathbb{R}^p$ are the state, input and output vectors, respectively, and $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$.

Theorem 1. (Kaczorek and Borawski, 2021; Kaczorek and Rogowski, 2015; Kailath, 1980) *The solution of Eqn. (1a) has the form*

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-\tau)}Bu(\tau)\,\mathrm{d}\tau, \quad x_0 = x(0). \tag{2}$$

Definition 1. (*Kaczorek and Borawski, 2021; Kaczorek and Rogowski, 2015; Kailath, 1980*) The system (1) is called (internally) *positive* if the state vector $x(t) \in \mathbb{R}^n_+$, the output vector $y(t) \in \mathbb{R}^p_+$ for $t \geq 0$ and all initial conditions $x(0) \in \mathbb{R}^n_+$, and all inputs $u(t) \in \mathbb{R}^m_+$ for $t \geq 0$.

Definition 2. (*Kaczorek and Borawski, 2021; Kaczorek and Rogowski, 2015; Kailath, 1980*) A real matrix $A = [a_{ij}] \in \mathbb{R}^{n\times n}$ is called a *Metzler matrix* if its off a diagonal entries are nonnegative, i.e., $a_{ij} \geq 0$ for $i \neq j$.

Lemma 1. (Kaczorek and Borawski, 2021; Kaczorek and Rogowski, 2015; Kailath, 1980) *Let $A \in \mathbb{R}^{n\times n}$. Then*

$$e^{At}x_0 \in \mathbb{R}^{n\times n}_+, \quad t \geq 0 \tag{3}$$

if and only if A is a Metzler matrix.

Theorem 2. (Kaczorek and Borawski, 2021; Kaczorek and Rogowski, 2015; Kailath, 1980) *The linear system (1) is positive if and only if*

$$A \in M_n, \quad B \in \mathbb{R}^{n\times m}_+, \quad C \in \mathbb{R}^{p\times n}_+, \tag{4}$$

where M_n is the set of Metzler matrices.

Definition 3. (*Kaczorek and Rogowski, 2015*) The positive system (1) is called *reachable* in time $[0, t_f]$ if there exists an input $u(t) \in \mathbb{R}^m_+$ for $t \in [0, t_f]$ which steers the state of the system from the zero initial condition $x(0) = 0$ to the final state $x_f = x(t_f)$.

Definition 4. (*Kaczorek and Borawski, 2021; Kaczorek and Rogowski, 2015; Kailath, 1980*) A square matrix is called *monomial* if its every column and its every row have only one positive entry and the remaining entries are zero.

Theorem 3. (Kaczorek and Borawski, 2021; Kaczorek and Rogowski, 2015; Kailath, 1980) *The positive system (1) is reachable if and only if the matrix*

$$R_f = \int_0^{t_f} e^{At}BB^Te^{At}dt, \quad t_f > 0 \tag{5}$$

is monomial.

Consider the linear discrete-time system

$$x_{i+1} = Ax_i + Bu_i, \tag{6a}$$
$$y_i = Cx_i, \tag{6b}$$

where $x_i \in \mathbb{R}^n$, $u_i \in \mathbb{R}^m$, $y_i \in \mathbb{R}^p$ are the state, input and output vectors, respectively, and $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$.

Theorem 4. (Kaczorek, 2002) *The solution of Eqn. (6a) has the form*

$$x_i = A^ix_0 + \sum_{k=0}^{i-1} A^{i-k-1}Bu_k, \quad i = 0, 1, \dots \tag{7}$$

Definition 5. (*Kaczorek and Borawski, 2021*) The system (6) is called (internally) *positive* if the state vector $x_i \in \mathbb{R}^n_+$, the output vector $y_i \in \mathbb{R}^p_+$ for $i = 0, 1, \dots$ for all initial conditions $x_0 \in \mathbb{R}^n_+$ and all inputs $u_i \in \mathbb{R}^m_+$ for $i = 0, 1, \dots$

Theorem 5. (Kaczorek and Borawski, 2021) *The linear system (6) is positive if and only if*

$$A \in \mathbb{R}^{n\times n}_+, \quad B \in \mathbb{R}^{n\times m}_+, \quad C \in \mathbb{R}^{p\times n}_+. \tag{8}$$

Definition 6. (*Kaczorek and Borawski, 2021; Kaczorek and Rogowski, 2015; Kailath, 1980*) The positive system (6) is called reachable in n steps if there exists an input sequence $u_i \in \mathbb{R}^m_+$ for $i = 0, 1, \dots, n-1$ which steers the state of the system from the zero initial condition to the final state $x_f = x_n$.

Theorem 6. (Kaczorek and Borawski, 2021; Kaczorek and Rogowski, 2015; Kailath, 1980) *The linear system (6) is reachable if*

$$\operatorname{rank} R = \operatorname{rank}\begin{bmatrix} B & AB & \dots & A^{n-1}B \end{bmatrix} = n \tag{9a}$$

and

$$R^T[RR^T] \in \mathbb{R}^{nm\times n}_+. \tag{9b}$$

Remark 1. The single input ($m = 1$) positive system (1) is reachable in n steps if the matrix

$$R = \begin{bmatrix} B & AB & \dots & A^{n-1}B \end{bmatrix} \in \mathbb{R}^{n\times n}_+ \tag{10}$$

is a permutation matrix.

Theorem 7. (Gantmacher, 1959 (Kronecker–Capelli, Rouch–Capelli)) *The matrix equation*

$$PX = Q, \quad P \in \mathbb{R}^{n\times p}, \quad Q \in \mathbb{R}^{n\times q} \tag{11}$$

has a solution X if and only if

$$\operatorname{rank}\begin{bmatrix} P & Q \end{bmatrix} = \operatorname{rank} P. \tag{12}$$

Theorem 8. (Gantmacher, 1959) *If the condition* (12) *is satisfied, then the solution $X \in \mathbb{R}^{p\times 1}$ of the matrix equation* (11) *is given by*

$$X = \Big\{P^T[PP^T]^{-1} + (I_q - P^T[PP^T]^{-1}P)K_1\Big\}Q \tag{13a}$$

or

$$X = K_2[PK_2]^{-1}Q, \tag{13b}$$

where K_1 and K_2 are real matrices.

3. Positive linear continuous-time systems with controllable pairs (A, B) in canonical forms

Problem 1. For a given pair (A, B) of the system (1) for $m = 1$ satisfying the condition

$$\operatorname{rank}\begin{bmatrix} A & B \end{bmatrix} = n, \tag{14}$$

find a nonsingular matrix

$$M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \in \mathbb{R}^{(n+1)\times(n+1)}, \quad M_{11} \in \mathbb{R}^{n\times n}, \quad M_{22} \in \mathbb{R}^{1\times 1} \tag{15}$$

such that

$$\begin{bmatrix} A & B \end{bmatrix} M = \begin{bmatrix} \bar{A} & \bar{B} \end{bmatrix}, \tag{16}$$

where the pair $(\bar{A}, \bar{B})$ is in controllable canonical form:

$$\bar{A} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 & \dots & -a_{n-1} \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \tag{17a}$$

or

$$\bar{A} = \begin{bmatrix} 0 & 0 & \dots & 0 & -a_0 \\ 1 & 0 & \dots & 0 & -a_1 \\ 0 & 1 & \dots & 0 & -a_2 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 & -a_{n-1} \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}. \tag{17b}$$

The following two cases will be analyzed.

Case 1. The matrix A is nonsingular,

$$\det A \neq 0. \tag{18}$$

Case 2. The matrix A is singular,

$$\det A = 0. \tag{19}$$

Case 1. It will be shown that, if the condition (18) is satisfied then we may assume $M_{21} = 0$. In this case,

$$\begin{bmatrix} A & B \end{bmatrix} \begin{bmatrix} M_{11} & M_{12} \\ 0 & M_{22} \end{bmatrix} = \begin{bmatrix} \bar{A} & \bar{B} \end{bmatrix} \tag{20}$$

and

$$AM_{11} = \bar{A}. \tag{21}$$

From (21) we have

$$M_{11} = A^{-1}\bar{A} \tag{22}$$

since $\det A \neq 0$.

From (20) we also have

$$\begin{bmatrix} A & B \end{bmatrix} \begin{bmatrix} M_{12} \\ M_{22} \end{bmatrix} = \bar{B}. \tag{23}$$

From (14) and Theorem 7 it follows that Eqn. (23) has a solution M_{22} since we may choose M_{12} such that

$$\operatorname{rank}\begin{bmatrix} B & \bar{B} - AM_{12} \end{bmatrix} = \operatorname{rank} B \tag{24}$$

and the equation

$$BM_{22} = \bar{B} - AM_{12} \tag{25}$$

has a solution M_{22} for given $\bar{B}$ and M_{12}.

Therefore, the following theorem has been proved.

Theorem 9. *If the condition* (18) *is satisfied, then the matrix* (15) *can be chosen in the form*

$$M = \begin{bmatrix} M_{11} & M_{12} \\ 0 & M_{22} \end{bmatrix}, \tag{26}$$

where the matrix M_{11} is given by (22) *and M_{22} is a solution of Eqn.* (25).

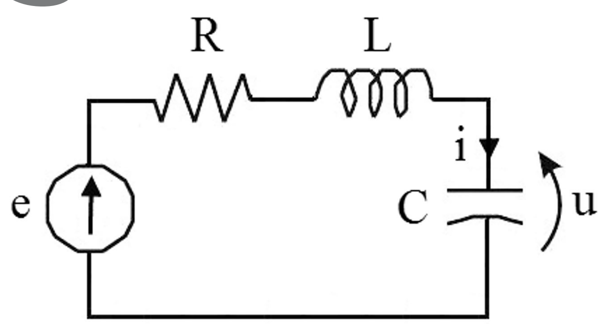

Fig. 1. Linear electrical circuit.

Theorem 10. *If* $\det A \neq 0$ *and* $\bar{B} = cB$, $c \in \mathbb{R}$, *then the matrix* M *has the block diagonal form*

$$M = \begin{bmatrix} M_{11} & 0 \\ 0 & M_{22} \end{bmatrix}. \tag{27}$$

Proof. By Theorem 9, if $\det A \neq 0$, then $M_{21} = 0$. From the equality (25) it follows that if $\bar{B} = cB$, then we may assume $M_{12} = 0$ and $M_{22} = cI_m$, where m is the number of columns of B and $\bar{B}$. ■

To compute the matrix (15) for given controllable pairs (A, B) and $(\bar{A}, \bar{B})$ satisfying (14), the following procedure can be used.

Procedure 1. Determining matrix M for Case 1.

Step 1. Check the condition (18) and the controllability of the pair $(\bar{A}, \bar{B})$.

Step 2. Using (22), compute the matrix M_{11} for given matrices A and $\bar{A}$.

Step 3. Choose the matrix M_{12} satisfying (24).

Step 4. Using (25) compute the matrices M_{22} and M.

Example 1. For the electrical circuit shown in Fig. 1 with resistance $R = 2$, inductance L =1, capacitance $C = 1$, find the nonsingular matrix (15) satisfying Eqn. (16).

Using Kirchhoff's laws we may write, for the electrical circuit, the equations

$$e = Ri + L\frac{\mathrm{d}i}{\mathrm{d}t} + u, \quad i = C\frac{\mathrm{d}u}{\mathrm{d}t}, \tag{28}$$

where e is the source voltage, i is the current, u is the voltage on the capacitor and R, L, C are the resistance, inductance and capacitance, respectively. As the state variables we choose current i and voltage u.

Equations (28) can be written in the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} i \\ u \end{bmatrix} = A_1 \begin{bmatrix} i \\ u \end{bmatrix} + B_1 e,$$
$$A_1 = \begin{bmatrix} -\frac{R}{L} & -\frac{1}{L} \\ \frac{1}{C} & 0 \end{bmatrix}, \tag{29a}$$
$$B_1 = \begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix}$$

or

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} u \\ i \end{bmatrix} = A_2 \begin{bmatrix} u \\ i \end{bmatrix} + B_2 e,$$
$$A_2 = \begin{bmatrix} 0 & \frac{1}{C} \\ -\frac{1}{L} & -\frac{R}{L} \end{bmatrix}, \tag{29b}$$
$$B_2 = \begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix}$$

Note that the matrices A_1 and A_2 are different but their characteristic polynomials

$$\det[I_2 s - A_1] = \begin{vmatrix} s + \frac{R}{L} & \frac{1}{L} \\ -\frac{1}{C} & s \end{vmatrix} = s^2 + \frac{R}{L}s + \frac{1}{LC} \tag{30a}$$

and

$$\det[I_2 s - A_2] = \begin{vmatrix} s & -\frac{1}{C} \\ \frac{1}{L} & s + \frac{R}{L} \end{vmatrix} = s^2 + \frac{R}{L}s + \frac{1}{LC} \tag{30b}$$

are equal.

Using Procedure 1, we compute the matrices separately for the pairs (A_1, B_1), (A_2, B_2) and

$$\bar{A} = \begin{bmatrix} -2 & 1 \\ 1 & -3 \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{31}$$

Using Procedure 1 for the pair (A_1, B_1), we obtain the following.

Step 1. The condition (18) is satisfied since $\det A_1 = 1/LC$ is nonzero and the pair (31) is controllable.

Step 2.Using (22), (31) and

$$A_1 = \begin{bmatrix} -\frac{R}{L} & -\frac{1}{L} \\ \frac{1}{C} & 0 \end{bmatrix} = \begin{bmatrix} -2 & -1 \\ 1 & 0 \end{bmatrix}, \tag{32}$$

we obtain

$$M_{11} = A_1^{-1}\bar{A} = \begin{bmatrix} -2 & -1 \\ 1 & 0 \end{bmatrix}^{-1} \begin{bmatrix} -2 & 1 \\ 1 & -3 \end{bmatrix} = \begin{bmatrix} 1 & -3 \\ 0 & 5 \end{bmatrix}. \tag{33}$$

Step 3. In this case, the matrix M_{12} satisfying (24) has the form

$$M_{12} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

since

$$B_1 M_{22} = \bar{B} - A_1 M_{12} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} - \begin{bmatrix} -2 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \tag{34}$$

Step 4. Using (34) and

$$B_1 = \begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

we obtain $M_{22} = [1]$. Therefore, the desired matrix M has the form

$$M = \begin{bmatrix} M_{11} & M_{12} \\ 0 & M_{22} \end{bmatrix} = \begin{bmatrix} 1 & -3 & 1 \\ 0 & 5 & -1 \\ 0 & 0 & 1 \end{bmatrix}. \tag{35}$$

For the pair (A_2, B_2) we obtain, respectively, the following.

Step 1. The condition (18) is satisfied since $\det A_2 = 1/LC = 1$ is nonsingular and the pair (31) is controllable.

Step 2. Using (18), (29) and

$$A_2 = \begin{bmatrix} 0 & \frac{1}{C} \\ -\frac{1}{L} & -\frac{R}{L} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}, \tag{36}$$

we obtain

$$\begin{aligned} M_{11} = A_2^{-1}\bar{A} &= \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}^{-1} \begin{bmatrix} -2 & 1 \\ 1 & -3 \end{bmatrix} \\ &= \begin{bmatrix} 3 & 1 \\ -2 & 1 \end{bmatrix}. \end{aligned} \tag{37}$$

Step 3. In this case,

$$\bar{B} = B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

and by Theorem 10,

$$M_{12} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

since

$$\begin{aligned} B_2 M_{22} &= \bar{B} - A_2 M_{12} \\ &= \begin{bmatrix} 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \end{aligned} \tag{38}$$

Step 4. Using (36) and

$$B_2 = \begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

from (38) we obtain $M_{22} = [1]$. Therefore, the desired matrix *M* in this case has the form

$$M = \begin{bmatrix} M_{11} & M_{12} \\ 0 & M_{22} \end{bmatrix} = \begin{bmatrix} 3 & 1 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{39}$$

and is different from the matrix (35). ♦

Case 2. If the matrix A is singular $(\det A = 0)$, then from (16) we have

$$\begin{bmatrix} A & B \end{bmatrix} \begin{bmatrix} M_{11} \\ M_{21} \end{bmatrix} = \bar{A} \tag{40a}$$

and

$$\begin{bmatrix} A & B \end{bmatrix} \begin{bmatrix} M_{12} \\ M_{22} \end{bmatrix} = \bar{B}. \tag{40b}$$

By Theorem 7 Eqns. (40) have solutions by the assumption

$$\operatorname{rank} \begin{bmatrix} A & B \end{bmatrix} = n. \tag{41}$$

In this case, using Theorem 8, we may find the solutions of Eqns. (40) and the desired nonsingular matrix (40).

Therefore, we have the following result.

Theorem 11. *If the matrix A is singular and the condition* (41) *is satisfied, then there exists a nonsingular matrix M such that the pair (A, B) is in controllable canonical form (17).*

In this case, the matrix M can be computed by the following procedure.

Procedure 2. Determining matrix M for Case 2.

Step 1. Using (13), find the general solution of Eqns. (40).

Step 2. Choose the matrix K_1 (or K_2) such that the matrix M is nonsingular.

Example 2. Given the matrices

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{42}$$

and

$$\bar{A} = \begin{bmatrix} -2 & 1 \\ 1 & -3 \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \tag{43}$$

compute the matrix (15) such that the condition (16) is satisfied.

Using Procedure 2, we obtain what follows.

Steps 1 and 2. Using (42), (43) and (40a), we obtain

$$\begin{aligned} \begin{bmatrix} A & B \end{bmatrix} \begin{bmatrix} M_{11} \\ M_{21} \end{bmatrix} &= \begin{bmatrix} 0 & 1 & 1 \\ 0 & 2 & 0 \end{bmatrix} \begin{bmatrix} M_{11} \\ M_{21} \end{bmatrix} \\ &= \bar{A} = \begin{bmatrix} -2 & 1 \\ 1 & -3 \end{bmatrix} \end{aligned} \tag{44}$$

and

$$\begin{bmatrix} M_{11} \\ M_{21} \end{bmatrix}$$
$$= \Big\{ [\ A \quad B\ ]^T [[\ A \quad B\ ][\ A \quad B\ ]^T]^{-1}$$
$$+ (I_3 - [\ A \quad B\ ]^T ([\ A \quad B\ ][\ A \quad B\ ]^T)^{-1}$$
$$\times [\ A \quad B\ ])K_1 \Big\} \bar{A}$$
$$= \left\{ \begin{bmatrix} 0 & 0 \\ 1 & 2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 2 \\ 2 & 4 \end{bmatrix}^{-1} + \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right. \right. \tag{45}$$
$$\left. - \begin{bmatrix} 0 & 0 \\ 1 & 2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 2 \\ 2 & 4 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 1 & 1 \\ 0 & 2 & 0 \end{bmatrix} \right)$$
$$\left. \times \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \\ k_{31} & k_{32} \end{bmatrix} \right\} \begin{bmatrix} -2 & 1 \\ 1 & -3 \end{bmatrix}$$
$$= \begin{bmatrix} k_{11} & 0 \\ 0.5 & -1.5 \\ -2.5 & 2.5 \end{bmatrix}$$

for $k_{11} \neq 0$.

Similarly, using (40b), (42) and (43) for $K = 0$, we obtain

$$\begin{bmatrix} M_{12} \\ M_{22} \end{bmatrix} = [\ A \quad B\ ]^T ([\ A \quad B\ ][\ A \quad B\ ]^T)^{-1},$$
$$\bar{B} = \begin{bmatrix} 0 & 0 \\ 1 & 2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 2 \\ 2 & 4 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$
$$= \begin{bmatrix} 0 \\ 0.5 \\ -0.5 \end{bmatrix}. \tag{46}$$

Therefore, the desired matrix *M* has the form

$$M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}$$
$$= \begin{bmatrix} k_{11} & 0 & 0 \\ 0.5 & -1.5 & 0.5 \\ -2.5 & 2.5 & -0.5 \end{bmatrix}. \tag{47}$$

♦

4. Positive linear systems with (A, C) in canonical forms

In this section the findings of Section 3 will be extended to continuous-time linear systems with the observable pairs (A, C) in canonical forms. To simplify the notation, we assume $p = 1$ (single-output systems).

Problem 1. For a given pair (A, C) of the system (1) for $p = 1$ satisfying the condition

$$\operatorname{rank} \begin{bmatrix} A \\ C \end{bmatrix} = n, \tag{48}$$

find a nonsingular matrix (15) such that

$$M \begin{bmatrix} A \\ C \end{bmatrix} = \begin{bmatrix} \hat{A} \\ \hat{C} \end{bmatrix}, \tag{49}$$

where the $(\hat{A}, \hat{C})$ is in observable canonical form:

$$\hat{A} = \begin{bmatrix} 0 & 0 & \dots & 0 & -a_2 \\ 1 & 0 & \dots & 0 & -a_1 \\ 0 & 1 & \dots & 0 & -a_2 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 & -a_{n-1} \end{bmatrix},$$
$$\hat{C} = [\ 0 \quad \dots \quad 0 \quad 1\ ] \tag{50a}$$

or

$$\hat{A} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 & \dots & -a_{n-1} \end{bmatrix},$$
$$\hat{C} = [\ 1 \quad 0 \quad \dots \quad 0\ ] \tag{50b}$$

Case 1. The matrix A is nonsingular—the condition (18) is satisfied.

Case 2. The matrix A is singular—the condition (19) is satisfied.

In Case 1, if $\det A \neq 0$, then we may assume $M_{12} = 0$, and from

$$\begin{bmatrix} M_{11} & 0 \\ M_{21} & M_{22} \end{bmatrix} \begin{bmatrix} A \\ C \end{bmatrix} = \begin{bmatrix} \hat{A} \\ \hat{C} \end{bmatrix} \tag{51}$$

we have

$$M_{11} A = \hat{A} \tag{52}$$

and

$$M_{11} = \hat{A} A^{-1}. \tag{53}$$

From (51) we also have

$$[\ M_{21} \quad M_{22}\ ] \begin{bmatrix} A \\ C \end{bmatrix} = \hat{C}. \tag{54}$$

From Theorem 7 it follows that Eqn. (54) has a solution M_{22} since we may choose M_{21} such that

$$\operatorname{rank} \begin{bmatrix} C \\ \hat{C} - M_{21} A \end{bmatrix} = \operatorname{rank} C, \tag{55}$$

and the equation

$$M_{22} C = \hat{C} - M_{21} A \tag{56}$$

has a solution M_{22} for given $\hat{C}$ and M_{21}.

Therefore, the following result has been proved.

Theorem 12. *If (50b) holds, then the matrix M can be chosen in the form*

$$M = \begin{bmatrix} M_{11} & 0 \\ M_{21} & M_{22} \end{bmatrix}, \tag{57}$$

where M_{11} is given by (53) *and M_{22} is a solution of Eqn.* (56)*.*

Theorem 13. *If (50a) holds and*

$$\operatorname{rank} C = \operatorname{rank} \begin{bmatrix} C \\ \hat{C} \end{bmatrix}, \tag{58}$$

then the matrix M has the block diagonal form

$$M = \begin{bmatrix} M_{11} & 0 \\ 0 & M_{22} \end{bmatrix}. \tag{59}$$

The proof is similar (dual) to that of Theorem 10.

In Case 2, if the matrix A is singular, then

$$\begin{bmatrix} M_{11} & M_{12} \end{bmatrix} \begin{bmatrix} A \\ C \end{bmatrix} = \hat{A} \tag{60a}$$

and

$$\begin{bmatrix} M_{21} & M_{22} \end{bmatrix} \begin{bmatrix} A \\ C \end{bmatrix} = \hat{C}. \tag{60b}$$

By Theorem 7, Eqns. (60) have solutions since

$$\operatorname{rank} \begin{bmatrix} A \\ C \end{bmatrix} = n \tag{61}$$

In this case, using Theorem 8 we may find the solutions of Eqns. (60) and the matrix M.

Therefore, we have the following result.

Theorem 14. *If the matrix A is singular and the condition* (61) *is satisfied, then there exists a nonsingular matrix M such that the pair (A, C) is in observable canonical form (50).*

The presented discussion for continuous-time linear systems can be easily extended to discrete-time linear ones described by Eqns. (6).

5. Concluding remarks

New approaches to the transformations of linear continuous-time systems to their positive asymptotically stable canonical controllable (observable) forms were proposed. Conditions were established under which the problems have solutions (Theorems 10–12, 12–14). Procedures for the computation of the transformation matrices were given and illustrated with simple numerical examples. The new approaches can be extended to linear discrete-time systems as well as fractional linear continuous-time and discrete-time ones. An open problem is an extension of this approach to fractional linear systems.

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