Int. J. Appl. Math. Comput. Sci., 2025, Vol. 35, No. 1, 69–81
DOI: 10.61822/amcs-2025-0006

ON ROBUSTNESS TO A TOPOLOGICAL PERTURBATION IN FLUID MECHANICS

MERWAN ABDELBARI [a], KHADRA NACHI [b], JAN SOKOŁOWSKI [c,d,*], ANTONIO ANDRE NOVOTNY [e]

[a] Laboratory of Rheology and Mechanics
Hassiba Ben Bouali University
Hay Salem, 02000 Chlef, Algeria
e-mail: m.abdelbari@univ-chlef.dz

[b] Laboratory of Mathematical Analysis and Applications
University of Oran 1—Ahmed Ben Bella
BP 1524, El M'naouer Oran, Algeria
e-mail: nachi.khadra@univ-oran1.dz

[c] Élie Cartan Institute
University of Lorraine
BP 239, 54506 Vandoeuvre lès Nancy cedex, France
e-mail: jan.sokolowski@univ-lorraine.fr

[d] Systems Research Institute
Polish Academy of Sciences
Newelska 6, 01-447 Warsaw, Poland

[e] Department of Computational and Applied Mathematics
National Laboratory for Scientific Computing (LNCC/MCT)
Av. Getúlio Vargas 333, 25651-075 Petrópolis, RJ, Brazil
e-mail: novotny@lncc.br

The robustness to topological perturbations in geometrical domains filled by a fluid flowing in Stokes–Darcy regime is considered. The cost functional is given by the energy dissipation in the fluid. The topological perturbation is carried out by the nucleation of an infinitesimal circular obstacle, which can be considered as a small measurement device. Our approach is based on the topological derivative method, which has been previously employed in the shape and topology optimization problems. The topological derivative (TD) measures the sensitivity of a given shape functional with respect to topological domain perturbations. The TD is used to determine the location of the small device placement, through a distributed control problem. By taking into account the effect of the disturbance term or uncertain input data in the TD expression, the problem of robustness to topological perturbation for the energy functional can be formulated as a minimax optimization problem with a pointwise observation. Numerical examples illustrate the efficiency of the proposed topological derivative method.

Keywords: topological derivative, shape optimization, optimal control, robust control.

1. Introduction

Problems of shape and topological sensitivity in fluids mechanics has been a topic of interest of several studies (see, e.g., Amstutz, 2006; Guillaume and Hassine, 2008; Dziri *et al.*, 2004; Moubachir and Zolesio, 2006; Dziri and Zolésio, 2011). Based on these works, we can conclude that the geometric design has a significant impact on relevant quantities describing the fluid flow, like vorticity,

*Corresponding author

energy of system and drag forces. In this paper, our objective is to desensitize the energy functional with respect to a topological perturbation, which in this context represents a small circular obstacle within the fluid flow. Instead of studying the problem in a domain with a topological singularity, our approach consists of acting on the topological derivative by a distributed control, and taking in consideration the effect of a disturbance source term in the system. This approach leads to a minimax problem defined on the unperturbed domain. Our model is governed by the Stokes–Darcy system which describes the fluid flow with a slow motion in porous media. The topological perturbation is performed by inserting a small inclusion in the geometrical domain, also known by the volume penalization method (see, e.g., Krzyzanowski *et al.*, 2024).

Closer works to our problem have been presented in the framework of insensitizing (or desensitizing) control, which goes back to the book (Lions, 1992), where the notion of the sentinel was introduced. Intuitively, an insensitizing control serves to neutralize a perturbation in some system according to a given cost functional. The sentinel method was generalized in several directions. We mention here the works of Dáger (2006), Guerrero (2007) and Gueye (2013), analyzing the problem of insensitizing for the wave equation, Stokes and Navier–Stokes systems, respectively. All these papers have dealt with perturbations in initial or boundary conditions. The case where the perturbation is prescribed on the boundary was discussed recently by Ervedoza *et al.* (2022) for the parabolic case. The authors applied the shape derivative to describe the sensitivity with respect to boundary variations (see Sokołowski and Zolésio, 1992). We point out that the existence of insensitizing control is equivalent to the problem of controllability of a coupled system which involves the state and its adjoint state. In this paper, we introduce a relaxed problem in the sense that instead of looking for an exact desensitizing control for the energy functional with respect to topological perturbations, we seek a control such that the topological derivative evaluated where the obstacle will be created becomes as close as possible to zero.

Since our approach is based on the topological derivative concept, let us recall some results in topological sensitivity analysis that are available in the literature. The topological derivative was rigorously introduced by Sokołowski and Żochowski (1999). Considering its applications in shape optimization, it has been the subject of study for many models. For semilinear elliptic problems, the reader is referred to the works of Amstutz (2006) and Iguernane *et al.* (2009) and more recently the paper by Sturm (2020). The topological sensitivity for the compliance functional in linear elasticity was discussed by Garreau *et al.* (2001). For variational inequalities and contact problems, the reader may refer to the paper by Giusti *et al.* (2015). In fluid mechanics, we mention the papers by Amstutz (2005), Hassine and Masmoudi (2004) or Guillaume and Hassine (2008), where the authors derive TD expression for Navier–Stokes, Stokes and quasi-Stokes systems, respectively. Applications of the TD in inverse problems of detecting an unknown geometric object from a given measurements are discussed by Caubet and Dambrine (2012) or Kovtunenko and Kunisch (2014). Concerning the theoretical development of the topological asymptotic analysis, the reader is referred to the monograph by Novotny and Sokołowski (2012). More recently, Novotny *et al.* (2019) discussed the numerical methods and applications of the topological derivative for several problems.

It is well know that the topological derivative depends on initial data of the system in consideration. Therefore, the question of robustness to parameters arises naturally in numerical methods of shape optimization. Let us mention some works in this context. Hlaváček *et al.* (2009) discuss the continuity of the topological derivative for an elasticity system, with respect to Lamé coefficients and traction forces. Recently, Leugering *et al.* (2022) demonstrated that the TD for the Helmholtz equation is robust to frequency and boundary conditions. The upper and lower bounds of the TD with respect to initial data can be interpreted respectively as the worst-case design and the maximum-range design. Therefore, in this paper we deal with robustness issues with respect to topological perturbations in geometrical domains filled by a fluid flowing in Stokes–Darcy regime.

The rest of this paper is structured as follows. In Section 2, we introduce some notation and we describe our model of optimal control. The existence of robust control is discussed in Section 3. In Section 4, we derive the optimality conditions in terms of the adjoint state for robust control. Numerical results are presented in Section 5. Finally, some conclusions are drawn in Section 6.

2. Problem formulation

Let us briefly recall the definition of the topological gradient and some preliminary results in topology optimization. Suppose that $\mathcal{J}(\Omega)$ is an integral functional depending on the solution of the boundary value problem defined in $\Omega \subset \mathbb{R}^N$. For a small parameter $\varrho > 0$, consider the perforated domain $\Omega_\varrho := \Omega \setminus \overline{B_\varrho(x_0)}$, where $\overline{B_\varrho(x_0)}$ is the closed ball of radius ϱ and center x_0, and with the boundary Γ_ϱ. The topological derivative of the functional $\mathcal{J}(\Omega)$ is defined by the following asymptotic expansion:

$$\mathcal{J}(\Omega_\varrho) = \mathcal{J}(\Omega) + f(\varrho)\mathcal{T}(x_0) + o(f(\varrho)), \qquad (1)$$

where $\mathcal{J}(\Omega)$ is the functional evaluated for the given original domain and $\mathcal{J}(\Omega_\varrho)$ for a perturbed domain obtained by introducing a topological perturbation of size ϱ. The term $f(\varrho) > 0$ is a regularizing function which depends on dimension N. The remainder $o(f(\varrho))$ contains all terms of higher order than $f(\varrho)$, i.e.,

$$\lim_{\varrho \to 0} \frac{o(f(\varrho))}{f(\varrho)} = 0.$$

The function $x_0 \mapsto \mathcal{T}(x_0)$ is called the topological derivative of $\mathcal{J}$ at x_0. The topological derivative $\mathcal{T}(x_0)$ provides information for creating a small hole located at x_0. Actually, if $\mathcal{T}(x_0) < 0$ then $\mathcal{J}(\Omega_\varrho) < \mathcal{J}(\Omega)$ for sufficiently small ϱ. Therefore, in order to decrease the functional $\mathcal{J}$, we have to create a hole (or an obstacle) inside the geometrical domain where $\mathcal{T}$ is most negative. More generally, the function $\mathcal{T}$ can be used as a descent direction in topology optimization and, unlike classical shape optimization, it allows us to modify the topology of the domain during the optimization process; see, for example, the works of Garreau *et al.* (2001) or Hassine and Masmoudi (2004) for applications in elasticity and fluid mechanics, and that of Caubet and Dambrine (2012) for applications in inverse problems of detecting an obstacle immersed in a fluid. Through the previous analysis, we deduce that the best position to place a hole B_ϱ in Ω, regarding the shape functional $\mathcal{J}$, corresponds to

$$\overline{x} = \arg\min_{x\in\Omega} \mathcal{T}(x).$$

Suppose that we have an arbitrary infinitesimal topological singularity $B_\varrho(x_0)$ not necessary located at $\overline{x}$, ($x_0 \neq \overline{x}$); we can take, for example,

$$x_0 = \arg\max_{x\in\Omega} \mathcal{T}(x),$$

which is the most undesirable situation. The issue that we will address in our paper is as follows:

How can we reduce the impact of the topological perturbation $B_\varrho(x_0)$ on the cost functional $\mathcal{J}$?

In other words, we look for a control that makes the shape functional $\mathcal{J}$ less sensitive with respect to the topological singularity at x_0. Note that the problem of insensitizing control has been the subject of several communications, specially when the perturbation is prescribed in the initial or boundary conditions (see, e.g., Ervedoza *et al.*, 2022; Guerrero, 2007; Gueye, 2013; Lions, 1992). The abstract problem given above will become clearer after fixing the boundary value problem, which is in our case the stationary Stokes–Darcy system.

Let Ω be the fluid domain in $\mathbb{R}^N$ ($N = 2$ or 3), with C^2 smooth boundary $\Gamma := \partial\Omega$. The fluid is described by its velocity y and pressure p satisfying the Stokes–Darcy equations

$$\begin{cases} -\mu\Delta y + \nabla p + \eta y = h\chi_\omega \\ \quad +u\chi_{\omega_1} + \tau\chi_{\omega_2} & \text{in}\,\Omega, \\ \operatorname{div}\, y = 0 & \text{in}\,\Omega, \\ y = 0 & \text{on}\,\Gamma, \end{cases} \tag{2}$$

where μ stands for the kinematic viscosity coefficient, η is the inverse permeability and h is a given source term. The control and the disturbance terms are given by u, τ, respectively. The characteristic functions of ω, ω_1, $\omega_2 \subset \Omega$ are respectively denoted by χ_ω, χ_{ω_1}, χ_{ω_2}. For simplicity, the following notation is used for functional spaces

$$\begin{aligned} \mathbf{L}^2(\Omega) &:= L^2(\Omega)^N, \\ \mathbf{H}^1_0(\Omega) &:= H^1_0(\Omega)^N, \\ \mathbf{C}(\overline{\Omega}) &:= C(\overline{\Omega})^N, \\ \mathbf{H}_{\mathrm{div}}(\Omega) &:= \{\varphi \in \mathbf{H}^1_0(\Omega),\ \operatorname{div}\varphi = 0\}, \\ L^2_0(\Omega) &:= \{\varphi \in L^2(\Omega),\ \int_\Omega \varphi \,\mathrm{d}x = 0\}. \end{aligned}$$

The existence of solutions for the Stokes–Darcy system is well known; one can check that for all h, u, $\tau \in \mathbf{L}^2(\Omega)$, there exists a unique pair $(y,p) \in \mathbf{H}_{\mathrm{div}}(\Omega) \times L^2_0(\Omega)$ which is a solution of (2) (see, e.g., Boyer and Fabrie, 2005; Galdi, 2011). The energy dissipation functional of the system (2) is defined as follows:

$$\mathcal{E}_{u,\tau}(\Omega) = \mu \int_\Omega |\nabla y|^2 \,\mathrm{d}x + \int_\Omega \eta |y|^2 \,\mathrm{d}x. \tag{3}$$

The topological perturbation of the geometrical domain is defined by inserting an inclusion $\overline{B_\varrho(x_0)}$ in Ω, where $\overline{B_\varrho(x_0)} \Subset \Omega \setminus (\omega \cup \omega_1 \cup \omega_2)$ is the closed ball of radius ϱ and center x_0, and with the boundary Γ_ϱ. More precisely Ω_ϱ is defined through the penalization coefficient $\eta_\varrho := \eta\gamma_\varrho$ (see Krzyżanowski *et al.*, 2024), where γ_ϱ is a piecewise constant function given by

$$\gamma_\varrho(x) = \begin{cases} 1 & \text{in } \Omega \setminus \overline{B_\varrho}, \\ \gamma & \text{in } B_\varrho. \end{cases} \tag{4}$$

Here $\gamma > 0$ is the contrast parameter. Therefore, as $\gamma \to +\infty$, we have $|y|_{|B_\varrho} \equiv 0$; see Fig. 1.

We write (y_ϱ, p_ϱ) for the unique solution of the system (2) with the perturbation η_ϱ:

$$\begin{cases} -\mu\Delta y_\varrho + \nabla p_\varrho + \eta_\varrho y_\varrho = h\chi_\omega \\ \quad +u\chi_{\omega_1} + \tau\chi_{\omega_2} & \text{in}\,\Omega, \\ \operatorname{div}\, y_\varrho = 0 & \text{in}\,\Omega, \\ y_\varrho = 0 & \text{on}\,\Gamma. \end{cases} \tag{5}$$

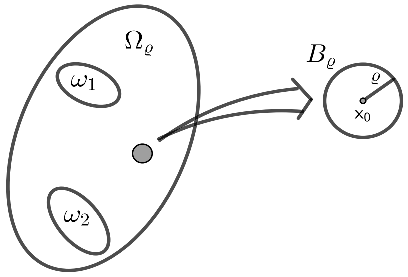


Fig. 1. Singular domain Ω_ϱ.

The topological derivative for the shape functional $\mathcal{E}$ is derived by Sá *et al.* (2016). To be more clear, in Appendix we will briefly present the proof of the following asymptotic expansion:

$$\mathcal{E}_{u,\tau}(\Omega_\varrho) - \mathcal{E}_{u,\tau}(\Omega) = f(\varrho)\mathcal{T}_{u,\tau}(x_0) + o(f(\varrho)), \quad (6)$$

where the topological derivative $\mathcal{T}_{u,\tau}$ and the function $f(\varrho)$ are given by

$$\begin{aligned} \mathcal{T}_{u,\tau}(x_0) &= (1-\gamma)\eta|y(x_0)|^2, \\ f(\varrho) &= \mathrm{meas}(B_\varrho). \end{aligned} \quad (7)$$

For $\gamma \geq 1$, we have $\mathcal{T}_{u,\tau} \leq 0$, which means that creating an infinitesimal inclusion inside Ω will decrease the energy functional $\mathcal{E}_{u,\tau}$. Suppose that τ is fixed in $\mathbf{L}^2(\omega_2)$. Our aim is to reduce the effect of singularity in the geometrical domain for energy dissipation $\mathcal{E}_{u,\tau}$, i.e., minimize the gap between $\mathcal{E}_{u,\tau}(\Omega_\varrho)$ and $\mathcal{E}_{u,\tau}(\Omega)$ by a distributed control u. This problem can be formulated as follow :

$$\min_{u \in L^2(\omega_1)} |(\mathcal{E}_{u,\tau}(\Omega_\varrho) - \mathcal{E}_{u,\tau}(\Omega))|.$$

Observe that if $|\mathcal{T}_{\overline{u},\tau}(x_0)| \leq |\mathcal{T}_{u,\tau}(x_0)|$, $\forall u \in L^2(\omega_1)$, then for ϱ small enough, we have

$$|\mathcal{E}_{\overline{u},\tau}(\Omega_\varrho) - \mathcal{E}_{\overline{u},\tau}(\Omega)| \leq |\mathcal{E}_{u,\tau}(\Omega_\varrho) - \mathcal{E}_{u,\tau}(\Omega)|. \quad (8)$$

The last remark motivates us to introduce the following cost functional:

$$\begin{aligned} \mathcal{J}(u,\tau) = \frac{1}{2}|y(x_0)|^2 + \frac{\alpha}{2}\int_{\omega_1} |u|^2 \,\mathrm{d}x \\ - \frac{\beta}{2}\int_{\omega_2} |\tau|^2 \,\mathrm{d}x, \end{aligned} \quad (9)$$

where $\alpha, \beta > 0$ are the regularization parameters. The cost functional $\mathcal{J}$ is simultaneously minimized with respect to the control u and maximized with respect to the disturbance term τ. The control u serves to influence the topological derivative at x_0 and make it as close as possible to zero, while τ is considered to increase the robustness of the control. Therefore, the worst-case disturbance corresponds to the maximum of $\mathcal{J}$ with respect to τ. The rest of this article is devoted to the analysis of the following minimax problem:

$$\begin{cases} \min\limits_{u \in \mathbf{L}^2(\Omega)} \max\limits_{\tau \in \mathbf{L}^2(\Omega)} \mathcal{J}(u,\tau) \\ \text{subject to (2).} \end{cases} \quad (10)$$

Remark 1. The regularity results for the Stokes system can be easily adapted to the system (2) (see, e.g., Galdi, 2011, Thm. IV.6.1.) thus according to the assumptions on Ω, for each $h \in \mathbf{L}^2(\omega)$, $u \in \mathbf{L}^2(\omega_1)$, and $\tau \in \mathbf{L}^2(\omega_2)$, we have $y \in \mathbf{H}^1_0(\Omega) \cap \mathbf{H}^2(\Omega)$, and by the embedding theorem we conclude that $y \in \mathbf{C}(\overline{\Omega})$. Therefore, the topological derivative $\mathcal{T}$, which is the pointwise term in the cost functional (9), is well defined.

It is well known that minimax problems are closely related to the existence of a saddle points. More precisely, we have the following definition for robust control.

Definition 1. The triple $(\overline{u}, \overline{\tau}, \overline{y})$, where $\overline{y} = y(\overline{u}, \overline{\tau})$, is a solution to the robust control problem if $(\overline{u}, \overline{\tau})$ is a saddle point of the cost functional $\mathcal{J}$, i.e.,

$$\mathcal{J}(\overline{u}, \tau) \leq \mathcal{J}(\overline{u}, \overline{\tau}) \leq \mathcal{J}(u, \overline{\tau}), \quad \forall (u,\tau) \in \mathbf{L}^2(\Omega),$$

or equivalently

$$\begin{aligned} \mathcal{J}(\overline{u}, \overline{\tau}) &= \min_{u \in \mathbf{L}^2(\Omega)} \max_{\tau \in \mathbf{L}^2(\Omega)} \mathcal{J}(u,\tau) \\ &= \max_{\tau \in \mathbf{L}^2(\Omega)} \min_{u \in \mathbf{L}^2(\Omega)} \mathcal{J}(u,\tau). \end{aligned}$$

3. Existence of robust control

In this section, we study the existence of a robust control for the problem (10). The first works dealing with robustness of control go back to the work by Bewley *et al.* (2000), who introduced a general framework for robust control for the Navier–Stokes problem. For optimal control problems with pointwise observations, the reader may refer to the PhD thesis of Brett (2014). The existence of a saddle point for the functional $\mathcal{J}$ is an application of the following proposition.

Proposition 1. (Ekeland and Temam, 1999, p. 173) *Let $\mathcal{J}$ be a functional defined on $\mathcal{U}_1 \times \mathcal{U}_2$, where $\mathcal{U}_1$, $\mathcal{U}_2$ are reflexive Banach spaces. If $\mathcal{J}$ satisfies the conditions*

1. *$\forall \kappa \in \mathcal{U}_2$, $\sigma \mapsto \mathcal{J}(\sigma,\kappa)$ is convex lower semicontinuous,*
2. *$\forall \sigma \in \mathcal{U}_1$, $\kappa \mapsto \mathcal{J}(\sigma,\kappa)$ is concave upper semicontinuous,*
3. $\exists\, \kappa_0 \in \mathcal{U}_2$, $\lim_{\|\sigma\| \to +\infty} \mathcal{J}(\sigma,\kappa_0) = +\infty$,
4. $\exists\, \sigma_0 \in \mathcal{U}_1$, $\lim_{\|\kappa\| \to +\infty} \mathcal{J}(\sigma_0,\kappa) = -\infty$,

then the functional $\mathcal{J}$ has at least one saddle point $(\overline{\sigma}, \overline{\kappa})$.

Our main result of this section is given by the following theorem.

Theorem 1. *For sufficiently large β, $(\beta > \beta_0 > 0)$, there exists at least one saddle point $(\overline{u}, \overline{\tau}) \in \boldsymbol{L}^2(\Omega) \times \boldsymbol{L}^2(\Omega)$ of the functional $\mathcal{J}$.*

Proof. We apply Proposition 1 with $\mathcal{U}_1 = \mathcal{U}_2 = \mathbf{L}^2(\Omega)$. We need to verify all the assumptions for the functional $\mathcal{J}$. First, we point out that the mappings $u \to y(u,\tau)$, $\tau \to y(u,\tau)$ are affine and continuous from $\mathbf{L}^2(\Omega)$ to $\mathbf{H}^2(\Omega)$. This is a direct result of the classical energy estimate:

$$\|y\|_{\mathbf{H}^2(\Omega)} + \|p\|_{L_0^2(\Omega)} \leq K(\|u\|_{\mathbf{L}^2(\omega_2)} + \|\tau\|_{\mathbf{L}^2(\omega_2)}).$$

Therefore, we deduce immediately that $\mathcal{J}$ is lower (and upper) semicontinuous with respect to u (and τ). For convexity and concavity, we can use the second derivatives of $\mathcal{J}$ with respect to u and τ; thus, we must check that $\mathrm{d}_u^2\mathcal{J}(\xi,\xi) > 0$ and $\mathrm{d}_\tau^2\mathcal{J}(\xi,\xi) < 0, \forall \xi \in \mathbf{L}^2(\Omega) \setminus \{0\}$. The operators $\tau \to y(u,\tau)$, $u \to y(u,\tau)$ are Fréchet differentiable and their derivatives $\theta_\xi := d_u y(u,\tau)(\xi)$ and $\psi_\xi := d_\tau y(u,\tau)(\xi)$ obey the systems

$$\begin{cases} -\mu\Delta\theta + \nabla v + \eta\,\theta = \xi\chi_{\omega_1} & \text{in } \Omega, \\ \operatorname{div}\,\theta = 0 & \text{in } \Omega, \\ \theta = 0 & \text{on } \Gamma, \end{cases} \tag{11}$$

$$\begin{cases} -\mu\Delta\psi + \nabla\kappa + \eta\,\psi = \xi\chi_{\omega_2} & \text{in } \Omega, \\ \operatorname{div}\,\psi = 0 & \text{in } \Omega, \\ \psi = 0 & \text{on } \Gamma. \end{cases} \tag{12}$$

Now the first and second derivatives of the functional $\mathcal{J}$ in the direction ξ are given by

$$\mathrm{d}_u\mathcal{J}(\xi) = y(x_0)\cdot\theta_\xi(x_0) + \alpha\langle u\,,\xi\rangle_{\mathbf{L}^2(\omega_1)}, \tag{13}$$
$$\mathrm{d}_u^2\mathcal{J}(\xi,\,\xi) = |\theta_\xi(x_0)|^2 + \alpha\|\xi\|^2_{\mathbf{L}^2(\omega_1)}, \tag{14}$$
$$\mathrm{d}_\tau\mathcal{J}(\xi) = y(x_0)\cdot\psi_\xi(x_0) - \beta\langle\tau\,,\xi\rangle_{\mathbf{L}^2(\omega_2)}, \tag{15}$$
$$\mathrm{d}_\tau^2\mathcal{J}(\xi,\,\xi) = |\psi_\xi(x_0)|^2 - \beta\|\xi\|^2_{\mathbf{L}^2(\omega_2)}. \tag{16}$$

From (14) we conclude that $\mathcal{J}$ is convex with respect to u. On the other hand, (ψ,κ) is the solution of the first Stokes system in (11), so we have the $\mathbf{H}^2$-regularity for the velocity ψ (see, e.g., Boyer and Fabrie, 2005; Galdi, 2011). Moreover, there exists $C_1 > 0$ depending only on Ω, such that

$$\|\psi_\xi\|_{\mathbf{H}^2(\Omega)} + \|\kappa\|_{L_0^2(\Omega)} \leq C_1\|\xi\|_{\mathbf{L}^2(\omega_2)}. \tag{17}$$

Recall that the space $\mathbf{H}^2(\Omega)$ is continuously embedded in $\mathbf{C}(\overline{\Omega})$. Thus, the following estimate holds:

$$|\psi_\xi(x_0)| \leq \|\psi_\xi\|_{\mathbf{C}(\overline{\Omega})} \leq C\|\xi\|_{\mathbf{L}^2(\omega_2)}, \tag{18}$$

where $C = C_1 \cdot C_2$, C_1 is given in (17) and C_2 is the embedding constant. By the expression of $\mathrm{d}_\tau^2\mathcal{J}(\xi,\xi)$ from (16) and the estimate (18), we deduce that

$$\mathrm{d}_\tau^2\mathcal{J}(\xi,\xi) \leq (C^2 - \beta)\|\xi\|^2_{\mathbf{L}^2(\omega_2)}.$$

Therefore, $\mathrm{d}_\tau^2\mathcal{J}(\xi,\xi) < 0$, for $\beta > \beta_1$, with $\beta_1 > C^2$. Taking $\tau = 0$, the coercivity of $\mathcal{J}(u,0)$ is a consequence of the following estimate:

$$\mathcal{J}(u,0) \geq \frac{\alpha}{2}\|u\|^2_{\mathbf{L}^2(\omega_1)}.$$

For the last condition, observe that we have the same estimate (18) for the state y, i.e.,

$$\begin{aligned} |y(0,\tau)(x_0)| &\leq \|y(0,\tau)\|_{\mathbf{C}(\overline{\Omega})} \\ &\leq C_0(\|\tau\|_{\mathbf{L}^2(\omega_2)} + \|h\|_{\mathbf{L}^2(\omega)}), \end{aligned}$$

which leads to

$$\begin{aligned} \mathcal{J}(0,\tau) \leq{}& \left(\frac{C_0^2}{2} - \frac{\beta}{2}\right)\|\tau\|^2_{\mathbf{L}^2(\omega_2)} \\ &+ C_0^2\|\tau\|_{\mathbf{L}^2(\omega_2)}\cdot\|h\|_{\mathbf{L}^2(\omega)} + \frac{C_0^2}{2}\|h\|^2_{\mathbf{L}^2(\omega)}. \end{aligned}$$

Thus for $\beta > \beta_2$, with $\beta_2 > C_0^2$, the last condition in Proposition 1 follows immediately. Finally, by setting $\beta_0 = \max\{\beta_1,\,\beta_2\}$, we recover both the concavity and coercivity of the cost functional $\mathcal{J}$ with respect to τ. ■

4. Optimality conditions

In this section, we formulate the first-order optimality conditions in terms of the adjoint state. Since we have no constraints on control u and disturbance term τ, the couple $(\overline{\tau}, \overline{u})$ is characterized by the Euler–Lagrange equations:

$$\mathrm{d}_{\overline{u}}\mathcal{J}(\xi) = 0 \ \text{ and } \ \mathrm{d}_{\overline{\tau}}\mathcal{J}(\xi) = 0.$$

The derivatives $\mathrm{d}_{\overline{u}}\mathcal{J}$, and $\mathrm{d}_{\overline{\tau}}\mathcal{J}$ are given by (13) and (15). The pointwise observation in the cost functional $\mathcal{J}$ leads to singular sources on the right-hand side of the adjoint equation

$$\begin{cases} -\mu\Delta v + \nabla q + \eta v = y\delta_{x_0} & \text{in } \Omega, \\ \operatorname{div}\, v = 0 & \text{in } \Omega, \\ v = 0 & \text{on } \Gamma. \end{cases} \tag{19}$$

Here δ_{x_0} represents the Dirac measure concentrated at x_0.

If we test the adjoint problem with the function θ, defined by the weak solution of the auxiliary system (11), we get

$$\int_\Omega \theta\cdot y\,\mathrm{d}\delta_{x_0} = \int_{\omega_1} v\cdot\xi\,\mathrm{d}x, \tag{20}$$

$$\int_\Omega \psi\cdot y\,\mathrm{d}\delta_{x_0} = \int_{\omega_2} v\cdot\xi\,\mathrm{d}x. \tag{21}$$

Replacing (20) and (21) in expressions of (13) and (15), respectively, we find

$$\mathrm{d}_u \mathcal{J} = (v + \alpha u)\big|_{\omega_1}, \tag{22}$$

$$\mathrm{d}_\tau \mathcal{J} = (v - \beta \tau)\big|_{\omega_2}. \tag{23}$$

The source term of the adjoint state (19) belongs to the space of bounded Borel measures in Ω, denoted by $\mathcal{M}(\Omega)^N$, which can be identified with the dual space of continuous functions. By the Sobolev embedding theorem it follows that $\delta_{x_0} \in \mathbf{W}^{-1,s}(\Omega)$ for $s < N/(N-1)$, where $\mathbf{W}^{-1,s}(\Omega)$ is the dual space of $\mathbf{W}_0^{1,s'}(\Omega)$, and s' is the Hölder conjugate of s, i.e., $\frac{1}{s} + \frac{1}{s'} = 1$. The Stokes problem with $\mathbf{W}^{-1,s}(\Omega)$ source term is discussed in Chapter 4 of the monograph by Galdi (2011). The result can immediately be generalized to the Stokes–Darcy system. Therefore, the existence and uniqueness for the adjoint state follow from the following result.

Lemma 1. (Galdi, 2011, p. 284) *Assume that* $1 < s < \infty$ *and* $f \in \boldsymbol{W}^{-1,s}(\Omega)$. *Then the problem*

$$\langle \nabla v : \nabla \varphi \rangle + \eta \langle v, \varphi \rangle - \langle q, \operatorname{div} \varphi \rangle + \langle \operatorname{div} v, \pi \rangle = \langle f, \varphi \rangle \tag{24}$$

has a unique weak solution $(v, q) \in \boldsymbol{W}_0^{1,s}(\Omega) \times L_0^s(\Omega)$, *for all* $(\varphi, \pi) \in \boldsymbol{W}^{1,s'}(\Omega) \times L^{s'}(\Omega)$.

Now, we can formulate the first-order necessary and sufficient optimality conditions as follows:

Proposition 2. *Suppose that* α *and* β *are sufficiently large and* $s \in \left(\frac{2N}{N+2}, \frac{N}{N-1}\right)$. *If* $(\overline{\tau}, \overline{u})$ *is a solution to the robust control problem (10), then*

$$\overline{u} = -\frac{1}{\alpha} v \chi_{\omega_1}, \quad \overline{\tau} = \frac{1}{\beta} v \chi_{\omega_2}, \tag{25}$$

where $((y, p), (v, q)) \in \left[\left(\boldsymbol{H}_0^1(\Omega) \cap \boldsymbol{H}^2(\Omega)\right) \times L_0^2(\Omega)\right] \times [\boldsymbol{W}_0^{1,s}(\Omega) \times L_0^s(\Omega)]$, *is the unique solution to the following coupled system:*

$$\begin{cases} -\mu \Delta y + \nabla p + \eta y = h \chi_\omega \\ \qquad\qquad - \dfrac{1}{\alpha} v \chi_{\omega_1} + \dfrac{1}{\beta} v \chi_{\omega_2} & \text{in } \Omega, \\ -\mu \Delta v + \nabla q + \eta v = y \delta_{x_0} & \text{in } \Omega, \\ \operatorname{div} y = 0, \ \operatorname{div} v = 0 & \text{in } \Omega, \\ \qquad y = v = 0 & \text{on } \Gamma. \end{cases} \tag{26}$$

Proof. From the Euler–Lagrange equations (22) and (23) and the embedding

$$\mathbf{W}^{1,s}(\Omega) \hookrightarrow \mathbf{L}^2(\Omega), \quad s \in \left(\frac{2N}{N+2}, \frac{N}{N-1}\right),$$

we can deduce the expression of the robust control given in (25). The existence of the solution for the optimality system (26) follows directly from Theorem 1. In addition, for α, β large enough, this solution is unique. Indeed, suppose that u_i, τ_i, $i = 1, 2$, be two solutions of problem (10), and (y_i, v_i) be the associated solution of system (26). By linearity, the difference $(y_1 - y_2, v_1 - v_2)$ also solves the optimality system for $h \equiv 0$. Moreover, by the stability estimate and (25) we get

$$\begin{aligned} \|y_1 - y_2\|_{\mathbf{H}^2(\Omega)} &\le C(\|u_1 - u_2\|_{\mathbf{L}^2(\omega_1)} \\ &\qquad + \|\tau_1 - \tau_2\|_{\mathbf{L}^2(\omega_2)}) \\ &\le C\Big(\frac{1}{\alpha} + \frac{1}{\beta}\Big)\|v_1 - v_2\|_{\mathbf{L}^2(\Omega)} \\ &\le C\Big(\frac{1}{\alpha} + \frac{1}{\beta}\Big)\|v_1 - v_2\|_{\mathbf{W}_0^{1,s}(\Omega)}, \end{aligned} \tag{27}$$

where C denotes a generic positive constant. For $v_1 - v_2$, we have the estimate

$$\begin{aligned} \|v_1 - v_2\|_{\mathbf{W}_0^{1,s}(\Omega)} &\le C|y_1(x_0) - y_2(x_0)| \\ &\le C\|y_1 - y_2\|_{\mathbf{C}(\Omega)} \\ &\le C\|y_1 - y_2\|_{\mathbf{H}^2(\Omega)}. \end{aligned} \tag{28}$$

Combining this with (27), we get

$$\|v_1 - v_2\|_{\mathbf{W}_0^{1,s}(\Omega)} \le C\Big(\frac{1}{\alpha} + \frac{1}{\beta}\Big)\|v_1 - v_2\|_{\mathbf{W}_0^{1,s}(\Omega)}. \tag{29}$$

Thus, for α, β large enough such that

$$C\Big(\frac{1}{\alpha} + \frac{1}{\beta}\Big) < 1,$$

the solution of problem (26) is unique. ■

Remark 2. From the relations (25), we deduce that the robust control can be evaluated by scaling the adjoint state with the regularized parameters $-1/\alpha$ and $1/\beta$. Therefore, if we set $\omega_1 = \omega_2$, the coefficient $\frac{1}{\beta} - \frac{1}{\alpha}$ is the one that yields the balance between control and disturbance directions.

5. Numerical examples

In this section, we present two numerical experiments to illustrate our theoretical findings. We recall that the numerical computation of the optimal control draws on two approaches, namely: *discretize-then-optimize* and *shapeoptimize-then-discretize*, (see, e.g., Tröltzsch, 2024). In our case, we will adopt the second path, and we will focus on the numerical solution of the optimality system (26). We employ the finite elements method to discretize the coupled system (26).

The numerical simulations to be presented are conducted in two dimension using the FEniCS package, (Logg *et al.*, 2012). We use a $\mathbb{P}_2 - \mathbb{P}_1$ Taylor-Hood element method to solve the Stokes–Darcy equations (Szabó and Babuška, 2011). For discretization, we use a family of triangulations $\{\mathcal{T}_h\}_{h>0}$ for the geometrical domain Ω, the mixed finite element approximation for the optimality system (26) can reduce to the following problem. Find $[(\tilde{y}, \tilde{p}), (\tilde{v}, \tilde{q})]$ in $[\mathcal{V} \times \mathcal{Q}]^2$ such that

$$\begin{aligned}&\langle \nabla\tilde{y}, \nabla\tilde{\varphi}\rangle + \langle \eta\tilde{y}, \tilde{\varphi}\rangle - \langle \tilde{p}, \operatorname{div}\tilde{\varphi}\rangle + \langle \operatorname{div}\tilde{y}, \tilde{r}\rangle \\ &\quad + \frac{1}{\alpha}\langle \tilde{v}, \tilde{\varphi}\rangle - \frac{1}{\beta}\langle \tilde{v}, \tilde{\varphi}\rangle + \langle \nabla\tilde{v}, \nabla\tilde{\varsigma}\rangle + \langle \eta\tilde{v}, \tilde{\varsigma}\rangle \\ &\quad - \langle q_h, \operatorname{div}\tilde{\varsigma}\rangle + \langle \operatorname{div}\tilde{v}, \tilde{\lambda}\rangle + \langle \tilde{y}(x_0), \tilde{\varsigma}(x_0)\rangle \\ &\quad\quad = \langle h, \tilde{\varphi}\rangle, \quad \forall [(\tilde{\varphi}, \tilde{r}), (\tilde{\varsigma}, \tilde{\lambda})] \in [\mathcal{V} \times \mathcal{Q}]^2.\end{aligned}$$

The finite dimensional subspaces $\mathcal{V}$ and $\mathcal{Q}$ of $\mathbf{H}_0^1(\Omega)$ and $L_0^2(\Omega)$ are defined by

$$\begin{aligned}\mathcal{V} &= \Big\{ y_h \in \mathbf{C}(\Omega),\ y_{h|_K} \in \mathbb{P}_2^2,\ \forall K \in \mathcal{T}_h \\ &\quad \text{and}\ y_{h|_\Gamma} = 0 \Big\}, \\ \mathcal{Q} &= \Big\{ p_h \in C(\Omega),\ p_{h|_K} \in \mathbb{P}_1,\ \forall K \in \mathcal{T}_h \\ &\quad \text{and} \int_\Omega p_h \, \mathrm{d}x = 0 \Big\},\end{aligned}$$

respectively. The singular term in the right-hand-side of adjoint state is handled using the following regularization:

$$\delta_{x_0} \approx \frac{\varepsilon}{\pi(\|x - x_0\|^2 + \varepsilon^2)},$$

where ε is sufficiently small. Finally, the kinematic viscosity μ and the inverse permeability η are equal to 1. In order to validate the theoretical results we have to follow these steps:

1. Solve the uncontrolled Stokes–Darcy system in reference domain Ω and compute the energy $\mathcal{E}(\Omega)$.
2. Solve the uncontrolled Stokes–Darcy system in perturbed domain Ω_ϱ and compute the energy $\mathcal{E}(\Omega_\varrho)$.
3. Solve the optimality system (26) and evaluate the the robust control $(\overline{u}, \overline{\tau})$ by the relations (25); then compute $\mathcal{E}_{\overline{u},\overline{\tau}}(\Omega)$.
4. Solve the Stokes–Darcy system in Ω_ϱ with robust control and compute $\mathcal{E}_{\overline{u},\overline{\tau}}(\Omega_\varrho)$.

Example 1. Consider the geometrical domain given by the unit square $\Omega =]0,1[\times]0,1[$. The control u is acting on $\omega_1 = B_{\varepsilon_1}(x_1)$, where $\varepsilon_1 = 0.25$, $x_1 = (0.7, 0.7)^\top$, and the disturbance term τ is supported in the subdomain $\omega_2 = B_{\varepsilon_2}(x_2)$, where $\varepsilon_2 = 0.25$, $x_2 = (0.3, 0.3)^\top$. The topological perturbation $B_\varrho(x_0)$ of size $\varrho = 0.01$, will be located at $x_0 = (0.3, 0.8)^\top$. The right hand-side $h = (h_1, h_2)^\top$ in system (2) is given by a rotational vector field of the form

$$h_1(x, y) = y, \qquad h_2(x, y) = -x,$$

whose support is given by $\omega := \omega_3 = B_{\varepsilon_3}(x_3)$, with $x_3 = (0.8, 0.2)^\top$ and $\varepsilon_3 = 0.15$. The geometrical domain and its discretization are represented in Fig. 2. The control and the disturbance parameters are given by $\alpha = 10^8$, $\beta = 10^5$. For the contrast parameter, we test three values: $\gamma = 10^{10}$, $\gamma = 10^5$ and $\gamma = 10^2$. The graphical representations are performed in the first case $\gamma = 10^{10}$.

Table 1. Comparison of the energy gap for different contrast parameters in Example 1.

Contrast	$\lvert\mathcal{E}(\Omega_\varrho) - \mathcal{E}(\Omega)\rvert$	$\lvert\mathcal{E}_{\overline{u},\overline{\tau}}(\Omega_\varrho) - \mathcal{E}_{\overline{u},\overline{\tau}}(\Omega)\rvert$
$\gamma = 10^{10}$	2.77452×10^{-5}	3.89782×10^{-10}
$\gamma = 10^5$	2.77960×10^{-5}	2.94058×10^{-10}
$\gamma = 10^2$	2.79900×10^{-5}	1.91397×10^{-12}

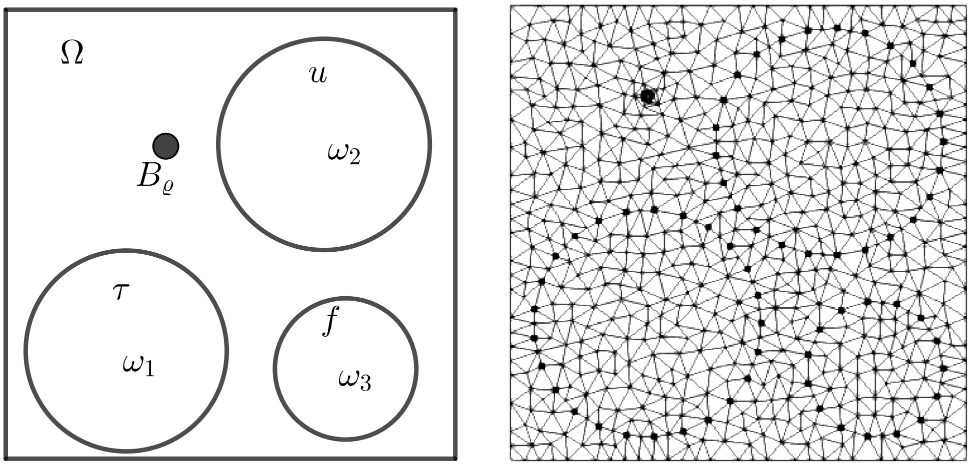
The mesh here is refined to 12681 cells. The flow in the perturbed domain is shown in Fig. 3 (a). As expected, if we plot $|y|$ over the line $x_2' = 0.8$ (x_2' is the vertical axis in the Cartesian coordinates), see Fig. 3 (b), we find $y_{|B_\varrho} \approx 0$. The optimal state $y(\overline{u}, \overline{\tau})$ and the adjoint state v given by the optimality system (26) are presented in Fig. 4. The control serves to bring the topological derivative as close as possible to zero at the point x_0, which is equivalent to drive the state to zero at x_0. In our case, we found the following result:

$$y(0.303, 0.809) = (3.89, 7.096)^\top \times 10^{-6}.$$

The last step consists in checking the energy gap, or the estimate (8). The energy functionals associated to ($u \equiv 0$, and $\tau \equiv 0$) in Ω and Ω_ϱ are respectively denoted by $\mathcal{E}(\Omega)$ and $\mathcal{E}(\Omega_\varrho)$. The quantitative results are summarized in Table 1. As intended, by a robust control $(\overline{u}, \overline{\tau})$, we can determine the minimum energy gap with respect to the topological perturbation. ♦

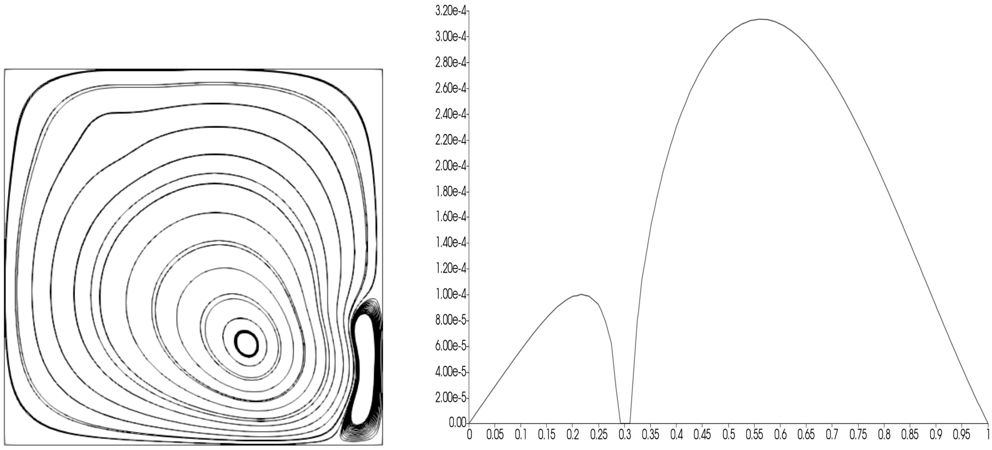
Example 2. Consider the enclosed Stokes–Darcy flow in the unit circle, $\Omega = B_R(\mathcal{O})$, $R = 1$ and $\mathcal{O} = (0, 0)^\top$. The control term acts in the ball $\omega_1 = B_{\varepsilon_1}(x_1)$, where $\varepsilon_1 = 0.4$ and $x_1 = (0, 0.5)^\top$. The disturbance and the source terms are supported in $\omega_2 = B_{\varepsilon_2}(x_2)$, where $\varepsilon_2 = 0.4$ and $x_2 = (0, -0.5)^\top$. The obstacle of size $\varrho = 0.01$ is located at $x_0 = (0.7, 0)^\top$. The corresponding design domain is shown in Fig. 5. The right hand-side $h = (h_1, h_2)^\top$ of system (2) is given by

$$h_1(x, y) = 0, \qquad h_2(x, y) = -1.$$



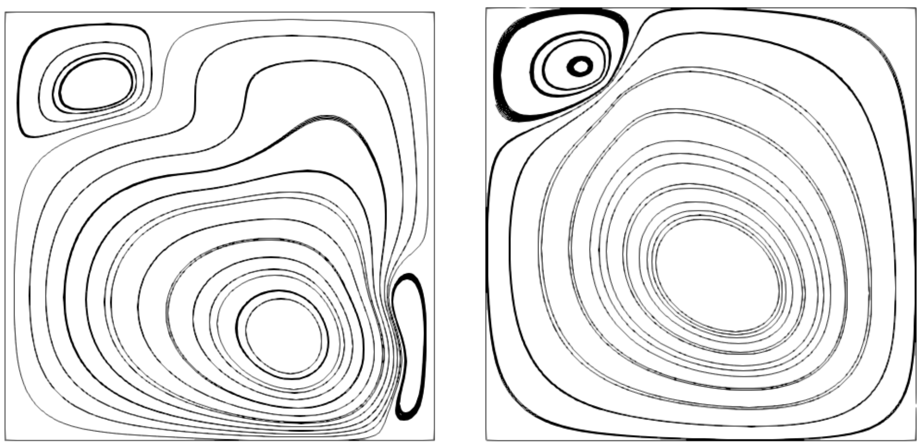
(a) (b)

Fig. 2. Example 1: geometrical domain Ω (a), plot of the mesh for Ω (b).



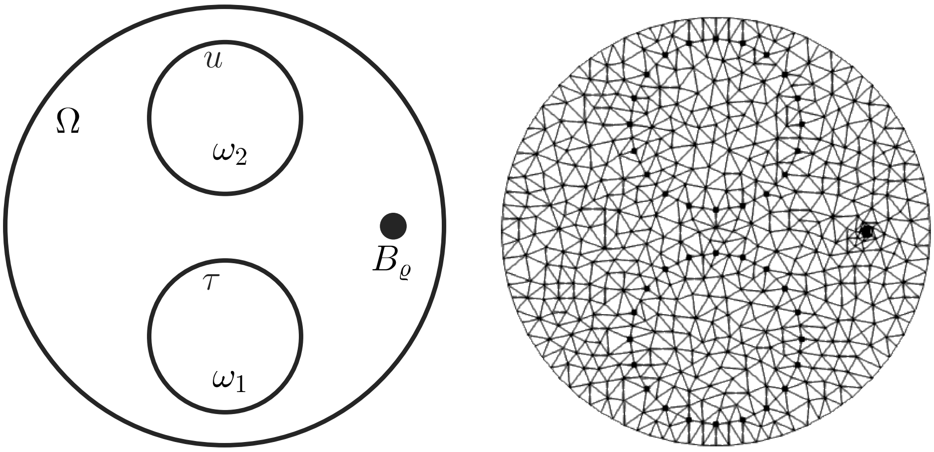
(a) (b)

Fig. 3. Example 1: streamline for the Stokes–Darcy flow in Ω_ϱ with $\gamma = 10^{10}$ (a), plot over the line $y = 0.8$ (b).



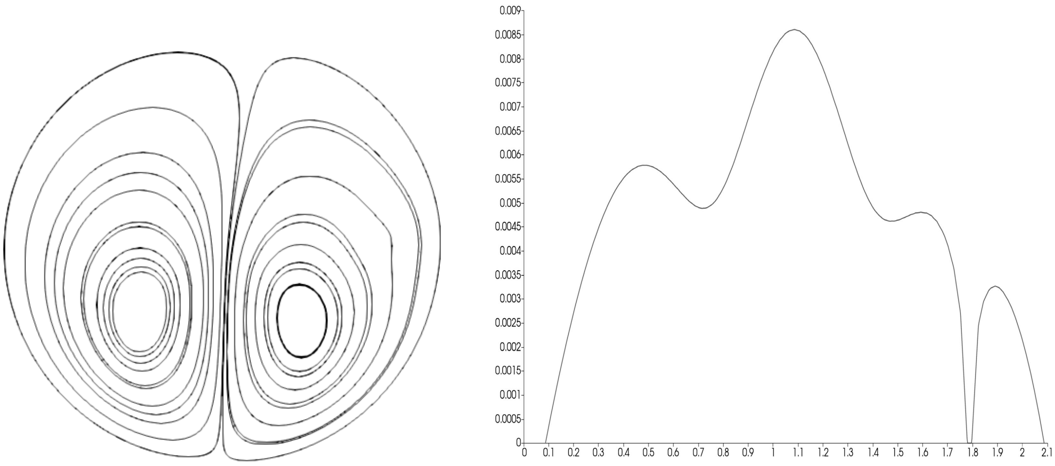
(a) (b)

Fig. 4. Example 1: streamline for the controlled state $y(\overline{u}, \overline{\tau})$ (a), streamline for the adjoint state v (b).



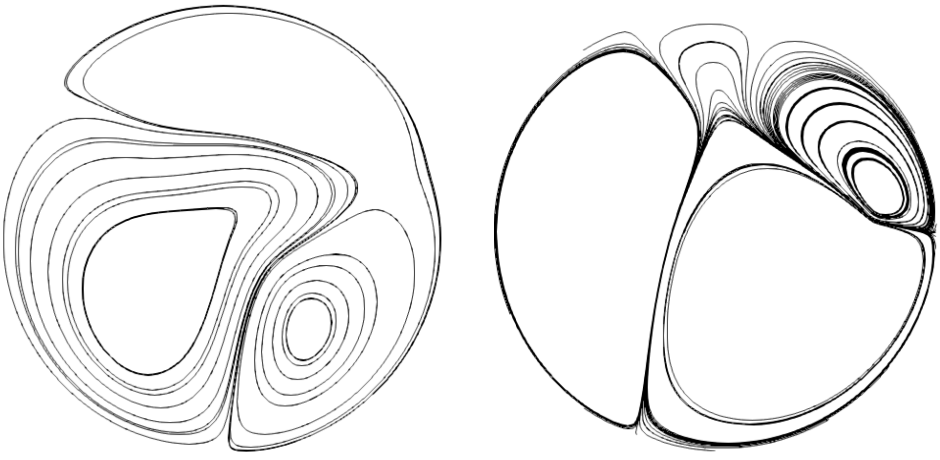
(a) (b)

Fig. 5. Example 2: geometrical domain Ω (a), plot of the mesh for Ω (b).



(a) (b)

Fig. 6. Example 2: streamline for the Stokes–Darcy flow in Ω_ϱ with $\gamma = 10^{10}$ (a), plot over the line $y = 0$ (b).



(a) (b)

Fig. 7. Example 2: streamline for the controlled state $y(\overline{u}, \overline{\tau})$ (a), streamline for the adjoint state v (b).

Table 2. Comparison of the energy gap for different contrast parameters in Example 2.

Contrast	$\lvert\mathcal{E}(\Omega_\varrho)-\mathcal{E}(\Omega)\rvert$	$\lvert\mathcal{E}_{\overline{u},\overline{\tau}}(\Omega_\varrho)-\mathcal{E}_{\overline{u},\overline{\tau}}(\Omega)\rvert$
$\gamma=10^{10}$	0.00149909	9.47862×10^{-6}
$\gamma=10^{5}$	0.00151717	7.52874×10^{-6}
$\gamma=10^{2}$	0.00159826	5.33838×10^{-8}

The penalization parameters are fixed as follows: $\alpha = 10^8, \beta = 10^5$. Again, three values for the contrast parameter are considered: $\gamma = 10^{10}, \gamma = 10^5$ and $\gamma = 10^2$. The mesh in Fig. 5 is refined to 16242 elements. 6(a) shows the streamline for the velocity field in the perturbed domain Ω_ϱ. The curve in Fig. 6(b), represents $|y|$ over the line $x_2' = 0$, and we observe that at the obstacle location $x_0 = (0.7, 0)^\top$ we have $y_{|B_\varrho} \approx 0$. Figure 7 presents the optimal state $y(\overline{u}, \overline{\tau})$ and the adjoint state v. The quantitative results obtained for the energy functional are reported in Table 2.

6. Conclusion

In this paper, a new method that leads to insensitivity of the energy functional with respect to topological perturbation is presented. The model problem in fluid mechanics is governed by the Stokes–Darcy equations. A penalization approach is used to deal with the no-slip boundary condition on the infinitesimal obstacle. Our approach is based on the resolution of a minimax auxiliary problem, where the cost functional involves point evaluations of the state. This study can naturally be generalized in several directions, in particular for elliptic linear problems, where the theory of topological sensitivity is successfully developed. For nonlinear problems, it is well known that the topological derivative depends on the direct and adjoint states and probably their gradients. In this case, our approach leads to an optimal control problem for a cascade system, i.e., a coupled system which includes the nonlinear state and its costate, with a control term acting partially through the state equation. This issue is currently under development and will be the subject for a forthcoming work. For the nonsteady case, several important questions arise, such as the asymptotic behavior of optimal control in long time horizon and the turnpike properties (see, e.g., Gugat and Lazar, 2023; Gugat and Sokołowski, 2023). For time-dependent problems, the strategy used in this article requires further analysis, especially the expression of the topological derivative which remains in question in the nonstationary case.

Acknowledgment

The research of A.A. Novotny and J. Sokołowski is supported by the project *Conception optimale de matériaux micro-architecturés* within the framework of the action CAPES-COFECUB Ph1035/24 (no. dossier Campus France: 50472UA). The authors want to thank the anonymous referees for their meticulous revision and useful comments that helped improve this paper.

References

Amstutz, S. (2005). The topological asymptotic for the Navier–Stokes equations, *ESAIM: Control, Optimization and Calculus of Variations* **11**(3): 401–425.

Amstutz, S. (2006). Topological sensitivity analysis for some nonlinear PDE systems, *Journal de Mathématiques Pures et Appliquées* **85**(4): 540–557.

Baumann, P. and Sturm, K. (2022). Adjoint-based methods to compute higher-order topological derivatives with an application to elasticity, *Engineering Computations* **39**(1): 60–114.

Bewley, T.R., Temam, R. and Ziane, M. (2000). A general framework for robust control in fluid mechanics, *Physica D: Nonlinear Phenomena* **138**(3–4): 360–392.

Bogachev, V.I. and Ruas, M.A.S. (2007). *Measure Theory*, Vol. 1, Springer, Berlin.

Boyer, F. and Fabrie, P. (2005). *Eléments d'analyse pour l'étude de quelques modèles d'écoulements de fluides visqueux incompressibles*, Springer, Berlin.

Brett, C.E. (2014). *Optimal Control and Inverse Problems Involving Point and Line Functionals and Inequality Constraints*, PhD thesis, University of Warwick, Coventry.

Caubet, F. and Dambrine, M. (2012). Localization of small obstacles in Stokes flow, *Inverse Problems* **28**(10): 105007.

Dáger, R. (2006). Insensitizing controls for the 1-D wave equation, *SIAM Journal on Control and Optimization* **45**(5): 1758–1768.

Dziri, R., Moubachir, M. and Zolésio, J.-P. (2004). Dynamical shape gradient for the Navier–Stokes system, *Comptes Rendus Mathematique* **338**(2): 183–186.

Dziri, R. and Zolésio, J.-P. (2011). Drag reduction for non-cylindrical Navier–Stokes flows, *Optimization Methods and Software* **26**(4–5): 575–600.

Ekeland, I. and Temam, R. (1999). *Convex Analysis and Variational Problems*, SIAM, Philadelphia.

Ervedoza, S., Lissy, P. and Privat, Y. (2022). Desensitizing control for the heat equation with respect to domain variations, *Journal de l'École Polytechnique Mathématiques* **9**: 1397–1429.

Galdi, G. (2011). *An Introduction to the Mathematical Theory of the Navier–Stokes Equations: Steady-State Problems*, Springer, New York.

Garreau, S., Guillaume, P. and Masmoudi, M. (2001). The topological asymptotic for PDE systems: The elasticity case, *SIAM Journal on Control and Optimization* **39**(6): 1756–1778.

Giusti, S.M., Sokołowski, J. and Stebel, J. (2015). On topological derivatives for contact problems in elasticity, *Journal of Optimization Theory and Applications* **165**: 279–294.

Guerrero, S. (2007). Controllability of systems of Stokes equations with one control force: Existence of insensitizing controls, *Annales de l'Institut Henri Poincaré C, Analyse non linéaire* **24**(6): 1029–1054.

Gueye, M. (2013). Insensitizing controls for the Navier–Stokes equations, *Annales de l'Institut Henri Poincaré C, Analyse non linéaire* **30**(5): 825–844.

Gugat, M. and Lazar, M. (2023). Optimal control problems without terminal constraints: The turnpike property with interior decay, *International Journal of Applied Mathematics and Computer Science* **33**(3): 429–438, DOI: 10.34768/amcs-2023-0031.

Gugat, M. and Sokołowski, J. (2023). An aspect of the turnpike property: Long time horizon behavior, *Serdica Mathematical Journal* **49**(1–3): 127–154.

Guillaume, P. and Hassine, M. (2008). Removing holes in topological shape optimization, *ESAIM: Control, Optimisation and Calculus of Variations* **14**(1): 160–191.

Hassine, M. and Masmoudi, M. (2004). The topological asymptotic expansion for the quasi-Stokes problem, *ESAIM: Control, Optimisation and Calculus of Variations* **10**(4): 478–504.

Hlaváček, I., Novotny, A., Sokołowski, J. and Żochowski, A. (2009). On topological derivatives for elastic solids with uncertain input data, *Journal of Optimization Theory and Applications* **141**(3): 569–595.

Iguernane, M., Nazarov, S.A., Roche, J.-R., Sokolowski, J. and Szulc, K. (2009). Topological derivatives for semilinear elliptic equations, *International Journal of Applied Mathematics and Computer Science* **19**(2): 191–205, DOI: 10.2478/v10006-009-0016-4.

Kovtunenko, V.A. and Kunisch, K. (2014). High precision identification of an object: Optimality-conditions-based concept of imaging, *SIAM Journal on Control and Optimization* **52**(1): 773–796.

Krzyżanowski, P., Malikova, S., Mucha, P. B. and Piasecki, T. (2024). Comparative analysis of obstacle approximation strategies for the steady incompressible Navier–Stokes equations, *Applied Mathematics & Optimization* **89**(2): 1–20.

Leugering, G., Novotny, A.A. and Sokołowski, J. (2022). On the robustness of the topological derivative for Helmholtz problems and applications, *Control and Cybernetics* **51**(2): 227–248.

Lions, J. (1992). *Sentinelles pour les Systmès Distribués à Données Incomplètes*, Masson, Paris.

Logg, A., Mardal, K.-A. and Wells, G. (2012). *Automated Solution of Differential Equations by the Finite Element Method: The FEniCS Book*, Springer, Berlin.

Moubachir, M. and Zolesio, J.-P. (2006). *Moving Shape Analysis and Control: Applications to Fluid Structure Interactions*, Chapman and Hall/CRC, Boca Raton.

Novotny, A.A. and Sokołowski, J. (2012). *Topological Derivatives in Shape Optimization*, Springer, Berlin.

Novotny, A.A., Sokołowski, J. and Żochowski, A. (2019). *Applications of the Topological Derivative Method*, Springer, Cham.

Sá, N.L., Amigo, R.R., Novotny, A.A. and Silva, N.E. (2016). Topological derivatives applied to fluid flow channel design optimization problems, *Structural and Multidisciplinary Optimization* **54**: 249–264.

Sokołowski, J. and Żochowski, A. (1999). On the topological derivative in shape optimization, *SIAM Journal on Control and Optimization* **37**(4): 1251–1272.

Sokołowski, J. and Zolésio, J.-P. (1992). *Introduction to Shape Optimization*, Springer, Berlin.

Sturm, K. (2020). Topological sensitivities via a Lagrangian approach for semilinear problems, *Nonlinearity* **33**(9): 4310.

Szabó, B. and Babuška, I. (2011). *Introduction to Finite Element Analysis: Formulation, Verification and Validation*, Wiley, Chichester.

Tröltzsch, F. (2024). *Optimal Control of Partial Differential Equations: Theory, Methods and Applications*, American Mathematical Society, Providence.

Merwan Abdelbari received his PhD degree in applied mathematics from the University of Oran, Algeria, in 2021. Currently, he is an assistant professor at the University of Hassiba Benbouali, Algeria. His research interests include shape and topology optimization, optimal control, fluid dynamics and numerical methods for PDEs.

Khadra Nachi received her PhD in mathematics at the University of Oran, Algeria. She is a full professor at University of Oran 1 and a member of the Laboratory of Mathematical Analysis and Applications. Her research focuses on continuous optimization, optimal control, fixed-point theory and set-valued analysis. She has published research papers in national and international journals and conference proceedings.

Jan Sokołowski is a professor emeritus of the University of Lorraine and a full professor at the Polish Academy of Sciences at the Systems Research Institute in Warsaw, Poland. His research concerns the theory and numerical methods for shape and topology optimization, optimal control and optimum design of distributed parameter systems governed by partial differential equations. He has authored six monographs and 200 papers.

Antonio Andre Novotny is a senior researcher at the National Laboratory for Scientific Computing, Petrópolis, Brazil. He has been trained as a mechanical engineer and an applied mathematician. His field of specialization is topological asymptotic expansions of shape functionals with PDE constraints in singularly perturbed geometrical domains.

Appendix

In this section, we will derive the asymptotic expansion (6). In general, several methods are devoted to calculate the topological derivative expression, for example the adjoint methods. We refer the reader to Baumann and Sturm (2022), who give a comprehensive review about this approach. The topological derivative can also be derived as a singular limit of the shape derivative. This method is widely used, especially for the energy functionals, see the monograph by Novotny and Sokołowski (2012) and the references therein. The common step among all of these methods is to perform the asymptotic analysis of the state with respect to the parameter ϱ. In our case, we will see that the penalization of the no-slip boundary condition enormously simplifies the asymptotic analysis for the y_ϱ and $\mathcal{E}(\Omega_\varrho)$ as well.

Lemma A1. *Let y and y_ϱ be the weak solutions to the systems (2) and (5), respectively. Then, we have the following estimate:*

$$\|y - y_\varrho\|_{\boldsymbol{H}_0^1(\Omega)} \le C\varrho^{\frac{N}{2}+s}, \tag{A1}$$

where $s \in]0,1]$, and C is a constant independent of the parameter ϱ.

Proof. The weak form for the Stokes–Darcy system (2) in reference domain Ω is given by

$$\mu\int_\Omega \nabla y\cdot\nabla\varphi + \int_\Omega \eta y\cdot\varphi - \int_\Omega m\cdot\varphi = 0, \quad \forall\varphi\in\mathbf{H}_{\mathrm{div}}(\Omega), \tag{A2}$$

where $m = h\chi_\omega + u\chi_{\omega_1} + \tau\chi_{\omega_2}$. In the perturbed domain Ω_ϱ, the weak form of the system (5) reads

$$\mu\int_{\Omega_\varrho} \nabla y_\varrho\cdot\nabla\varphi + \int_{\Omega_\varrho} \eta_\varrho y_\varrho\cdot\varphi - \int_{\Omega_\varrho} m\cdot\varphi = 0, \quad \forall\varphi\in\mathbf{H}_{\mathrm{div}}(\Omega). \tag{A3}$$

Let us now subtract (A3) from (A2), to obtain

$$\mu\int_\Omega \nabla(y_\varrho - y)\cdot\nabla\varphi + \int_\Omega (\eta_\varrho y_\varrho - \eta y)\cdot\varphi = 0, \quad \forall\varphi\in\mathbf{H}_{\mathrm{div}}(\Omega). \tag{A4}$$

Setting $\varphi = y_\varrho - y$ in (A4), we get

$$\mu\int_\Omega |\nabla(y_\varrho - y)|^2 + \int_\Omega (\eta_\varrho y_\varrho - \eta y)\cdot(y_\varrho - y) = 0. \tag{A5}$$

The coefficient η_ϱ is defined piecewise in Ω_ϱ, hence we have the following decomposition for the last integral:

$$\begin{aligned}\int_\Omega &(\eta_\varrho y_\varrho - \eta y)\cdot(y_\varrho - y)\\ &= \int_{\Omega\setminus B_\varrho} \eta|y_\varrho - y|^2 + \int_{B_\varrho} \eta(\gamma y_\varrho - y)\cdot(y_\varrho - y)\\ &= \int_\Omega \eta_\varrho|y_\varrho - y|^2 - \int_{B_\varrho}(1-\gamma)\eta y\cdot(y_\varrho - y).\end{aligned} \tag{A6}$$

After replacing in (A5), we get

$$\int_\Omega |\nabla(y_\varrho - y)|^2 + \int_\Omega \eta_\varrho|y_\varrho - y|^2 = \int_{B_\varrho}(1-\gamma)\eta y\cdot(y_\varrho - y). \tag{A7}$$

Using the Cauchy–Schwarz inequality and the Lebesgue differentiation theorem (Bogachev and Ruas, 2007), p. 351, we find the following estimate:

$$\mu\int_\Omega |\nabla(y_\varrho - y)|^2 + \int_\Omega \eta_\varrho|y_\varrho - y|^2 \le C\varrho^{\frac{N}{2}}\|y_\varrho - y\|_{\mathbf{L}^2(B_\varrho)}. \tag{A8}$$

On other side, by the Hölder inequality and the Sobolev embedding $\mathbf{H}_0^1(B_\varrho)\hookrightarrow\mathbf{L}^2(B_\varrho)$, we get

$$\begin{aligned}\|y_\varrho - y\|_{\mathbf{L}^2(B_\varrho)} &\le C\varrho^{\frac{N}{2q}}\left(\int_{B_\varrho}|y_\varrho - y|^{2p}\right)^{\frac{1}{2p}}\\ &= C\varrho^{\frac{N}{2q}}\|y_\varrho - y\|_{\mathbf{L}^{2p}(B_\varrho)}\\ &\le C\varrho^{s}\|y_\varrho - y\|_{\mathbf{H}_0^1(\Omega)},\end{aligned} \tag{A9}$$

where $\frac{1}{p}+\frac{1}{q} = 1$, $q \ge \frac{N}{2}$, and $s = \frac{N}{2q}$. Using (A8) and (A9), we obtain

$$\mu\int_\Omega |\nabla(y_\varrho - y)|^2 + \int_\Omega \eta_\varrho|y_\varrho - y|^2 \le C\varrho^{\frac{N}{2}+s}\|y_\varrho - y\|_{\mathbf{H}_0^1(\Omega)}. \tag{A10}$$

Finally, the estimate (A1) can be derived directly from the coercivity inequality, i.e.,

$$c\|y_\varrho - y\|^2_{\mathbf{H}_0^1(\Omega)} \le \mu\int_\Omega |\nabla(y_\varrho - y)|^2 + \int_\Omega \eta_\varrho|y_\varrho - y|^2. \tag{A11}$$

■

Now, let us go back to the shape functional $\mathcal{E}(\Omega)$. By setting $\varphi = y_\varrho - y$ in the weak forms (A2) and (A3), we obtain

$$\mathcal{E}(\Omega) = \mu\int_\Omega \nabla y_\varrho\cdot\nabla y + \int_\Omega \eta_\varrho y_\varrho\cdot y, \tag{A12}$$

$$\mathcal{E}(\Omega_\varrho) = \mu\int_\Omega \nabla y_\varrho\cdot\nabla y + \int_\Omega \eta y_\varrho\cdot y. \tag{A13}$$

In addition, we have

$$\begin{aligned}\mathcal{E}(\Omega_\varrho) - \mathcal{E}(\Omega) \\ &= \int_{B_\varrho} (1-\gamma)\eta y_\varrho \cdot y \\ &= \int_{B_\varrho} (1-\gamma)\eta |y|^2 + (1-\gamma)\eta (y_\varrho - y) \cdot y\end{aligned} \tag{A14}$$

The first term gives the TD of $\mathcal{E}$, while the second term is a remainder of order $o(\varrho^N)$. More precisely, we have

$$\begin{aligned}\int_{B_\varrho} \eta(y_\varrho - y)\cdot y &\le C\varrho^{\frac{N}{2}} \|y_\varrho - y\|_{\mathbf{H}_0^1(B_\varrho)} \\ &\le C\varrho^{\frac{N}{2}} \|y_\varrho - y\|_{\mathbf{H}_0^1(\Omega)} = o(\varrho^N),\end{aligned} \tag{A15}$$

$$\mathcal{E}(\Omega_\varrho) - \mathcal{E}(\Omega) = \int_{B_\varrho} (1-\gamma)\eta|y|^2 + o(\varrho^N). \tag{A16}$$

Again, using the Lebesgue differentiation theorem we find the topological derivative expression at x_0, namely,

$$\begin{aligned}\mathcal{E}(\Omega_\varrho) - \mathcal{E}(\Omega) = \operatorname{meas}(B_\varrho)(1-\gamma)\eta|y(x_0)|^2 \\ + o(\varrho^N).\end{aligned} \tag{A17}$$

Received: 29 May 2024
Revised: 14 October 2024
Accepted: 10 December 2024