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DISCRETE–TIME LINEAR SYSTEMS WITH DESIRED POLES AND ZEROS OF THE TRANSFER MATRICES

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Methods for the design of discrete-time linear systems with desired poles and zeros of their transfer matrices are proposed. Conditions for the existence of the solution to the problem and the procedures for computation of the desired matrices are given. Reduction of the systems with controllable and observable pairs to those with nilpotent matrices is analysed. The procedures are illustrated by simple numerical examples of linear discrete-time systems.

Keywords: design method, linear system, discrete-time system, nilpotent system, pole, zero, transfer matrix.

1. Introduction

The concepts of controllability and observability introduced by Kalman (1960; 1963) have been the basic notions of the modern control theory. It well known that if a linear system is controllable then by state feedback it is possible to modify the dynamical properties of the closed-loop system (Antsaklis and Michel, 1997; Hautus and Heymann 1978; Kaczorek, 1992, Kailath, 1980; Klamka, 1991; 2018; Mitkowski, 2019, Zak, 2003). If a linear system is observable then it is possible to design an observer which reconstructs the state vector of the system (Antsaklis and Michel, 1997; Kaczorek, 1992; Mitkowski, 2019; Emirsajłow, 2021).

Kaczorek (2021) proposed a method for pole-zero assignment by state feedback in positive linear systems. He then introduced a method for eigenvalue assignment in uncontrollable linear systems by state feedback (Kaczorek, 2022) and a method for eigenvalue assignment in descriptor linear systems by state-derivative feedback (Kaczorek, 2023a). Further transformations of the matrices of linear systems to their canonical form with desired eigenvalues were also proposed (Kaczorek, 2023b) and a method for transformations of linear standard systems to positive asymptotically stable linear systems was presented (Kaczorek, 2024).

Global stability of discrete-time feedback nonlinear systems with descriptor positive linear parts and interval state matrices was considered by Kaczorek and Ruszewski (2022). Veselić *et al*. (2020) analyzed the discrete–time sliding mode control of linear systems with input saturation. A decentralized static output feedback controller design for linear interconnected systems was introduced by Fadhilah *et al*. (2023).

In this paper, in much the same way as in the works of Kaczorek (2023b; 2024), the design of linear systems with desired stable poles and zeros of their transfer matrices will be extended to linear discrete-time systems. The previously mentioned papers dealt witch continuous-time linear systems and they were focused on systems with singular state matrices as well as positive systems. This paper addresses SISO and MIMO discrete-time standard (nonpositive) linear systems as well as reduction of the discrete-time systems with controllable and observable pairs to systems with nilpotent matrices.

In Section 2 some basic definitions and theorems concerning the controllability and observability of linear discrete-time systems and of the matrix equations with nonsquare matrices and their solutions are recalled. The proposed approach for SISO systems is presented in Section 3 and for MIMO linear systems in Section 4. In Section 5 the reduction of the discrete-time systems with controllable and observable pairs to systems with nilpotent matrices is analyzed. In Section 6 a new method based on the matrix decomposition is proposed

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and illustrated by simple numerical examples. Concluding remarks are given in Section 7.

The following notation will be used: $\mathbb{R}$, the set of real numbers; $\mathbb{R}^{n\times m}$, the set of $n\times m$ real matrices; I_n, the $n\times n$ identity matrix.

2. Basic definitions and theorems for linear discrete-time systems

Consider the linear discrete-time system

$$x_{i+1} = Ax_i + Bu_i, \tag{1}$$
$$y_i = Cx_i + Du_i \tag{2}$$

where $x_i \in \mathbb{R}^n$, $u_i \in \mathbb{R}^m$, $y_i \in \mathbb{R}^p$ are the state, input and output vectors, respectively, and $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$, $D \in \mathbb{R}^{p\times m}$.

Theorems 1–3 as well as Definitions 1–4 are standard and can be found, e.g., in the works of Antsaklis and Michel (1997), Kaczorek (1992), Kailath (1980), Mitkowski (2019) or Zak (2003).

Theorem 1. *The solution of (1) has the form*

$$x_{i+1} = A^i x_i + \sum_{k=0}^{i-1} A^{i-k-1}Bu_i, \quad i \in \mathbb{Z}_+. \tag{3}$$

Definition 1. The linear discrete-time system (1)–(2) is called *controllable* in the range from 0 to q if there exists an input u_i for $i = 0, 1, \dots, q-1$ which steers the state of the system from the initial condition $x_0 \in \mathbb{R}^n$ to the final state $x_f = x_q \in \mathbb{R}^n$.

Theorem 2. *The linear system (1) is controllable if and only if*

$$\operatorname{rank}[\begin{array}{cccc} B & AB & \dots & A^{n-1}B \end{array}] = n, \tag{4}$$

$$\operatorname{rank}[\begin{array}{cc} I_n z - A & B \end{array}] = n, \quad z \in \mathbb{C}, \tag{5}$$

where $\mathbb{C}$ is the field of complex numbers.

Definition 2. The discrete-time linear system (1)–(2) is called *observable* if knowing its input u_i and output y_i in the interval range from 0 to $q-1$ it is possible to find its unique initial condition x_0.

Theorem 3. *The discrete-time linear system (1)–(2) is observable if and only if one the following conditions is satisfied:*

(i)

$$\operatorname{rank}\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} = n, \tag{6}$$

(ii)

$$\operatorname{rank}\begin{bmatrix} I_n z - A \\ C \end{bmatrix} = n, \quad z \in \mathbb{C} \tag{7}$$

where $\mathbb{C}$ is the field of complex numbers.

Definition 3. The pair (A, B) for $m = 1$ is in the *controllable canonical Frobenius* if

$$A = \begin{bmatrix} 0 & 0 & \dots & 0 & -a_0 \\ 1 & 0 & \dots & 0 & -a_1 \\ 0 & 1 & \dots & 0 & -a_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & -a_{n-1} \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{8}$$

or

$$A = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -a_0 & -a_1 & -a_2 & \dots & -a_{n-1} \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}. \tag{9}$$

Definition 4. The pair (A, C) for $p = 1$ is in the *observable canonical Frobenius* if

$$A = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -a_0 & -a_1 & -a_2 & \dots & -a_{n-1} \end{bmatrix}, \quad C = [\begin{array}{cccc} 1 & 0 & \dots & 0 \end{array}] \tag{10}$$

or

$$A = \begin{bmatrix} 0 & 0 & \dots & 0 & -a_0 \\ 1 & 0 & \dots & 0 & -a_1 \\ 0 & 1 & \dots & 0 & -a_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & -a_{n-1} \end{bmatrix}, \quad C = [\begin{array}{cccc} 0 & \dots & 0 & 1 \end{array}]. \tag{11}$$

The canonical forms of controllable pair (A, B) for $m > 1$ and the canonical forms of the observable pairs (A, C) for $p > 1$ are given by Antsaklis and Michel (1997), Kaczorek (1992), Kailath (1980), Mitkowski (2019) and Zak (2003).

The transfer matrix of the system (1)–(2) has the form

$$T(z) = C[I_n z - A]^{-1}B + D. \tag{12}$$

Definition 5. The matrix $A \in \mathbb{R}^{n\times n}$ is called *nilpotent* if there exists an integer $0 < q \leq n$ such that $A^q = 0$.

Theorem 4. (Gantmacher, 1959) *The matrix equation*

$$PX = Q, \quad P \in \mathbb{R}^{n\times p}, \quad Q \in \mathbb{R}^{n\times q} \tag{13}$$

has a solution X if and only if

$$\operatorname{rank}[\begin{array}{cc} P & Q \end{array}] = \operatorname{rank} P. \tag{14}$$

Theorem 5. (Gantmacher, 1959) *If the condition (14) is satisfied then the solution $X \in \mathbb{R}^{p\times q}$ of the matrix equation (13) for $P \in \mathbb{R}^{n\times p}$ is given by*

$$\begin{aligned} X = \{&P^T[PP^T]^{-1} \\ &+(I_q - P^T[PP^T]^{-1}P)K_1\}\,Q \end{aligned} \tag{15}$$

or

$$X = K_2[PK_2]^{-1}Q, \tag{16}$$

where K_1 and K_2 are real matrices.

Definition 6. (*Kaczorek, 1992*) The following operations are called the *elementary operations* on the real matrix $A \in \mathbb{R}^{n\times n}$:

- multiplication of the i-th row (column) by a real $a \neq 0$,
- addition of the j-th row (column), multiplied by a number $b \neq 0$, to the i-th row (column),
- interchange of any two rows (columns).

The elementary operations do not change the rank of the matrix A (Kaczorek, 1992).

3. Proposed method for analysis of discrete-time SISO linear systems

Let $\bar{x}_i \in \mathbb{R}^n$, $\bar{u}_i \in \mathbb{R}^m$, $\bar{y}_i \in \mathbb{R}^p$ be respectively the new state, input and output vectors of the discrete-time system (1)–(2) and

$$\begin{bmatrix} \bar{x}_{i+1} \\ \bar{y}_i \end{bmatrix} = M \begin{bmatrix} x_{i+1} \\ y_i \end{bmatrix}, \qquad \det M \neq 0, \quad i = 0, 1, \dots \tag{17}$$

and

$$\begin{bmatrix} x_i \\ u_i \end{bmatrix} = N \begin{bmatrix} \bar{x}_i \\ \bar{u}_i \end{bmatrix}, \qquad \det N \neq 0, \quad i = 0, 1, \dots \tag{18}$$

where $M \in \mathbb{R}^{(n+p)\times(n+p)}$, $N \in \mathbb{R}^{(n+m)\times(n+m)}$.

From (17) and (18) we have

$$\begin{aligned} \begin{bmatrix} \bar{x}_{i+1} \\ \bar{y}_i \end{bmatrix} &= M \begin{bmatrix} x_{i+1} \\ y_i \end{bmatrix} = M \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} x_i \\ u_i \end{bmatrix} \\ &= M \begin{bmatrix} A & B \\ C & D \end{bmatrix} N \begin{bmatrix} \bar{x}_i \\ \bar{u}_i \end{bmatrix} \\ &= \begin{bmatrix} \bar{A} & \bar{B} \\ \bar{C} & \bar{D} \end{bmatrix} \begin{bmatrix} \bar{x}_i \\ \bar{u}_i \end{bmatrix}, \quad i = 0, 1, \dots \end{aligned} \tag{19}$$

where

$$\begin{bmatrix} \bar{A} & \bar{B} \\ \bar{C} & \bar{D} \end{bmatrix} = M \begin{bmatrix} A & B \\ C & D \end{bmatrix} N. \tag{20}$$

Case 1: $M = I_{n+p}$. In this case (20) has the form

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} N = \begin{bmatrix} \bar{A} & \bar{B} \\ \bar{C} & \bar{D} \end{bmatrix} \tag{21}$$

and it has a solution if the condition of Theorem 4 is satisfied.

Case 2: $N = I_{n+m}$. In this case (20) has the form

$$\begin{bmatrix} \bar{A} & \bar{B} \\ \bar{C} & \bar{D} \end{bmatrix} = M \begin{bmatrix} A & B \\ C & D \end{bmatrix} \tag{22}$$

and after transposition we obtain

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^T M^T = \begin{bmatrix} \bar{A} & \bar{B} \\ \bar{C} & \bar{D} \end{bmatrix}^T. \tag{23}$$

The solution M^T of (23) can be found using Theorem 5. Therefore, Case 2 has been reduced to Case 1.

Knowing the desired poles and zeros of the transfer matrix, we choose the corresponding matrices $\bar{A}, \bar{B}, \bar{C}, \bar{D}$. The choice of these matrices $\bar{A}, \bar{B}, \bar{C}$ in canonical Frobenius form is recommended.

The details of this approach will be shown on the following simple numerical examples.

Example 1. Consider the system (1)–(2) with the matrices

$$\begin{aligned} A &= \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, & B &= \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \\ C &= [\begin{array}{cc} 1 & 0 \end{array}], & D &= [0]. \end{aligned} \tag{24}$$

The desired asymptotically stable system has stable poles $z_1 = -0.6$ and $z_2 = 0.2$, and zero $z = 0.3$. In this case the desired matrices have the forms

$$\begin{aligned} \bar{A} &= \begin{bmatrix} 0 & 1 \\ 0.12 & 0.4 \end{bmatrix}, & \bar{B} &= \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \\ \bar{C} &= [\begin{array}{cc} 0.3 & 1 \end{array}], & \bar{D} &= [0]. \end{aligned} \tag{25}$$

Compute the matrix N. In this case the desired transfer function of the system has the form

$$\begin{aligned}\bar{T}(z) &= \bar{C}[I_2 z - \bar{A}]^{-1}\bar{B} + \bar{D}\\ &= \begin{bmatrix} 0.3 & 1 \end{bmatrix}\begin{bmatrix} z & -1 \\ -0.12 & z-0.4 \end{bmatrix}^{-1}\begin{bmatrix} 0 \\ 1 \end{bmatrix}\\ &\quad + [0]\\ &= \frac{z-0.3}{z^2+0.4z-0.12}.\end{aligned} \tag{26}$$

Using (21), (24) and (25), we obtain

$$\begin{aligned}N &= \begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1}\begin{bmatrix} \bar{A} & \bar{B} \\ \bar{C} & \bar{D} \end{bmatrix}\\ &= \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}^{-1}\begin{bmatrix} 0 & 1 & 0 \\ 0.12 & 0.4 & 1 \\ 0.3 & 1 & 0 \end{bmatrix}\\ &= \begin{bmatrix} 0.3 & 1 & 0 \\ 0.9 & -0.6 & 1 \\ -0.9 & 1.6 & -1 \end{bmatrix}.\end{aligned} \tag{27}$$

♦

Example 2. Given the system (1)–(2) with the matrices (24). The desired transfer function of the system has poles $z_1 = -0.2$, $z_2 = 0.3$ and zero $z = 0.2$. The matrices of the desired system have the forms

$$\bar{A} = \begin{bmatrix} 0 & 0.1 \\ 1 & 0.06 \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \bar{C} = \begin{bmatrix} 0.2 & 1 \end{bmatrix}, \quad \bar{D} = [0]. \tag{28}$$

since $(z-z_1)(z-z_2) = (z+0.2)(z-0.3) = z^2 - 0.1z - 0.06$.

Using (24) and (28), we obtain

$$\begin{aligned}M &= \begin{bmatrix} \bar{A} & \bar{B} \\ \bar{C} & \bar{D} \end{bmatrix}\begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1}\\ &= \begin{bmatrix} 0 & 0.1 & 1 \\ 1 & 0.06 & 0 \\ 0.2 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}^{-1}\\ &= \begin{bmatrix} 1 & -0.9 & 0.9 \\ 0 & 0.06 & 0.94 \\ 0 & 1 & -0.8 \end{bmatrix}.\end{aligned} \tag{29}$$

The considerations can be easily extended to the case with nonzero matrices D and $\bar{D}$. ♦

4. Extension to discrete-time MIMO linear systems

Consider the discrete-time linear system (1)–(2) with the transfer matrix (12) and unstable poles and zeros. Let the transfer matrix

$$\bar{T}(z) = \bar{C}[I_n z - \bar{A}]^{-1}\bar{B} + \bar{D} \tag{30}$$

of the system

$$\bar{x}_{i+1} = \bar{A}\bar{x}_i + \bar{B}\bar{u}_i, \tag{31}$$

$$\bar{y}_i = \bar{C}\bar{x}_i + \bar{D}\bar{u}_i \tag{32}$$

where $\bar{A} \in \mathbb{R}^{n\times n}$, $\bar{B} \in \mathbb{R}^{n\times m}$, $\bar{C} \in \mathbb{R}^{p\times n}$ and $\bar{D} \in \mathbb{R}^{p\times m}$, have the desired stable pole and zeros.

We are looking for a matrix $N \in \mathbb{R}^{(n+m)\times(n+m)}$ satisfying the matrix equation (21). From Theorem 4 the matrix equation (21) has a solution N if

$$\operatorname{rank}\begin{bmatrix} A & B & \bar{A} & \bar{B} \\ C & 0 & \bar{C} & \bar{D} \end{bmatrix} = \operatorname{rank}\begin{bmatrix} A & B \\ C & 0 \end{bmatrix}. \tag{33}$$

Therefore, the following result has been proven.

Theorem 6. *Given matrices A, B, C, D and $\bar{A}, \bar{B}, \bar{C}, \bar{D}$ a matrix N satisfying the matrix equation (21) exists if and only if the condition (33) is satisfied.*

The desired matrix N satisfying (21) can be computed using of the following procedure.

Procedure 1.

Step 1. Knowing the transfer matrix $T(z)$ compute its matrices A, B, C in the Frobenius canonical forms and the matrix D.

Step 2. Knowing the transfer matrix $\bar{T}(z)$ compute its matrices $\bar{A}, \bar{B}, \bar{C}$ in the Frobenius canonical forms and the matrix $\bar{D}$.

Step 3. Using (21) compute the desired matrix N.

Example 3. For the matrix

$$T(z) = \begin{bmatrix} \frac{1}{z+2} & \frac{1}{z-1} \\ \frac{1}{z-2} & \frac{1}{z+1} \end{bmatrix} \tag{34}$$

find a matrix N such that the desired transfer matrix with stable poles and zeros has the form

$$\bar{T}(z) = \begin{bmatrix} \frac{1}{z+0.2} & \frac{1}{z+0.3} \\ \frac{1}{z+0.4} & \frac{1}{z-0.2} \end{bmatrix}. \tag{35}$$

Using Procedure 1, we obtain what follows.

Step 1. The matrices A, B, C, D of the transfer matrix (34) written in the form

$$T(z) = \begin{bmatrix} \frac{1}{z^2+z-2} & 0 \\ 0 & \frac{1}{z^2-z-2} \end{bmatrix}\begin{bmatrix} z-1 & z+2 \\ z+1 & z-2 \end{bmatrix} \tag{36}$$

are

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} -1 & 1 & 1 & 2 \\ 1 & 1 & -2 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}. \tag{37}$$

Step 2. The matrices $\bar{A}, \bar{B}, \bar{C}, \bar{D}$ of the transfer matrix (35) written in the form

$$\bar{T}(z) = \begin{bmatrix} \frac{1}{z^2+0.5z+0.06} & 0 \\ 0 & \frac{1}{z^2+0.2z-0.08} \end{bmatrix} \times \begin{bmatrix} z+0.3 & z+0.2 \\ z-0.2 & z+0.4 \end{bmatrix} \quad (38)$$

are

$$\bar{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -0.06 & -0.5 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0.08 & -0.2 \end{bmatrix},$$
$$\bar{B} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad (39)$$
$$\bar{C} = \begin{bmatrix} 0.3 & 1 & -0.2 & 1 \\ -0.2 & 1 & 0.4 & 1 \end{bmatrix},$$
$$\bar{D} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Step 3. The matrix equation (21) in this case has the form

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 2 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 & 1 \\ -1 & 1 & 1 & 2 & 0 & 0 \\ 1 & 1 & -2 & 1 & 0 & 0 \end{bmatrix} N = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -0.06 & -0.5 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0.08 & -0.2 & 0 & 1 \\ 0.3 & 1 & -0.2 & 1 & 0 & 0 \\ -0.2 & 1 & 0.4 & 1 & 0 & 0 \end{bmatrix} \quad (40)$$

and its solution

$$N = \begin{bmatrix} -0.4 & 0 & 0 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -0.1 & 0 & -0.2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0.74 & 0.5 & 0 & -4 & 1 & 0 \\ 0.2 & 0 & 0.48 & -3.2 & 0 & 1 \end{bmatrix} \quad (41)$$

is nonsingular, i.e., $\det N \neq 0$. ♦

5. Reduction of discrete-time linear systems with controllable (A, B) and observable (A, C) to systems with nilpotent matrices

To simplify the notation, the proposed method will be presented for the SISO systems with matrices in the following canonical forms:

$$A = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -a_0 & -a_1 & -a_2 & \dots & -a_{n-1} \end{bmatrix} \in \mathbb{R}^{n\times n},$$
$$B = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \in \mathbb{R}^{n\times 1},$$
$$C = [\, 1 \quad 0 \quad \dots \quad 0 \,] \in \mathbb{R}^{1\times n}. \quad (42)$$

Note that the above pair (A,B) is controllable and the pair (A,C) is observable.

The desired matrices $\bar{A}, \bar{B}, \bar{C}$ of the nilpotent system have the forms

$$\bar{A} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix} \in \mathbb{R}^{n\times n},$$
$$\bar{B} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \in \mathbb{R}^{n\times 1}, \quad (43)$$
$$\bar{C} = [\, 1 \quad 0 \quad \dots \quad 0 \,] \in \mathbb{R}^{1\times n}.$$

For matrices (42) and (43) compute a nonsingular matrix $N \in \mathbb{R}^{(n+1)\times(n+1)}$ satisfying the equation

$$\begin{bmatrix} \bar{A} & \bar{B} \\ \bar{C} & 0 \end{bmatrix} N = \begin{bmatrix} A & B \\ C & 0 \end{bmatrix}. \quad (44)$$

It is easy to check that for matrices (42) and (43)

$$\operatorname{rank} \begin{bmatrix} \bar{A} & \bar{B} \\ \bar{C} & \bar{D} \end{bmatrix} = \operatorname{rank} \begin{bmatrix} A & B \\ C & 0 \end{bmatrix} = n+1. \quad (45)$$

By Theorem 4 the matrix equation (44) has a unique nonsingular solution N if the condition (45) is satisfied. Therefore, from (21) and (44) we have the following theorem.

Theorem 7. *If matrices A, B, C have the canonical form (42) and matrices $\bar{A}, \bar{B}, \bar{C}$ the canonical form (43) then Eqn. (44) has unique solution*

$$N = \begin{bmatrix} A & B \\ C & 0 \end{bmatrix}^{-1} \begin{bmatrix} \bar{A} & \bar{B} \\ \bar{C} & 0 \end{bmatrix}. \quad (46)$$

Knowing the matrices (42) and the desired matrices (43) the matrix N can be computed using the following procedure.

Procedure 2.

Step 1. For given matrices (42) compute the matrix

$$\begin{bmatrix} A & B \\ C & 0 \end{bmatrix} \in \mathbb{R}^{(n+1)\times(n+1)}. \tag{47}$$

Step 2. Knowing the matrices (43) compute the matrix

$$\begin{bmatrix} \bar{A} & \bar{B} \\ \bar{C} & 0 \end{bmatrix} \in \mathbb{R}^{(n+1)\times(n+1)}. \tag{48}$$

Step 3. Using (46) compute the matrix $N \in \mathbb{R}^{(n+1)\times(n+1)}$.

Example 4. For the following matrices in canonical forms

$$A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix} \tag{49}$$

and the matrices

$$\bar{A} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}. \quad \bar{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \bar{C} = \begin{bmatrix} 1 & 0 \end{bmatrix} \tag{50}$$

compute the matrix N.

In this case the condition (45) is satisfied since

$$\begin{aligned} &\operatorname{rank} \begin{bmatrix} A & B & \bar{A} & \bar{B} \\ C & 0 & \bar{C} & 0 \end{bmatrix} \\ &= \operatorname{rank} \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 0 \\ -2 & -3 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \\ &= \operatorname{rank} \begin{bmatrix} \bar{A} & \bar{B} \\ \bar{C} & 0 \end{bmatrix} \\ &= \operatorname{rank} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} = 3. \end{aligned} \tag{51}$$

Using Procedure 2 and (49), (50) we obtain what follows.

Step 1. Using (47) and (49) we obtain the nonsingular matrix

$$\begin{bmatrix} A & B \\ C & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -2 & -3 & 1 \\ 1 & 0 & 0 \end{bmatrix}. \tag{52}$$

Step 2. Using (48) and (50) we obtain the nonsingular matrix

$$\begin{bmatrix} \bar{A} & \bar{B} \\ \bar{C} & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}. \tag{53}$$

Step 3. Using (46), (52) and (53) we obtain the nonsingular matrix

$$\begin{aligned} N &= \begin{bmatrix} A & B \\ C & 0 \end{bmatrix}^{-1} \begin{bmatrix} \bar{A} & \bar{B} \\ \bar{C} & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 & 0 \\ -2 & -3 & 1 \\ 1 & 0 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 3 & 1 \end{bmatrix}. \end{aligned} \tag{54}$$

The considerations can be extended to the case of $m > 1$ and $p > 1$. ♦

6. Matrix decomposition approach

Consider the matrix equation

$$XAY = B, \tag{55}$$

for given nonsingular matrices $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times n}$ and unknown matrices $X \in \mathbb{R}^{n\times n}$, $Y \in \mathbb{R}^{n\times n}$. For given matrices A and B the problem consists in finding matrices X and Y.

The presented solution method is based on the factorization of matrices A and B,

$$A = A_1A_2, \qquad B = B_1B_2, \tag{56}$$

where

$$\begin{aligned} A_1 &= \begin{bmatrix} \bar{a}_{11} & 0 & \dots & 0 & 0 \\ \bar{a}_{21} & \bar{a}_{22} & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ \bar{a}_{n,1} & \bar{a}_{n,2} & \dots & \bar{a}_{n,n-1} & \bar{a}_{n,n} \end{bmatrix}, \\ A_2 &= \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1,n-1} & a_{1,n} \\ 0 & a_{22} & \dots & a_{2,n-1} & a_{2,n} \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & a_{n,n} \end{bmatrix}, \end{aligned} \tag{57}$$

$$\begin{aligned} B_1 &= \begin{bmatrix} \bar{b}_{11} & 0 & \dots & 0 & 0 \\ \bar{b}_{21} & \bar{b}_{22} & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ \bar{b}_{n,1} & \bar{b}_{n,2} & \dots & \bar{b}_{n,n-1} & \bar{b}_{n,n} \end{bmatrix}, \\ B_2 &= \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1,n-1} & b_{1,n} \\ 0 & b_{22} & \dots & b_{2,n-1} & b_{2,n} \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & b_{n,n} \end{bmatrix} \end{aligned} \tag{58}$$

Using (56), Eqn. (55) can be written in the form

$$XA_1A_2Y = B_1B_2 \tag{59}$$

and then

$$XA_1 = B_1, \quad A_2Y = B_2, \tag{60}$$

where the matrices A_1 and A_2 are nonsingular.

Solving the matrix equations (60), we obtain

$$X = B_1 A_1^{-1} \qquad Y = A_2^{-1} B_2. \tag{61}$$

From the above considerations we deduce the following procedure for computation of the solution X and Y for given matrices A and B.

Procedure 3.

Step 1. Using the above procedure compute the matrices A_1, A_2 and B_1, B_2 satisfying (56).

Step 2. Using (61) compute the solution X and Y of Eqn. (60).

The following results will be used in the further considerations.

Lemma 1. *Every nonsingular matrix $A \in \mathbb{R}^{n\times n}$ can be decomposed into the product $A = A_1 A_2$ of the lower triangular matrix*

$$A_1 = \begin{bmatrix} \bar{a}_{11} & 0 & \dots & 0 & 0 \\ \bar{a}_{21} & \bar{a}_{22} & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ \bar{a}_{n,1} & \bar{a}_{n,2} & \dots & \bar{a}_{n,n-1} & \bar{a}_{n,n} \end{bmatrix} \tag{62}$$

and upper triangular matrix

$$A_2 = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1,n-1} & a_{1,n} \\ 0 & a_{22} & \dots & a_{2,n-1} & a_{2,n} \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & a_{n,n} \end{bmatrix} \tag{63}$$

The lemma will be illustrated by the following simple numerical example.

Example 5. Decompose the nonsingular matrix

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \\ 1 & 1 & 2 \end{bmatrix} \tag{64}$$

into lower and upper triangular matrices. ♦

Applying elementary row operations to the matrix (64), we obtain

$$\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & -1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \\ 1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & -3 & 0 \\ 0 & 0 & 3 \end{bmatrix} \tag{65}$$

and the desired decomposition of the matrix (64) is

$$\begin{aligned} &\begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \\ 1 & 1 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & -1 & 3 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 2 & 1 \\ 0 & -3 & 0 \\ 0 & 0 & 3 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 0.33 & 0.33 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 0 & -3 & 0 \\ 0 & 0 & 3 \end{bmatrix}. \end{aligned} \tag{66}$$

Lemma 2. *The product of two lower (upper) triangular matrices is also a lower (upper) triangular matrix.*

Example 6.

$$\begin{aligned} A = A_1 A_2 &= \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 2 & 0 & 0 \\ 3 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}. \end{aligned} \tag{67}$$

♦

Lemma 3. *The inverse matrix of the nonsingular triangular matrix (62) or (63) has the same triangular form*

$$\begin{aligned} A_1^{-1} &= \begin{bmatrix} \hat{a}_{11} & 0 & \dots & 0 & 0 \\ \hat{a}_{21} & \hat{a}_{22} & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ \hat{a}_{n,1} & \hat{a}_{n,2} & \dots & \hat{a}_{n,n-1} & \hat{a}_{n,n} \end{bmatrix}, \\ A_2^{-1} &= \begin{bmatrix} \tilde{a}_{11} & \tilde{a}_{12} & \dots & \tilde{a}_{1,n-1} & \tilde{a}_{1,n} \\ 0 & \tilde{a}_{22} & \dots & \tilde{a}_{2,n-1} & \tilde{a}_{2,n} \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & \tilde{a}_{n,n} \end{bmatrix} \end{aligned} \tag{68}$$

Example 7.

$$\begin{aligned} A_1^{-1} &= \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 3 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 5 & -3 & 1 \end{bmatrix}, \\ A_2^{-1} &= \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}. \end{aligned} \tag{69}$$

♦

Example 8. For given nonsingular matrices

$$A = \begin{bmatrix} 3 & 2 \\ 2 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 13 & 9 \\ 12 & 12 \end{bmatrix} \tag{70}$$

compute the solution $X \in \mathbb{R}^{2\times 2}$ and $Y \in \mathbb{R}^{2\times 2}$ of Eqn. (55).

Using Procedure 3 in this case we obtain what follows.

Step 1. For the matrix A given by (70) we obtain

$$\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ 2 & 2 \end{bmatrix}=\begin{bmatrix} 1 & 0 \\ 2 & 2 \end{bmatrix} \tag{71}$$

and

$$\begin{aligned} A=\begin{bmatrix} 3 & 2 \\ 2 & 2 \end{bmatrix}&=\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}^{-1}\begin{bmatrix} 1 & 0 \\ 2 & 2 \end{bmatrix} \\ &=A_1A_2, \end{aligned} \tag{72}$$

where

$$A_1=\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad A_2=\begin{bmatrix} 1 & 0 \\ 2 & 2 \end{bmatrix}. \tag{73}$$

In a similar way, for the matrix B we obtain

$$B_1=\begin{bmatrix} 2 & 3 \\ 0 & 4 \end{bmatrix}, \quad B_2=\begin{bmatrix} 2 & 0 \\ 3 & 3 \end{bmatrix}. \tag{74}$$

Step 2. The solution of $XA_1=B_1$ for given A_1 and B_1 has the form

$$\begin{aligned} X=B_1A_1^{-1}&=\begin{bmatrix} 2 & 3 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^{-1} \\ &=\begin{bmatrix} 2 & 1 \\ 0 & 4 \end{bmatrix} \end{aligned} \tag{75}$$

and the solution of $A_2Y=B_2$ has the form

$$\begin{aligned} Y=A_2^{-1}B_2&=\begin{bmatrix} 1 & 0 \\ 2 & 2 \end{bmatrix}^{-1}\begin{bmatrix} 2 & 0 \\ 3 & 3 \end{bmatrix} \\ &=\begin{bmatrix} 2 & 0 \\ -0.5 & 1.5 \end{bmatrix}. \end{aligned} \tag{76}$$

It is easy to check that X and Y given by (75) and (76) constitute the solution of (55) for matrices (70) since

$$\begin{aligned} XAY&=\begin{bmatrix} 2 & 1 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ -0.5 & 1.5 \end{bmatrix} \\ &=\begin{bmatrix} 13 & 9 \\ 12 & 12 \end{bmatrix} \end{aligned} \tag{77}$$

♦

For given matrices

$$S=\begin{bmatrix} A & B \\ C & D \end{bmatrix} \tag{78}$$

and

$$\bar{S}=\begin{bmatrix} \bar{A} & \bar{B} \\ \bar{C} & \bar{D} \end{bmatrix} \tag{79}$$

the desired matrices M and N satisfying the equation

$$MSN=\bar{S} \tag{80}$$

can be computed using the following procedure.

Procedure 4.

Step 1. For the matrix (78) compute the matrices S_1, S_2 satisfying the equation

$$S=S_1S_2, \tag{81}$$

where

$$\begin{aligned} S_1&=\begin{bmatrix} \hat{s}_{11} & 0 & \dots & 0 & 0 \\ \hat{s}_{21} & \hat{s}_{22} & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ \hat{s}_{n,1} & \hat{s}_{n,2} & \dots & \hat{s}_{n,n-1} & \hat{s}_{n,n} \end{bmatrix}, \\ S_2&=\begin{bmatrix} s_{11} & s_{12} & \dots & s_{1,n-1} & s_{1,n} \\ 0 & s_{22} & \dots & s_{2,n-1} & s_{2,n} \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & s_{n,n} \end{bmatrix} \end{aligned} \tag{82}$$

Step 2. For the matrix (79) compute the matrices $\bar{S}_1, \bar{S}_2$ satisfying the equation

$$\bar{S}=\bar{S}_1\bar{S}_2, \tag{83}$$

where

$$\begin{aligned} \bar{S}_1&=\begin{bmatrix} \hat{\bar{s}}_{11} & 0 & \dots & 0 & 0 \\ \hat{\bar{s}}_{21} & \hat{\bar{s}}_{22} & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ \hat{\bar{s}}_{n,1} & \hat{\bar{s}}_{n,2} & \dots & \hat{\bar{s}}_{n,n-1} & \hat{\bar{s}}_{n,n} \end{bmatrix}, \\ \bar{S}_2&=\begin{bmatrix} \bar{s}_{11} & \bar{s}_{12} & \dots & \bar{s}_{1,n-1} & \bar{s}_{1,n} \\ 0 & \bar{s}_{22} & \dots & \bar{s}_{2,n-1} & \bar{s}_{2,n} \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & \bar{s}_{n,n} \end{bmatrix} \end{aligned} \tag{84}$$

Step 3. Compute the matrices

$$M=\bar{S}_1S_1^{-1} \quad , \quad N=S_2^{-1}\bar{S}_2. \tag{85}$$

Example 9. For given matrices

$$\begin{aligned} A&=\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad B=\begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \\ C&=\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & 0 \end{bmatrix}, \quad D=\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \end{aligned} \tag{86}$$

and

$$\begin{aligned} \bar{A}&=\begin{bmatrix} 1 & -0.06 & 0 & 0 \\ 0 & -0.5 & 0 & 0 \\ 0 & 0 & 1 & -0.05 \\ 0.3 & 0 & -0.2 & 0.4 \end{bmatrix}, \\ \bar{B}&=\begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \\ \bar{C}&=\begin{bmatrix} 0 & 0.5 & 0 & 0.1 \\ 0 & 0 & 0 & -0.2 \end{bmatrix}, \\ \bar{D}&=\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \end{aligned} \tag{87}$$

Compute matrices M and N satisfying Eqn. (80).

Using Procedure 4, we obtain what follows.

Step 1. The matrix (78) for (86) has the form

$$S = \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 & 0 \end{bmatrix} \quad (88)$$

and can be decomposed to

$$S_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 \end{bmatrix}, \quad S_2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \quad (89)$$

Step 2. The matrix (79) for (87) has the form

$$\bar{S} = \begin{bmatrix} \bar{A} & \bar{B} \\ \bar{C} & \bar{D} \end{bmatrix} = \begin{bmatrix} 1 & -0.06 & 0 & 0 & 0 & 0 \\ 0 & -0.5 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -0.05 & 0 & 0 \\ 0.3 & 0 & -0.2 & 0.4 & 0 & 0 \\ 0 & 0.5 & 0 & 0.1 & 0 & 0 \\ 0 & 0 & 0 & -0.2 & 0 & 1 \end{bmatrix} \quad (90)$$

and can be decomposed to

$$\bar{S}_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0.3 & -0.036 & -0.2 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0.256 & 1 & 0 \\ 0 & 0 & 0 & -0.513 & 0.019 & 1 \end{bmatrix}, \quad \bar{S}_2 = \begin{bmatrix} 1 & -0.06 & 0 & 0 & 0 & 0 \\ 0 & -0.5 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -0.05 & 0 & 0 \\ 0 & 0 & 0 & 0.39 & 0.036 & 0 \\ 0 & 0 & 0 & 0 & 0.99 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad (91)$$

Step 3. Matrices (85) have the form

$$M = \bar{S}_1 S_1^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0.3 & -0.036 & -0.2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0.256 & 1 & 0 \\ 0 & 0.019 & 1 & -0.513 & 0.019 & 1 \end{bmatrix},$$
$$N = S_2^{-1} \bar{S}_2 = \begin{bmatrix} 1 & -0.06 & 0 & 0 & 0 & 0 \\ 0 & -0.5 & 0 & 0 & -0.009 & 0 \\ 0 & 0 & 0.5 & -0.025 & 0 & -0.5 \\ 0 & 0 & 0 & 0.39 & 0.036 & 0 \\ 0 & 0 & 0 & 0 & 0.99 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad (92)$$

and they satisfy (77). In this example the LU factorization method has been used (Schwarzenberg-Czerny, 1995). ♦

7. Concluding remarks

A new approach to design discrete-time linear systems with desired poles and zeros of their transfer matrices has been proposed. Conditions have been established under which the transfer matrices have the desired stable poles and zeros. Procedures for computation of the matrices of the system with desired poles and zeros of the transfer matrices have been proposed and illustrated by simple numerical examples. The reduction of the discrete-time systems with controllable pair (A, B) and observable pairs (A, C) with nilpotent matrix A has been also considered.

The proposed approach can be easily implemented in practice. It can be extended to continuous-time and discrete-time fractional linear systems (Kaczorek, 1992; Klamka, 1991).

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