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STRATEGIC SENSORS AND SPY SENSORS

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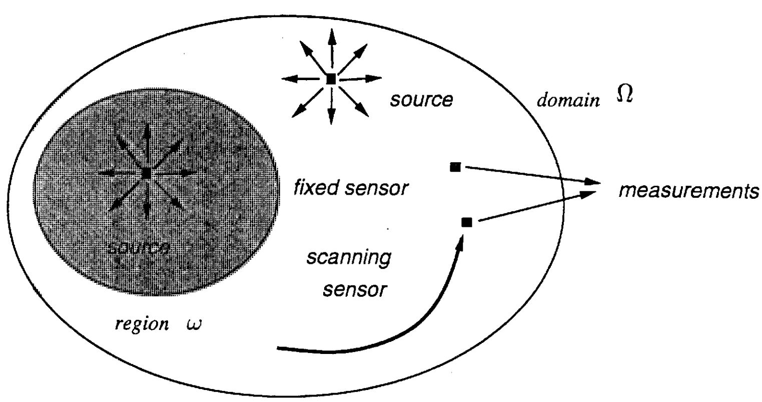
The purpose of this paper is to give general results on regional observation and regional detection of distributed systems. This leads to the so called strategic sensors and spy sensors. Characterizations of such sensors are also given as well as the relationship between these two notions. The scanning sensor case and the minimum time detection problem are also examined. Various examples illustrate the different results and an application to a bidimensional diffusion system is given.

1. Introduction

For a given distributed system, the detectability concept (see El Jai and Afifi, 1994) is defined as the possibility of reconstruction of a source from the knowledge of the output of the system under consideration. In practical situations, the output is given by means of sensors. A sensor which allows a unique reconstruction of the source is said to be a spy sensor. If the source is approximatively located in a subregion ω, then we have a regional detectability problem and a regional spy sensors.



On the other hand, the observability of a system is the possibility of the reconstruction of its state supposed to be unknown. This concept is closer to practical situations when we are only interested in the knowledge of the state in a given subregion of the domain, or if the system is not observable on the whole domain.

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The regional observability was introduced recently by A. El Jai and the results are finer than those on the usual observability concept, (Amouroux *et al.*, to appear; El Jai *et al.*, 1993; 1994). A sensor which allows a unique reconstruction of the state of the system (or on a considered subregion) is said to be strategic (or regionally strategic).

The observability and the detectability are two different concepts and then the notions of (regional) strategic sensors and (regional) spy sensors are also different. In this paper, we show that there are various relations between regional strategic sensors and regional spy sensors.

The next section is devoted to the presentation of the system under consideration and preliminaries. In sections 3 and 4, we study respectively regional observability and regional detectability of the system in the case where the output is given by means of fixed or moving sensors. We also give a characterization of these concepts in term of sensors structures. In section 5, we give the relationship between the two concepts and hence between (regional) strategic sensors and (regional) spy sensors. In the last section an application to the case of a bidimensional diffusion system is considered. Examples on various situations are also given along the paper.

2. The System Under Consideration — Preliminaries

Let Ω be an open bounded subset of $\mathbb{R}^n$ with a sufficiently regular boundary $\Gamma = \partial\Omega$. We consider the system described by the state equation

$$\begin{cases} \dot{x}(t) = \mathcal{A}x(t) + \mathcal{W}e(t), & 0 < t < T \\ x(0) = x_0 \in X \end{cases} \tag{1}$$

where $X = L^2(\Omega)$ is the state space and the operator $\mathcal{A}$ is linear and generates on X a strongly continuous semi-group $(S(t))_{t\geq 0}$ defined by

$$S(t)z = \sum_{n\geq 1} e^{\lambda_n t} \sum_{j=1}^{r_n} \langle z, \phi_{n,j}\rangle_X \; \phi_{n,j} \tag{2}$$

$(\phi_{n,j})_{n,j}$ is an orthonormal basis of eigenfunctions of $\mathcal{A}$ (with a certain boundary conditions) associated to the eigenvalues $(\lambda_n)_n$ with multiplicity r_n,

$$\begin{cases} \mathcal{A}\phi_{n,j} = \lambda_n \phi_{n,j} \\ \|\phi_{n,j}\|_X^2 = 1 \\ n \geq 1, \qquad 1 \leq j \leq r_n \end{cases} \tag{3}$$

We assume that

$$r = \sup_{n\geq 1} r_n < \infty \tag{4}$$

The operator $\mathcal{W}$ characterizes the nature of the source which excites the system and $e(t)$ is related to the intensity of the exciting source and will be defined precisely later. The state of the system $x \in L^2(0,T;V)$ where V is a Hilbert space such that $X \subset V$ with continuous injection and if we identify X to its dual X', we have

$$V' \subset X \subset V \tag{5}$$

with continuous injections. The state regularity depends on the nature of the source and $x(t)$ can be in the Hilbert space V (case of pointwise source for example). The output equation is given by

$$y(t) = \mathcal{C}x(t), \qquad 0 < t < T \tag{6}$$

where Y is the observation Hilbert space, $\mathcal{C} : X \longrightarrow Y$ is linear and $y \in L^2(0,T;Y)$. In concrete situations, the output is given by means of q sensors (see El Jai and Amouroux, 1990; El Jai *et al.*, 1994; El Jai and Pritchard, 1988), in this case we have $Y = \mathbb{R}^q$. Let us recall the following definitions

Definition 1. A sensor is a couple (D, g) where

i) D is the support of the sensor, $D \subset \Omega$,

ii) $g \in L^2(\Omega)$ is the spatial distribution of the sensor on D.

A sensor (D, g) is said to be

i) *zone* if D is a subdomain of Ω,

ii) *pointwise* if D is reduced to a single point, $D = \{c\}$.
In this case, we have

$$g = \delta(. - c) \tag{7}$$

and the sensor will be denoted by (c, δ_c)

iii) *boundary* (zone or pointwise) if $D \subset \Gamma$,

iv) *moving* or scanning (zone, pointwise or boundary) if D depends on time, $D = D(t)$, in this case g also depends on time and $g(t,.)$ is the spatial distribution of the sensor on $D(t)$ at time t.

If the output is given by means of q sensors $(D_i, g_i)_{1 \le i \le q}$, we have

$$\mathcal{C} = \begin{pmatrix} \langle g_1, \cdot \rangle \\ \vdots \\ \langle g_q, \cdot \rangle \end{pmatrix} \tag{8}$$

and in the case of moving sensors, the operator $\mathcal{C} = \mathcal{C}(t)$ depends on time variable.

Let ω be a given subregion of Ω; the present paper is focused on the following problems.

Problem 1. *Regional Observability*

Given the autonomous system associated to (1) and output (6), characterize the sensors which allow the regional observability in ω. This case has been extensively studied particularly by El Jai and Zerrik in (Amouroux *et al.*, to appear; El Jai *et al.*, 1993; 1994). Various results on fixed sensors structure have been established (see El Jai *et al.*, 1994; El Jai *et al.*, 1993) and applied to a thermal process (see Amouroux *et al.*, to appear).

Problem 2. *Regional Detection*

Given the excited system (1) and output (6), characterize the sensors which allow a unique reconstruction (detection) of the source supposed to be located in a given subregion ω. Basic results were established by El Jai and Afifi in (1994).

In this paper we recall the most important of these results and extend them to the case of scanning sensors. We also show the relationship between regional strategic sensors and regional spy sensors. The first problem is a more general situation of the usual state reconstruction, whereas the second problem can be interpreted as an observation problem of an unknown control exciting the system.

3. Regional Observability and Strategic Sensors

In this section we assume that system (1) is autonomous

$$\begin{cases} \dot{x}(t) = Ax(t), & 0 < t < T \\ x(0) = x_0 \end{cases} \tag{9}$$

where x_0 is supposed to be unknown. We have

$$x(t) = S(t)x_0, \qquad 0 \leq t < T$$

and the output equation (6) becomes

$$\begin{cases} y(t) = CS(t)x_0 = K(t)x_0 \\ 0 < t < T \end{cases} \tag{10}$$

where $K : z \in L^2(\Omega) \longrightarrow Kz = CS(.)z \in L^2(0,T;Y)$. The adjoint K^* of K is defined by

$$K^*y = \int_0^T S^*(t)C^*y(t)\,\mathrm{d}t$$

In the following subsections we recall some results on regional observability and strategic sensors and then we give new results on the case of strategic scanning sensors.

3.1. Observability

Let ω be a nonempty subregion of Ω and p the restriction operator to ω

$$p : f \in L^2(\Omega) \longrightarrow f|_\omega \in L^2(\omega) \tag{11}$$

the adjoint of p is $p^* = i$, the natural injection $L^2(\omega) \longrightarrow L^2(\Omega)$.

Definition 2. System (9) with the output (10) is

i) $\omega-$exactly observable (or exactly observable on ω) if

$$\mathrm{Im}(p \circ K^*) = L^2(\omega) \tag{12}$$

ii) $\omega-$weakly observable (or weakly observable on ω) if

$$\mathrm{Ker}(K \circ i) = \{0\} \tag{13}$$

this is equivalent to

$$\overline{\mathrm{Im}(p \circ K^*)} = L^2(\omega) \tag{14}$$

Remark. Definition (2) can be extended to the case where the operator $\mathcal{C}$ depends on time (case of moving sensors). ■

We have the following properties

- The exact observability on $\omega \Longrightarrow$ the weak observability on ω.
- For $\omega_1 \subset \omega_2$ $(\subset \Omega)$ with $\omega_1 \neq \emptyset$, the exact (weak) observability on $\omega_2 \Longrightarrow$ the exact (weak) observability on ω_1.
- The system can be regionally observable but not observable on the whole domain. The following example illustrates this result.

Example 1. Consider the one-dimensional diffusion system defined in $\Omega =]0,1[$ by the equation

$$\left\{ \begin{array}{l} \dfrac{\partial x}{\partial t}(\xi, t) = \dfrac{\partial^2 x}{\partial \xi^2}(\xi, t) \\ x(0,t) = x(1,t) = 0 \\ x(.,0) = x_0 \end{array} \right. \tag{15}$$

and assume that measurements are given by means of a pointwise sensor located in $c \in]0,1[$, hence the output equation is

$$y(t) = x(c,t), \qquad t \in]0,T[\tag{16}$$

The operator $\mathcal{A} = \dfrac{\partial^2}{\partial x^2}$ generates a strongly continuous semi-group defined by

$$S(t)x = \sum_{n \geq 1} e^{\lambda_n t} \langle x, \phi_n \rangle_X \ \phi_n$$

with $\lambda_n = -n^2\pi^2$ and $\phi_n(\xi) = \sqrt{2}\sin(n\pi\xi)$, $\xi \in]0,1[$. If $c \in Q$, system (15)–(16) is not observable on $\Omega =]0,1[$ (see El Jai and Pritchard, 1988). As an example, the initial state $x_0(.) = \sqrt{2}\sin(n\pi.)$ is not observable in $]0,1[$ $(x_0 \in \mathrm{Ker}K)$ but is regionally observable on $\omega = \left[\frac{1}{4}, \frac{3}{4}\right]$ $\left(x_0 \notin \mathrm{Ker}(K \circ i)\right)$. ■

3.2. Strategic Sensors

We suppose that the output is given by q pointwise sensors $(c_i, \delta_{c_i})_{1 \leq i \leq q}$, then (10) becomes

$$y(t) = \begin{pmatrix} x(c_1,t) \\ \vdots \\ x(c_q,t) \end{pmatrix}, \qquad 0 < t < T \tag{17}$$

Definition 3. The sensors $(c_i, \delta_{c_i})_{1 \leq i \leq q}$ are strategic (regionally strategic on ω or ω-strategic) if system (9)–(17) is weakly observable on Ω (on ω).

For $n \geq 1$, let

$$M_n = \begin{pmatrix} \phi_{n,1}(c_1) & \cdots & \phi_{n,r_n}(c_1) \\ \phi_{n,1}(c_2) & \cdots & \phi_{n,r_n}(c_2) \\ \vdots & & \vdots \\ \phi_{n,1}(c_q) & \cdots & \phi_{n,r_n}(c_q) \end{pmatrix} \tag{18}$$

and

$$\gamma_n(\omega) = \begin{pmatrix} \gamma_{n,1}(\omega) \\ \vdots \\ \gamma_{n,r_n}(\omega) \end{pmatrix} \tag{19}$$

with

$$\gamma_{n,j}(\omega) = \Big(\gamma_{n,j\,;\,m,k}(\omega)\Big), \qquad k = 1, r_m; \;\; m \geq 1$$

where

$$\gamma_{n,j\,;\,m,k}(\omega) = \int_\omega \phi_{n,j}(\xi)\phi_{m,k}(\xi)\,\mathrm{d}\xi$$

We have the following result on the characterization of regional strategic sensors, see El Jai *et al.* (1994).

Proposition 1. *The sensors $(c_i, \delta_{c_i})_{1 \leq i \leq q}$ are ω-strategic if and only if*

$$q \geq \sup_{n \geq 1} r_n \tag{20}$$

and

$$\operatorname{rank}\{M_n \gamma_n(\omega)\} = r_n, \qquad \forall n \geq 1 \tag{21}$$

Remark. Definition 3 is the same for zone sensors $(D_i, g_i)_{1 \leq i \leq q}$ and Proposition 1 is also valid with

$$M_n = \begin{pmatrix} \langle g_1, \phi_{n,1}\rangle & \cdots & \langle g_1, \phi_{n,r_n}\rangle \\ \langle g_2, \phi_{n,1}\rangle & \cdots & \langle g_2, \phi_{n,r_n}\rangle \\ \vdots & & \vdots \\ \langle g_q, \phi_{n,1}\rangle & \cdots & \langle g_q, \phi_{n,r_n}\rangle \end{pmatrix} \tag{22}$$

and

$$\langle g_i, \phi_{n,k}\rangle = \int_\omega g_i(\xi)\phi_{n,k}(\xi)\,\mathrm{d}\xi$$

■

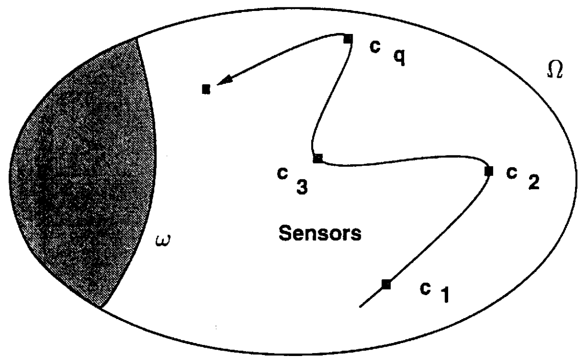
In the next section we extend the previous result to the case of scanning sensor.

3.3. Strategic Scanning Sensor

We suppose that measurements are given by means of a scanning pointwise sensor describing a curve γ where $\gamma = \text{Im}(c)$ with

$$c : \rho \in \mathcal{D} \longrightarrow c(\rho) \in \gamma \tag{23}$$

is given and sufficiently regular.



The output operator becomes

$$\mathcal{C} = \mathcal{C}(\rho) = \delta\left(. - c(\rho)\right), \qquad \rho \in \mathcal{D} \tag{24}$$

then the observation becomes

$$y(\rho, t) = \mathcal{C}(\rho)S(t)x_0 = K(\rho, t)x_0 \tag{25}$$

and the sensor $\left(c(\rho), \delta_{c(\rho)}\right)_{\rho \in \mathcal{D}}$ is ω-strategic if and only if

$$K(\rho, t)x_0 = 0 \ \forall \rho,\ t \implies x_0 = 0 \ \text{ in } \ \omega \tag{26}$$

Proposition 2. (El Jai and Afifi, 1994) *If γ crosses q points $c_1, c_2, \cdots, c_q$ such that the associated pointwise sensors $(c_i, \delta_{c_i})_{1 \leq i \leq q}$ are ω-strategic, then the scanning sensor $\left(c(\rho), \delta_{c(\rho)}\right)_{\rho \in \mathcal{D}}$ is an ω-strategic sensor.*

In the case where γ is parametrized by the time variable t,

$$t \in \mathcal{I} \subset]0, T[\longrightarrow c\left(\rho(t)\right) \in \gamma \tag{27}$$

the observation becomes

$$y(t) = \mathcal{C}\left(\rho(t)\right)S(t)x_0 = K(t)x_0 \tag{28}$$

and the scanning sensor $\left(c\left(\rho(t)\right), \delta_{c(\rho(t))}\right)_{t \in \mathcal{I}}$ is ω-strategic if and only if

$$K(t)x_0 = 0 \ \forall\, t \in \mathcal{I} \implies x_0 = 0 \ \text{in} \ \omega \tag{29}$$

Let $I_1, ..., I_q$ time intervals $\subset \mathcal{I}$ such that $I_i \bigcap I_j = \emptyset$ for $i \neq j$ and $c_j = c(\rho(t))$ on I_j with $c_i \neq c_j$ for $i \neq j$. Then we have the following result.

Proposition 3. *If the values $(c_i)_{1 \leq i \leq q}$ can be associated to q pointwise sensors locations and if the associated sensors $(c_i, \delta_{c_i})_{1 \leq i \leq q}$ are ω-strategic, then the scanning sensor $\Big(c\big(\rho(t)\big), \delta_{c(\rho(t))}\Big)_{t \in \mathcal{I}}$ is ω-strategic.*

Remark. The results can be easily extended to the case of q scanning sensors. ■

4. Detection of Sources and Spy Sensors

The following definitions (sources and detection) are general and do not depend on the linearity of the system.

4.1. Sources

Definition 4. A source s is defined by (Σ, g, I), where

i)
$$\Sigma(.) : t \in I \longrightarrow \Sigma(t) \subset \Omega$$
defines the support of the source at time t and is assumed to be varying in time.

ii)
$$g(t, .) : \xi \in \Sigma(t) \longrightarrow g(t, \xi)$$
defines the intensity of the excitement in ξ at time t.

iii)
$$I = \Big\{t / g(t, .) \neq 0 \text{ on } \Sigma(t)\Big\}$$
is the support of the function g and defines the life duration of the source s.

As we focus our attention on the detection of the source (this can be done separately from its life duration), we consider the source as a couple $s = (\Sigma, g)$. The set of such sources will be denoted $\mathcal{E}$, we have

$$\mathcal{E} \subset \mathcal{F}\Big(0, T; \mathcal{P}(\Omega)\Big) \times \mathcal{F}(0, T; V) \tag{30}$$

where $\mathcal{P}(\Omega)$ is the set of parts of Ω and $\mathcal{F}(0, T; Z)$ is the space of functions f : $(0, T) \longrightarrow Z$ ($Z = \mathcal{P}(\Omega)$ or V). The space $\mathcal{E}$ may be considered as a vector space with convenient addition and scalar product operations.

Remarks

- I is generally connexe (i.e. a time interval). It may happen that I is a union of several intervals, then the detection problem can be studied as if the system were excited by consecutive sources.
- The support of the source may also be non connexe $\Sigma(t) = \bigcup_i \Sigma_i(t)$. In this case the source is said to be a multi-source.

- For $s = (\Sigma, g) \in \mathcal{E}$, one can extend g and Σ as follows

$$g(t,.) = \begin{cases} g(t,.) & \text{on } I \\ 0 & \text{elsewhere} \end{cases}$$

$$\Sigma(t) = \begin{cases} \Sigma(t) & \text{on } I \\ \emptyset & \text{elsewhere} \end{cases} \qquad (31)$$

then $s = (\Sigma, g)$ is well defined on the time interval $]0, T[$. ■

Definition 5. A source s is said to be

i) pointwise if $\Sigma(t)$ is reduced to a single point for all t in I:

$$\Sigma(t) = \{b(t)\}, \quad \forall t \in I$$

It is the case of a moving pointwise source.

ii) zone if $\Sigma(t)$ is a region $\subset \Omega$, for all t in I.
It is the case of a moving zone source.

iii) fixed (or motionless) if Σ does not depend on time,

$$\Sigma(t) = \Sigma_0, \quad \forall t \in I$$

It is the case of a fixed source which may be zone or pointwise.

Remarks

- The case of a pointwise fixed source $\Sigma(t) = \{b\} \in \Omega$, $\forall\, t$, is the most common situation.
- The source may be on the boundary Γ of Ω, in this case we have $\Sigma(t) \subset \Gamma$, $\forall\, t \in I$ and we can define by the same a pointwise, zone, fixed or moving boundary source. ■

Definition 6.

i) The duration of a source is the length $\mu(I)$ of I.

ii) A source is persistent if $\mu(I) > 0$.

iii) A source is intantaneous (or non persistent) if $\mu(I) = 0$. In this case the source can be active only at times t_1, t_2, $\ldots t_N$.

Remark. For a pointwise instantaneous source, we have

$$g(t, \xi) = \delta(t - t_0)\, \delta(\xi - b) \qquad (32)$$

and (32) is considered as a product of generalized functions, see (Colombeau, 1983; 1986; Egorov, 1990). ■

4.2. Detection

In this section we suppose without loss of generality that the system is excited by a pointwise fixed source $s_0 = (b, e)$, then $\mathcal{W} = \delta(. - b)$ and system (1) becomes

$$\begin{cases} \dot{x}(t) = \mathcal{A}x(t) + \delta(. - b)e(t) \\ x(0) = x_0 \end{cases} \tag{33}$$

We assume that the source location is approximatively located in a given subregion ω of Ω. Note that for N sufficiently large, the function e can be written

$$e(t) = \sum_{i=1}^{N} \beta_i \, \chi_{[t_i, t_{i+1}[}(t) \tag{34}$$

with $t_1 = 0 < t_2 < \cdots < t_{N+1} = T$ and $\beta_i \geq 0$; $1 \leq i \leq N$. If the system is observed via q pointwise sensors $(c_i, \delta_{c_i})_{1 \leq i \leq q}$, then the output equation is

$$\tilde{y}(t) = \begin{pmatrix} \tilde{y}_1(t) \\ \vdots \\ \tilde{y}_q(t) \end{pmatrix} \in Y = \mathbb{R}^q \tag{35}$$

where

$$\begin{aligned} \tilde{y}_i(t) = & \sum_{n \geq 1} e^{\lambda_n t} \sum_{j=1}^{r_n} \langle x_0, \phi_{n,j} \rangle \, \phi_{n,j}(c_i) \\ & + \sum_{n \geq 1} \int_0^t e^{\lambda_n (t-\tau)} e(\tau) \, \mathrm{d}\tau \sum_{j=1}^{r_n} \phi_{n,j}(b) \phi_{n,j}(c_i) \end{aligned} \tag{36}$$

Let

$$\mathcal{E}_\omega = \Big\{ (a, f) \in \mathcal{E} \mid a \in \omega \Big\} \tag{37}$$

and

$$Q_\omega : s \in \mathcal{E}_\omega \longrightarrow y \in L^2(0, T; Y) \tag{38}$$

Definition 7. A source s is said to be ω-detectable (detectable in ω) on $]0, T[$ if the knowledge of the system together with the output (35) is sufficient to make the associated operator Q_ω injective.

In this case one has to find a reconstruction operator $R_\omega : L^2(0, T; Y) \longrightarrow \mathcal{E}_\omega$ such that

$$s = R_\omega y \tag{39}$$

and if Q_ω is invertible, then $Q_\omega^{-1} = R_\omega$. The detection of a source will obviously depend on ω and the output operator $\mathcal{C}$ and then on the nature of the sensors. We introduce the following definition:

Definition 8. A spy sensor (ω-spy sensor) is a sensor which allows a unique detection of the source s (in ω).

A spy sensor may be multipoint, zone or scanning. In practice, the reconstruction of all the parameters of a source may be difficult (or impossible). So we can try to detect (or reconstruct) only some of the parameters of the source (location for example). In this case we shall say that the source is ω-partially detectable. The associated sensor will be partially spy. The previous definitions can be extended to the case of zone or moving sources, see El Jai and Afifi (1994).

Remarks

- If $\omega_1 \subset \omega_2 \subset \Omega$, then a source which is ω_2-detectable is ω_1-detectable.
- If a source is ω-detectable, it is partially ω-detectable.
- A source can be regionally or partially detectable, but not detectable. To illustrate this result, we consider the following example. ∎

Example 2. Consider the one-dimensional parabolic system supposed to be disturbed by a pointwise fixed source $s_0 = (b, e)$ which is located in a subregion $\omega \subset]0,1[$,

$$\begin{cases} \dfrac{\partial x}{\partial t}(\xi, t) = \dfrac{\partial^2 x}{\partial x^2}(\xi, t) + \delta(. - b)e(t) \\ x(0,t) \ = x(1,t) = 0 \\ x(.,0) \ = x_0 \end{cases} \tag{40}$$

In this case, we have $V = H^{-1-\varepsilon}(\Omega)$ with $0 < \varepsilon << 1$. We assume that measurements are given by means of a pointwise sensor located in $c \in]0,1[$, hence the output function is

$$y(t) = x(c,t), \qquad t \in]0,T[\tag{41}$$

then we have, for $c = \frac{1}{2}$ and $b \neq \frac{1}{2}$.

i) If $1 - b \notin \omega$, the source is detectable in ω, but not detectable in Ω,

ii) The source is partially detectable (with respect to e), but not detectable.

The detection of an unknown source depends also on the different parameters of the system amongst which is the length of the time interval.

4.3. Minimum Time Detection Problem

This section is devoted to the statement of the minimum time detection problem. It is clear that a source may be detected in a minimum time, that is equivalent to find the minimum time interval which makes the operator Q_ω (38) injective. Suppose without loss of generality that a source $s_0 = (b, e)$ is active on a certain time interval $[0, T_0]$. The choice of T depends on T_0 and one has to consider $T > T_0$ (or $0 < T < T_0$, for large T_0) because if the source is ω-detectable on $]0, \hat{T}[$ then it is ω-detectable on $]0, T[$, for all $T > \hat{T}$. Let

$$\Theta(s_0) = \Big\{ T > 0 \text{ such that } s_0 \text{ is } \omega\text{-detectable on }]0, T[\Big\} \tag{42}$$

For a fixed source s_0, the minimum time problem can be stated in the form

$$\inf_T \Theta(s_0) \tag{43}$$

the solution of which also depends on different parameters as ω, the output operator $\mathcal{C}$ (sensors) and the source location. The case of finite speed propagation systems must be considered with more care. The solution of (43) is denoted by $T^*_\omega(\mathcal{C})$.

Using the previous results, it is easy to show

Proposition 3.

i)
$$T^*_\omega(\mathcal{C}) \geq \tau_0 \tag{44}$$

where τ_0 is defined by

$$\tau_0 = \inf_t \left\{ t > 0 \;/\; \tilde{y}(t) \neq y(t) \right\}$$

and $y(t) = K(t)x_0$ is the observation corresponding to the unexcited system (1) $(e(.) = 0)$ and $\tilde{y}(t)$ is defined in (35).

ii) For fixed ω and $c_1, \cdots, c_q \in \Omega$, let $\mathcal{C}_p$ be the output operator corresponding to the pointwise sensors $(c_i, \delta_{c_i})_{1 \leq i \leq p}$, with $1 \leq p \leq q$. Then if the source is ω-detectable on $]0, T[$ with respect to $\mathcal{C}_p$, it is ω-detectable on $]0, T[$ with respect to $\mathcal{C}_q$, and hence

$$T^*_\omega(\mathcal{C}_q) \leq T^*_\omega(\mathcal{C}_p) \tag{45}$$

iii) For a given output operator $\mathcal{C}$ and $\omega_1 \subset \omega_2 (\subset \Omega)$ with $b \in \omega_1$, then if the source is ω_2-detectable on $]0, T[$, it is also ω_1-detectable on $]0, T[$, then

$$T^*_{\omega_1}(\mathcal{C}) \leq T^*_{\omega_2}(\mathcal{C}) \tag{46}$$

iv) For any subregion ω such that $b \in \omega$, we have

$$T^*_\omega(\mathcal{C}) \longrightarrow \tau_0 \;\; when \;\; \omega \searrow \{b\} \tag{47}$$

Remarks

- In the case of partial detection of the source, let $T^*_\omega(\mathcal{C})_{par}$ be the corresponding minimum time partial detection. Then we have

$$T^*_\omega(\mathcal{C})_{par} \leq T^*_\omega(\mathcal{C}) \tag{48}$$

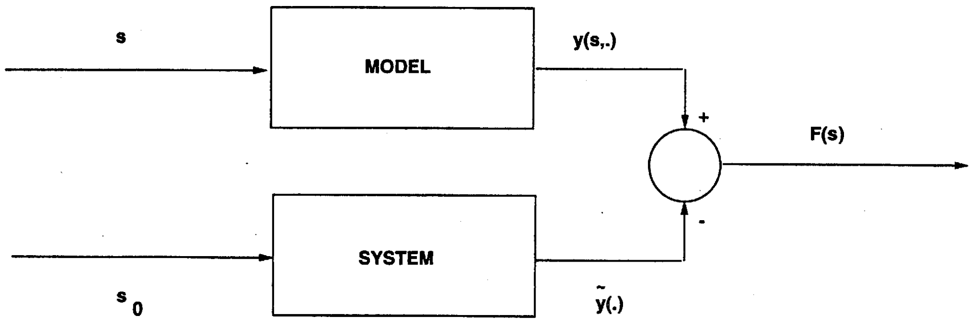
- The results can be extended to zone sources and zone sensors. ■

4.4. Characterization of Spy Sensors

The characterization of regional spy sensors can be stated as a minimization problem. For $s = (a, f) \in \mathcal{E}_\omega$, let

$$F(s) = \|y(s, .) - \tilde{y}\|^2_{L^2(0,T;\mathbb{R}^q)} \tag{49}$$

where $y(s,.)$ is the observation corresponding to the source s and $\tilde{y}$ is given by (35) $\left(\tilde{y}(.) = y(s_0,.)\right)$.



The source detection problem is then equivalent to the minimization problem

$$\begin{cases} \inf\limits_{s} F(s) \\ s \in \mathcal{E}_\omega \end{cases} \tag{50}$$

Let

$$\mathcal{S}_\omega = \left\{ \bar{s} \in \mathcal{E}_\omega \ / \ F(\bar{s}) = \inf_{s \in \mathcal{E}_\omega} F(s) \right\} \tag{51}$$

we have $s_0 \in \mathcal{S}_\omega$ and

Proposition 4. *The sensors* $(c_i, \delta_{c_i})_{1 \le i \le q}$ *are* ω*-spy sensors if and only if*

$$\mathcal{S}_\omega = \{s_0\} \tag{52}$$

Consider the matrix given by (18) and

$$f_n : \xi \in \Omega \longrightarrow \begin{pmatrix} \phi_{n,1}(\xi) \\ \vdots \\ \phi_{n,r_n}(\xi) \end{pmatrix} \in \mathbb{R}^{r_n} \tag{53}$$

then we have the result (see El Jai and Afifi, 1994)

Proposition 5. *Characterization of spy sensors.*
The sensors $(c_i, \delta_{c_i})_{1 \le i \le q}$ *are* ω*-spy sensors if and only if*

$$\begin{cases} \alpha, \beta > 0; \ \ \xi, \mu \in \omega \\ \alpha f_n(\xi) - \beta f_n(\mu) \in \operatorname{Ker} M_n \implies \alpha = \beta \ \text{ and } \ \xi = \mu \\ \forall n \ge 1 \end{cases} \tag{54}$$

Remark. The result is also true for zone sensors with M_n given by (22) and the method considered here can be extended to the zone sources and boundary sensors cases.

■

4.5. Scanning Spy Sensor

We consider the case where the measurements are given by one pointwise moving sensor discribing a parametrized curve γ. We have similar results to those of scanning strategic sensors. Suppose that $\gamma = \mathrm{Im}(c)$ where

$$c : \rho \in \mathcal{D} \longrightarrow c(\rho) \in \gamma$$

and $c(.)$ sufficiently regular.

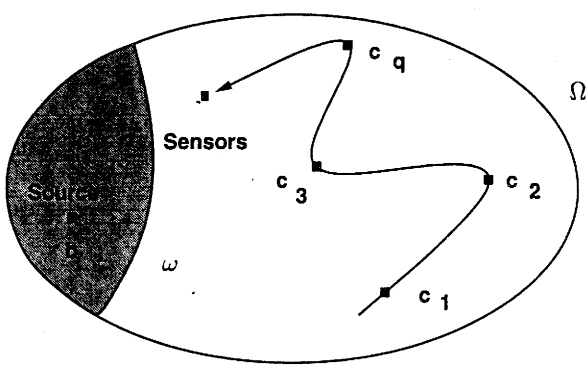
Proposition 6. *If γ crosses q points $c_1, c_2, \cdots, c_q$ such that the $(c_i, \delta_{c_i})_{1 \le i \le q}$ are ω-spy sensors, then the scanning sensor $(c(\rho), \delta_{c(\rho)})_{\rho \in \mathcal{D}}$ is an ω-spy-sensor.*

In the case where γ is parametrized by time variable

$$t \in \mathcal{I} \longrightarrow c(\rho(t)) \in \gamma$$

with $c \circ \rho$ sufficiently regular, then we have the following result

Proposition 7. *Let $I_1, ..., I_q$ be time intervals $\subset \mathcal{I}$ and $c_1, c_2, \cdots, c_q \in \Omega$ such that $I_i \cap I_j = \emptyset$ and $c_i \neq c_j$ for $i \neq j$, with $c(\rho(t)) = c_j$ for $t \in I_j$. Then if $(c_i, \delta_{c_i})_{1 \le i \le q}$ are ω-spy sensors, the scanning sensor $(c(\rho(t)), \delta_{c(\rho(t))})_{t \in \mathcal{I}}$ is an ω-spy sensor.*

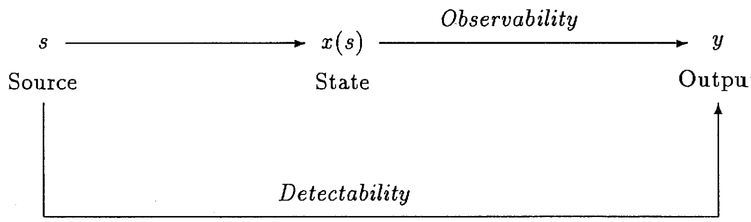


Remarks. In the detectability (or observability) case,

- $\mathcal{I}$ does not necessarily have to be equal to $]0, T[$.
- In the case of q scanning sensors, they are ω-spy sensors (ω-strategic sensors) if the trajectory of each of them crosses a pointwise fixed ω-spy sensor (ω-strategic sensor) location.
- This method can be extended to a moving zone sensor.
- The result also means that q ω-spy sensors (ω-strategic sensors) may be alternatively activated in time and ensure the ω-detectability of the system (this is to be considered when one does not know the minimum number of sensors which can ensure the ω-detection). ■

5. Strategic Sensors and Spy Sensors

The state observation and the detection problem as seen in the previous sections are two concepts which can be represented with the following diagram.



Since $s \longrightarrow x(s)$ is injective, we have the general result

Proposition 8. *If the sensors are ω-strategic, then they are ω-spy sensors.*

The above result is valid for any type of sensors (fixed or moving). The following example shows that the converse is not true.

Example 3. In the case of system (15) with the output (16),

i) if $c \in Q$, it is well known (see El Jai and Pritchard, 1988) that the corresponding sensor (c, δ_c) is not strategic, but for $c \neq \frac{1}{2}$, (c, δ_c) is a spy sensor, see El Jai and Afifi (1994),

ii) for $\omega = [\alpha, \beta]$ and $c \neq \frac{1}{2}$ such that $\dfrac{c-\alpha}{\beta-\alpha} \in Q$, then (c, δ_c) is an ω-spy sensor but not ω-strategic. ■

Remarks

- If the sensors are not ω-spy sensors, one can slightly modify the region ω and make the sensors ω-strategic, hence ω-spy sensors, see (Amouroux *et al.*, to appear; El Jai *et al.*, 1993).
- The results are true in the zone source and zone sensors cases and can be extended to the boundary case.
- A necessary condition for q sensors to be strategic is that

$$q \geq r = \sup_n r_n \tag{55}$$

In the case of spy sensors (55) is not necessary. As an illustrative example we consider the system described in $\Omega =]0,1] \times]0,1[$ by the equation

$$\begin{cases} \dfrac{\partial x}{\partial t}(\xi, t) = \dfrac{\partial^2 x}{\partial \xi^2}(\xi, t) + \delta(. - b)e(t) \\ x|_\Gamma = 0 \\ x(., 0) = x_0 \end{cases} \tag{56}$$

with an output given by q pointwise sensors located in $c_1, \cdots, c_q \in \Omega$. For any integer $q \geq 1$, the sensors $(c_i, \delta_{c_i})_{1 \leq i \leq q}$ can not be strategic (Berrahmoune and El Jai, 1983) but for $q = 2$ and $c_1 = (\frac{1}{2}, \frac{1}{3})$ and $c_2 = (\frac{1}{3}, \frac{1}{2})$, the sensors $(c_i, \delta_{c_i})_{1 \le i \le 2}$ are spy sensors. ■

Example 4. Consider now the excited system (40) with an output given by a pointwise scanning sensor and describing a curve γ parametrized by (23).
Let $\gamma = [a, b]$ with $a < b$, then γ crosses a point c_0 such that the associated sensor (c_0, δ_{c_0}) is strategic, then the sensor $(c(\rho), \delta_{c(\rho)})_{\rho \in \mathcal{D}}$ is strategic and hence a spy sensor.
In the case where γ is parametrized by (27), if $c(\rho(t)) = c_0$ on a subinterval I_0 of $\mathcal{I}$, the scanning sensor $(c(\rho(t)), \delta_{c(\rho(t))})_{t \in \mathcal{I}}$ is strategic, then a spy sensor. ■

6. Application to a Bidimensional Diffusion Process

In various physical problems, symmetry considerations lead to one or two space dimensions. The one-dimension case has been developed through examples and in (El Jai and Afifi, 1994). In the present paper, we consider a more general situation which regards the mathematical development where $\Omega =]0, a_1[\times]0, a_2[$ with $a_1, a_2 > 0$, $\Gamma = \partial\Omega$ and $c \in \Omega$. and the bidimensional system defined in Ω by the diffusion equation

$$\begin{cases} \dfrac{\partial x}{\partial t} = \Delta x + \delta(. - b)e(t) \\ x|_\Gamma = 0 \\ x(.,0) = x_0 \end{cases} \tag{57}$$

with the output

$$y(t) = x(c, t) \tag{58}$$

From the previous characterizations we derive results for (57). We suppose that one unknown pointwise source $s_0 = (b, e)$ is to be detected by means of one pointwise sensor assuming that the excitement is piecewise constant as defined in (34).

In this case, we have $V = H^{-1-\varepsilon}(\Omega)$ with $0 < \varepsilon << 1$ and the eigenvalues and eigenfunctions of the operator Δ with the Dirichlet homogeneous boundary conditions are given by

$$\begin{cases} \lambda_{m,n} = -\left(\dfrac{m^2}{a_1^2} + \dfrac{n^2}{a_2^2}\right)\pi^2 \\ \phi_{m,n}(\xi, \zeta) = \dfrac{2}{\sqrt{a_1 a_2}} \sin\left(m\pi\dfrac{\xi}{a_1}\right) \sin\left(n\pi\dfrac{\zeta}{a_2}\right) \end{cases}$$

Notice that $r = \sup r_{m,n}$, where $r_{m,n}$ is the multiplicity of $\lambda_{m,n}$, can be infinite (case where Ω is a square).

6.1. Case of a Fixed Sensor

In this section we assume that the output is given by a pointwise fixed sensor and that $\dfrac{a_1^2}{a_2^2} \notin Q$, then the multiplicity of the eigenvalues is $r_{m,n} = 1, \ \forall m, n \ge 1$.

Let $b = (b_1, b_2)$, $c = (c_1, c_2)$ and

$$B_1 = (b_1, b_2), \qquad B_2 = (b_1, a_2 - b_2)$$
$$B_3 = (a_1 - b_1, b_2), \qquad B_4 = (a_1 - b_1, a_2 - b_2)$$

Moreover, if we suppose that the source is located in a given subregion $\omega = [\alpha_1, \alpha_2] \times [\beta_1, \beta_2] \subset \Omega$ such that $b \in \omega$, then we have the following results.

Proposition 9.

a) The sensor (c, δ_c) is ω-strategic if and only if

$$\frac{1}{a_1} \frac{c_1 - \alpha_1}{\alpha_2 - \alpha_1} \notin Q \quad and \quad \frac{1}{a_2} \frac{c_2 - \beta_1}{\beta_2 - \beta_1} \notin Q$$

then (c, δ_c) is an ω-spy sensor.

b) If $\frac{1}{a_1} \frac{c_1 - \alpha_1}{\alpha_2 - \alpha_1} \in Q$ or $\frac{1}{a_2} \frac{c_2 - \beta_1}{\beta_2 - \beta_1} \in Q$, then the sensor (c, δ_c) is not ω-strategic but is a partial spy sensor with respect to e.

$$\mathcal{S}_\Omega \subset \Big\{(B_1, e), (B_2, e), (B_3, e), (B_4, e)\Big\}$$

and

$$\mathcal{S}_\omega \subset \Big\{(B_i, e) \;/\; B_i \in \omega\Big\}$$

i) If $c_1 \neq \frac{a_1}{2}$ and $c_2 \neq \frac{a_2}{2}$ the sensor (c, δ_c) is a spy sensor, and hence an ω-spy sensor,

ii) If $c_1 = \frac{a_1}{2}$ and $c_2 \neq \frac{a_2}{2}$, then

- *if $b_1 = \frac{a_1}{2}$ ($B_1 = B_3$), the sensor (c, δ_c) is a spy sensor ,*
- *if $b_1 \neq \frac{a_1}{2}$, (c, δ_c) is not a spy sensor, and if $B_3 \notin \omega$, it is an ω-spy sensor.*

iii) If $c_1 \neq \frac{a_1}{2}$ and $c_2 = \frac{a_2}{2}$, the result is analogous to the previous case.

iv) If $c = \left(\frac{a_1}{2}, \frac{a_2}{2}\right)$, then $\mathcal{S}_\Omega = \Big\{(B_i, e),\; 1 \leq i \leq 4\Big\}$ and hence

- *(c, δ_c) is a spy sensor $\Longleftrightarrow$ $b = c$*
- *(c, δ_c) is an ω-spy sensor $\Longleftrightarrow$ $\mathcal{S}_\omega = \Big\{(b, e)\Big\}$.*

In all the cases the sensor (c, δ_c) is a partial spy sensor (with respect to e), that is to say that it can detect the intensity e of the source. Notice that,

— If $b_1 = \frac{a_1}{2}$, then $B_1 = B_3$ and $B_2 = B_4$,

— If $b_2 = \frac{a_2}{2}$, then $B_1 = B_2$ and $B_3 = B_4$,

— If $b_1 = \frac{a_1}{2}$ and $b_2 = \frac{a_2}{2}$, then $B_1 = B_2 = B_3 = B_4$.

Remarks

- Suppose that the source location $b \in \omega \subset]0, \frac{a_1}{2}[\times]0, \frac{a_2}{2}[$, then the sensor (c, δ_c) is an ω-spy sensor.
- If $c_1 = \frac{a_1}{2}$, $c_2 \neq \frac{a_2}{2}$ and $b \in \omega \subset]0, \frac{a_1}{2}[\times]0, a_2[$, then the sensor (c, δ_c) is an ω-spy sensor.
- The result is analogous for $c_1 \neq \frac{a_1}{2}$ and $c_2 = \frac{a_2}{2}$. ■

Case of a square domain. In the case where the domain Ω is a square $(a_1 = a_2)$, it is known that any finite number of sensors cannot be strategic (see Curtain and Pritchard, 1978; El Jai and El Yacoubi, 1993; El Jai and Pritchard, 1988). For the detection problem, it is easy to show the following result.

Proposition 10. *The couple of sensors located in* $\left(\frac{a_1}{2}, \alpha\right)$ *and* $\left(\beta, \frac{a_1}{2}\right)$, *with* $\alpha \neq \frac{a_1}{2}$ *and* $\beta \neq \frac{a_1}{2}$, *are spy sensors.*

Remark. In some specific situations (depending on the subregion ω) even one sensor may be a ω-spy sensor. ■

Case of a finite order approximation. In practice we consider an approximation of the problem using truncations depending on the nature of the problem. Consider the case where $\omega = \Omega$ and a truncation up to the order N, then the condition $\frac{c_i}{a_i} \notin Q$ for $i = 1, 2$ becomes

$$\frac{c_i}{a_i} \notin F_N \equiv \left\{\frac{k}{n}, \quad 1 \leq k < n \leq N\right\}$$

and one has to avoid a reduced number of points. For example if $N = 3$, we have (see Fig. 1)

$$F_N = \left\{\frac{1}{3}, \frac{1}{2}, \frac{2}{3}\right\}$$

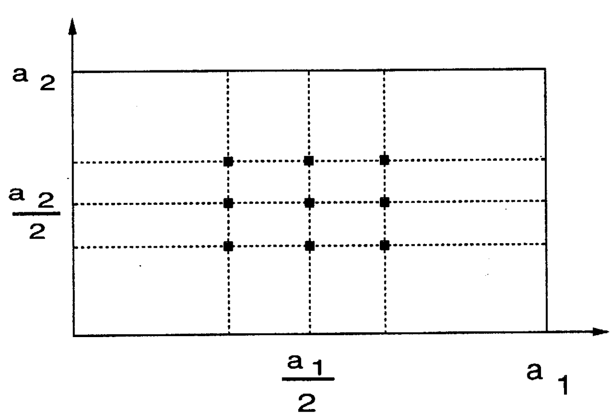


Fig. 1.

6.2. Case of a Scanning Sensor

In this section we suppose that the output is given by a pointwise scanning sensor describing a curve γ parametrized by (23) or (27), we have the following results

Corollary 1. *Suppose that* $a_1 \neq a_2$ *with* $\frac{a_1^2}{a_2^2} \notin Q$, *then*

i) If γ *is parametrized by (23) and crosses a point* $c_0 = (c_0^1, c_0^2)$ *such that*

$$\frac{c_0^i}{a_i} \notin Q, \qquad i = 1, 2 \tag{59}$$

then $(c(\rho), \delta_{c(\rho)})_{\rho \in \mathcal{D}}$ *is strategic and hence a spy sensor,*

ii) If γ *is parametrized by (27), and there exists* $c_0 = (c_0^1, c_0^2)$ *verifying (59) and a subinterval* I_0 *of* $\mathcal{I}$ *such that* $c(\rho(t)) = c_0$ *on* I_0, *then the scanning sensor* $\left(c(\rho(t)), \delta_{c(\rho(t))}\right)_{t \in \mathcal{I}}$ *is strategic and hence a spy sensor.*

In the case where the domain is a square $(a_1 = a_2)$, we have the following result

Corollary 2.

i) If γ *is parametrized by (23) and crosses two points* $c_1 = \left(\frac{a_1}{2}, \alpha\right)$ *and* $c_2 = \left(\beta, \frac{a_1}{2}\right)$ *with* $\alpha \neq \frac{a_1}{2}$ *and* $\beta \neq \frac{a_1}{2}$, *the sensor* $\left(c(\rho), \delta_{c(\rho)}\right)_{\rho \in \mathcal{D}}$ *is a spy sensor.*

ii) If γ *is parametrized by (27) and if there exist two time intervals* $I_1, I_2 \subset \mathcal{I}$ *and* c_1, c_2 *as defined in i) such that* $\left(c(\rho(t)\right) = c_i$ *on* I_i *for* $i = 1$ *or* 2, *then the scanning sensor* $\left(c(\rho(t)), \delta_{c(\rho(t))}\right)_{t \in \mathcal{I}}$ *is a spy sensor.*

In the case of a finite order approximation, we have

i) For a rectangular domain, if the trajectory [1] $\gamma = \mathrm{Im}(c)$ of the sensor is such that
- $\mathrm{Im}\left(\frac{c_i(.)}{a_i}\right)$ is not reduced to a single point of F_N, for [2] $i = 1$ or 2, the scanning sensor is strategic and then is a spy sensor (see Fig. 2 in the case where $N = 3$).
- $\mathrm{Im}\left(\frac{c_i(.)}{a_i}\right)$ is not reduced to $\frac{1}{2}$ for $i = 1$ or 2, then the scanning sensor is a spy sensor (see Fig. 3 for $N = 3$).

ii) In the case of a square domain, if γ crosses the axes $x = \frac{a_1}{2}$ and $y = \frac{a_1}{2}$ at two points different from $\left(\frac{a_1}{2}, \frac{a_1}{2}\right)$, then the scanning sensor is a spy sensor (see Fig. 4).

iii) Similar results can be achieved in the case where Ω is a disk.

[1] γ parametrized by (23) or (27)
[2] $c(.) = (c_1(.), c_2(.))$

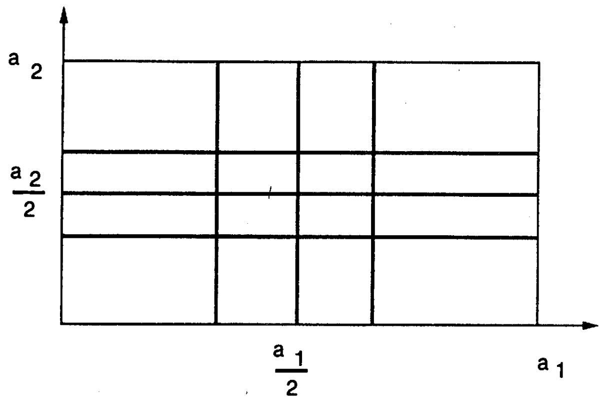


Fig. 2.

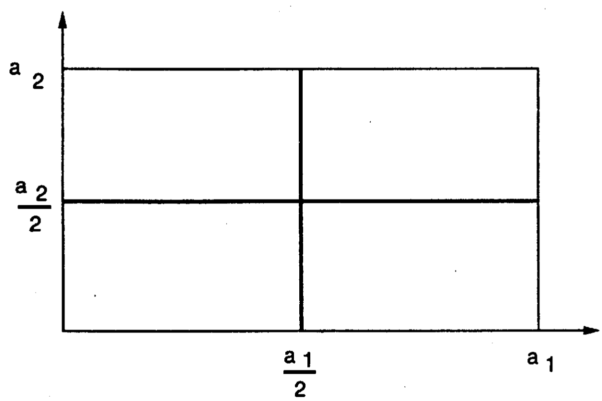


Fig. 3.

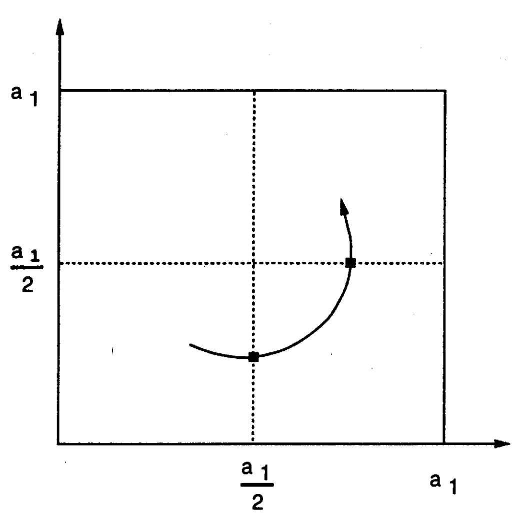


Fig. 4.

7. Conclusion

In this paper we give the principal results on the regional observability and detectability of a class of distributed systems, as well as the characterization of regional strategic sensors and spy sensors. We show the difference between the two concepts and we compare the results obtained in each of the cases. The detection problem naturally leads to the minimum time problem which was examined. Examples on various situations are given.

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