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CONTROLLABILITY OF NON-LINEAR MODELS IN MODELLING POPULATION DYNAMICS

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In the paper a mathematical model of two interacting biological populations is considered. Using techniques adopted from functional analysis such as Schauder's fixed-point theorem, a sufficient condition for local controllability is formulated and proved. Moreover, some additional remarks and comments on controllability problem are given.

1. Introduction

Controllability is one of the fundamental properties of general dynamic systems. In recent years controllability theory for various kinds of dynamical systems has been the subject of considerable interest to many research scientists (Klamka, 1991b). It has been motivated by the wide range of applications in different areas of science and engineering. This paper is intended to provide information about research results obtained in the field of two interacting populations (Freeman, 1980).

The analysis of the populations dynamics equations is of considerable interest and importance to many fields such as ecology, immunology and etc. (Freeman, 1980; Freeman and Waltman, 1975). There are many models available in the literature for two interacting populations dynamics (Albrecht *et al.*, 1976; Freeman, 1980; Freeman and Waltman, 1975; Joshi and Goerge, 1992; Klamka, 1991a). Yet despite the current interest in ecological problems, very few studies have appeared in which control theory has been used to treat the control of interacting populations (Joshi and Goerge, 1992; Klamka, 1991a).

In this paper, we shall consider a general, non-linear mathematical model of two interacting populations dynamics. This mathematical model represents a set of two ordinary differential equations defined in a given finite time interval.

The main purpose of this paper is to investigate the problem of local controllability in finite time interval for two interacting populations. Roughly speaking, the concept of controllability means that we can steer our dynamical system from a given initial state to the desired final state in finite time using the so-called admissible controls (Klamka, 1991b). Therefore, controllability is the fundamental property of dynamical systems and can be used in many different aspects of ecology and biology as e.g. in birth of species, dynamics of epidemics, etc. The controllability property may be also useful in computing the steering function that transfers the dynamical system to its desired final state.

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The paper extends the results recently presented in the papers (Joshi and Goerge, 1992) and (Klamka, 1991a). In order to solve our controllability problems we shall extensively use the techniques of functional analysis, especially the Schauder's fixed-point theorem (Klamka, 1976a). Moreover, some additional remarks and comments concerning various aspects of controllability will be presented.

2. Mathematical Model

Let us consider a dynamical system in which two kinds of species are living and the food required for the survival of one species (predator) is the other species (prey). Generally, it is possible to steer the population densities of the prey and predator using many different kinds of control inputs, e.g. harvesting or killing a certain amount of species and breeding other species.

In the sequel, we shall consider non-linear mathematical model of two interacting populations. This mathematical model can be represented by the following set of two non-linear scalar ordinary differential equations (Albrecht *et al.*, 1976; Freeman, 1980; Freeman and Waltman, 1975; Joshi and Goerge, 1989; 1992; Klamka, 1991a):

$$\dot{x}_1(t) = a_{11}(x_1(t))x_1(t) - a_{12}(x_1(t))x_2(t) + b_1(x_1(t))u(t) + f_1(x_1(t), x_2(t)) \tag{1a}$$

$$\dot{x}_2(t) = a_{21}(x_2(t))x_1(t) - a_{22}(x_2(t))x_2(t) + b_2(x_2(t))u(t) + f_2(x_1(t), x_2(t)) \tag{1b}$$

defined on a fixed time interval $[0, t_1]$, where $x_1(t)$ and $x_2(t)$ are the population densities of the prey and predator, respectively.

The other functions, which are generally assumed to be continuous have the following meaning:

$a_{11}(x_1) > 0$ is the specific growth rate of the prey,

$a_{12}(x_1) > 0$ is the predator response function by which the specific growth rate of the prey is diminished,

$a_{21}(x_2) > 0$ is the specific extinction rate of the predator population in the absence of the prey,

$a_{22}(x_2) > 0$ is the growth rate of the predator when the population density of the prey is x_1

The terms $b_1(x_1)u(t)$ and $b_2(x_2)u(t)$ represent the effect of controls on the prey and predator populations growth rate, respectively. The functions $f_1(x_1, x_2)$ and $f_2(x_1, x_2)$ are the returns (increasing or decreasing) of the prey and predator populations growth rates, respectively.

Mathematical model (1) is a special case of general non-linear non-autonomous dynamical systems considered in the papers (Klamka, 1976a; 1975a; 1975b; 1978).

For simplicity of notation, let us rewrite the mathematical model (1) in a vector-matrix form:

$$\dot{x}(t) = A(x(t))x(t) + B(x(t))u(t) + F(x(t)), \qquad t \in [0, t_1] \tag{2}$$

where

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \qquad A(x(t)) = \begin{bmatrix} a_{11}(x_1(t)) & -a_{12}(x_1(t)) \\ a_{21}(x_2(t)) & -a_{22}(x_2(t)) \end{bmatrix}$$

$$B(x(t)) = \begin{bmatrix} b_1(x_1(t)) \\ b_2(x_2(t)) \end{bmatrix}, \qquad F(x(t)) = \begin{bmatrix} f_1(x_1(t), x_2(t)) \\ f_2(x_1(t), x_2(t)) \end{bmatrix}$$

For the dynamical models (1) or equivalently (2), we can formulate the following so-called controllability problem: find sufficient condition on the system parameters which will guarantee the existence of a control $u(t)$, $t \in [0, t_1]$, which transfers any given initial populations density to the desired final populations density. Controllability problem is one of the fundamental concepts of modern control theory (Klamka, 1991b).

Now we shall give a formal definition of the local controllability for the dynamical system (1) or, equivalently the dynamical system (2).

Definition 1. Dynamical system (1) is said to be *locally controllable* on $[0, t_1]$ in the domain $D \subset \mathbb{R}^2$ if for each pair of vectors x^0, $x^1 \in D$, there exists a control $u \in C([0,t_1], \mathbb{R})$, such that the solution $x \in C([0,t_1], D)$ of (1) satisfies the initial condition $x(0) = x^0$ and the final condition $x(t_1) = x^1$

In other words, local controllability means, that it is possible to steer dynamical system (1) from an arbitrary initial state $x^0 \in D$ to an arbitrary final state $x^1 \in D$.

In many applications, the domain $D \subset \mathbb{R}^2$ is a closed ball centered at x^0 with radius $r > 0$, i.e.

$$D = \{x \in \mathbb{R}^2 : \|x - x^0\| \leq r\}$$

Moreover, it should be noted, that since the set $D \subset \mathbb{R}^2$ is convex, the set of continuous functions $C([0,t_1], D)$ is also convex.

This fact has an essential meaning in the proof of the main controllability theorem.

3. Controllability Condition

Controllability problems for various kinds of non-linear dynamical systems have been studied in many papers (see (Klamka, 1991b) for extensive bibliography). Among different techniques, the methods based on the fixed-point theorems are very useful. Using Schauder's fixed-point theorem in the papers (Davison and Kunze, 1970; Joshi and Goerge, 1989; 1992; Klamka, 1975a; 1975b; 1991a; 1991b) many sufficient conditions for various kinds of controllability for different types of non-linear dynamical systems have been formulated and proved. Moreover, these results have been extended to cover the case of dynamical systems with various kinds of delays in the control (Klamka, 1976a; 1976b; 1976c; 1978).

Now, let us recall some fundamental results concerning controllability of the dynamical system (1), (see e.g. (Klamka, 1976a; 1978) for more details).

In order to establish the sufficient conditions for local controllability of the dynamical system (1) we shall consider for each fixed function $z \in C([0,t_1]),D)$, a ***linear*** dynamical system given by the following equation (Klamka, 1976a):

$$\dot{x}(t) = A(z(t))x(t) + B(z(t))u(t) + F(z(t)), \qquad t \in [0,t_1] \tag{3}$$

In the dynamic system (3) the functional argument $x(t)$ in the matrix $A(x(t))$ and vectors $B(x(t))$, $F(x(t))$ have been replaced by a specific function $z \in C([0,t_1],D)$.

Let us denote by $G(t;z), t \in [0,t_1]$ the state transition matrix of the homogeneous part of the linear differential equation (3). Hence, using the standard methods applicable for linear dynamical systems, we may define the so-called *controllability matrix* $W(t_1;z)$ for system (3), (see (Klamka, 1976b; 1976c; 1978; 1991 – Chapter 1) for more details)

$$W(t_1;z) = \int_0^{t_1} G(t;z)B(z)B^T(z)G^T(t;z)\,\mathrm{d}t \tag{4}$$

where the symbol T denotes the transposition.

In order to use Schuder's fixed-point theorem and specially Arzela-Ascoli compactness theorem we need some additional technical assumptions regarding the system parameters. Hence, in the sequel, without loss of generality, we shall assume that all the functional parameters are uniformly bounded functions in the domain D, i.e. there exist positive constants K, M, N such that:

$$|a_{ij}(x)| \le M \quad \text{for} \quad i,j = 1,2, \ \ x \in D$$

$$|b_i(x)| \le N \quad \text{for} \quad i = 1,2, \ \ x \in D$$

$$|f_i(x)| \le K \quad \text{for} \quad i = 1,2, \ \ x \in D$$

Under the above, rather technical assumptions it is possible to formulate and prove the following sufficient condition for local controllability of the dynamical system (1).

Theorem 1. *Dynamical system (1) is locally controllable on* $[0,t_1]$ *in the domain* D *if there exists a constant* $c > 0$ *such that*

$$\inf_{z \in C([0,t_1],D)} \det W(t_1;z) \ge c \tag{5}$$

Proof. First of all, let us define the control $u(t;z)$, $t \in [0,t_1]$, which steers the linear dynamical system (3) from a given initial state $x^0 \in D$ to the desired final state $x^1 \in D$. This control function depends strongly on the choice of the function $z \in C([0,t_1],D)$ and has the following form (Klamka, 1975a; 1975b):

$$u(t;z) = B^T(z(t))G^T(t;z)W^{-1}(t_1;z)q(x^0,x^1;z), \ \ t \in [0,t_1] \tag{6}$$

where the vector $q(x^0,x^1;z) \in \mathbb{R}^2$ for a given function $z \in C([0,t_1],D)$ depends only on a given initial state $x^0 \in D$ and the desired final state $x^1 \in D$.

Moreover, it should be pointed out, that the vector $q(x^0, x^1; z)$ does not depend on time t and is given by the following formula (Klamka, 1975a; 1975b):

$$q(x^0, x^1; z) = x^1 - G(t_1; z)x^0 + \int_0^{t_1} G(t; z)F(z)\,\mathrm{d}t \tag{7}$$

For any fixed function $z \in C([0, t_1], D)$ the solution of the linear differential equation (3) for $t \geq 0$ has the following form (Klamka, 1975a; 1975b; 1991b; Mirza and Womack, 1972):

$$x(t; z) = G(t; z)x^0 + \int_0^t G(s; z)B(z)u(s; z)\,\mathrm{d}s + \int_0^t G(s; z)F(z)\,\mathrm{d}s \tag{8}$$

Substituting (6) into (8) it can be easily verified that the control $u(t; z)$, $t \in [0, t_1]$ steers the linear dynamical system (3) from a given initial state $x^0 \in D$ to the desired final state $x^1 \in D$ at time t_1.

Now, let us consider the right hand side of formula (8) as a non-linear operator $P(z)(t)$, which maps a closed convex set $C([0, t_1], D)$ into itself (Klamka, 1975a; 1975b). Hence, it is possible to write equality (8) in the form:

$$x(t; z) = P(z)(t) \tag{9}$$

Let us observe that the non-linear operator P, given by formula (9) is continuous with respect to the norm in the Banach space $C([0, t_1], \mathbb{R}^2)$. Therefore, using the Arzela – Ascoli theorem on compactness in the space $C([0, t_1], \mathbb{R}^2)$ it can be deduced that the non-linear operator P maps a closed convex set of the Banach space $C([0, t_1], \mathbb{R}^2)$ into its compact subset. Hence, by the classical Schauder's fixed-point theorem we conclude that the non-linear operator P has at least one fixed-point in the set $C([0, t_1], D)$. Therefore, there always exists at least one function $z^*(t) \in C([0, t_1], D)$ such that

$$z^*(t) = x^*(t) = P(z^*)(t) \tag{10}$$

It can be shown, by differentiating with respect to the variable t, that $x^*(t)$ is the solution of the dynamical system (3) for the control $u(t; z^*)$ given by formula (6). Hence our theorem follows. ■

It should be stressed that the most difficult problem in practical applications of Theorem 1 is to compute the controllability matrix $W(t_1; z)$. Practically, it is possible only for very narrow special classes of dynamical systems (1).

4. Concluding Remarks

In the paper the local controllability problem has been investigated. Using the functional analysis techniques, a sufficient condition for the local controllability in finite time interval has been formulated. Proof of this controllability condition is based on Schauder's fixed-point theorem and the Arzela-Ascoli compactness theorem (Klamka, 1975a; 1975b; 1978).

It should be mentioned that in our controllability problem it is possible to use other fixed-point theorems as for example the Banach fixed-point theorem or

Darboux fixed-point theorem. However, it needs more assumptions, which are in a sense more restrictive. The discussion of such possibilities can be found in the monograph (Klamka, 1991b) where an extensive bibliography concerning controllability problems for various kinds of dynamical systems is given.

The controllability results presented in this paper can be extended to more general non-autonomous, non-linear dynamical systems with time-variable delays in the control (Klamka, 1976b). Such an extension requires modification of the controllability matrix and formulae (6) and (7). Another possible extension is to consider the so-called distributed delays in the control (Klamka, 1978). In this case the delayed control are under the integral sign. This modification leads to a very complicated form of the controllability matrix (Klamka, 1978). However, it should be stressed that in both the extensions the methods of proofs are always the same and are based on Schauder's fixed-point theorem (Klamka, 1978).

The so-called constrained controllability, i.e. controllability with constrained controls (Klamka, 1991b) still remains open and to be solved. However, constrained controllability results concerning prey-predator dynamical systems have been recently presented in the paper (Klamka, 1991a).

Finally, it should be mentioned that a similar non-linear technique of analysis can be used in controllability investigations of other types of mathematical models of population dynamics (Freeman and Waltman, 1975).

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