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ON LIMITING FORMS OF THE RELIABILITY FUNCTIONS SEQUENCE OF THE PARALLEL-SERIES SYSTEM

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In the investigation of large scale systems the problem of the complexity of their reliability functions appears. This problem may be approximately solved on the assumption that the number of the system components tends to infinity and by finding the limit reliability function of this system. In this paper a ten-element closed class of limit reliability functions for parallel-series system with dissimilar components is given. This system is such that at least the number of its series or the number of its parallel components tends to infinity. The result is obtained on the assumption that lifetimes of the particular components are independent random variables. The fixed class is more extensive than up to now known class of limit distributions of maximin statistics of independent random variables with the same distributions. The result can be useful in the reliability evaluation of large systems.

1. Introduction

The results obtained in (Chernoff and Teicher, 1965) and also presented in (Barlow and Proschan, 1975) allow us to state that the only possible limit reliability functions of the parallel-series system with independent, identical components and equal numbers of series and parallel components are: $\mathbb{R}_1(x) = \exp(-(-x)^{-\alpha})$ for $x < 0$, where $\alpha > 0$, $\mathbb{R}_2(x) = \exp(-x^{-\alpha})$ for $x > 0$, where $\alpha > 0$ and $\mathbb{R}_3(x) = \exp(-\exp x)$ for $x \in (-\infty, \infty)$.

In a natural way the problem of the existence of limit reliability functions for the parallel-series system with not equal numbers of series and parallel components and not identical components arises. This problem is solved for the systems with identical components in (Kołowrocki, 1993a; 1994a; 1994c) and summarized in (Kołowrocki, 1994b). This paper is an effort to widen and to summarize the results given in (Kołowrocki, 1994d; 1994e; 1994f; 1995) and to put an end to the current state of the problem of the existence of limit reliability functions for any parallel- series system with unalike components.

2. Essential Notions and Theorems

Let E_{ij}, where $i = 1, 2, ..., k$, $j = 1, 2, ...l$, be components of a parallel-series system S and X_{ij} be independent random variables representing the lifetimes of the components E_{ij}.

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Definition 1. A system S is called *non-homogeneous* if the set

$$\left\{\{X_{ij} : j = 1, 2, ..., l\} : i = 1, 2, ..., k\right\}$$

consists of a, $1 \leq a \leq k$, kinds of subsets of random variables and the frequency of the ν-th kind subset is equal to q_ν, where $\sum_{i=1}^{a} q_\nu = 1$ and $q_\nu > 0$. Besides, the ν-th kind subset consists of e_ν, $1 \leq e_\nu \leq l$, kinds of random variables with distribution functions $F^{(\nu,v)}(x) = 1 - R^{(\nu,v)}(x)$, $v = 1, 2, ..., e_\nu$, where $R^{(\nu,v)}(x)$ is a reliability function, and the frequency of the v-th kind random variable in this subset is equal to $p_{\nu v}$, where $\sum_{v=1}^{e_\nu} p_{\nu v} = 1$ and $p_{\nu v} > 0$.

The above definition means that the system is composed of different kinds of subsystems. The frequency q_ν means that there are $q_\nu k$ parallel-series subsystems of the ν-th kind in the system. The frequency $p_{\nu v}$ means that there are $p_{\nu v} l$ identical components in each parallel system of the v-th kind of the ν-th subsystem.

Aassuming in Definition 1 $k = k_n$ and $l = l_n$, where n tends to infinity and k_n and l_n are sequences of natural numbers such that at least one of them tends to infinity, we obtain sequences of the non-homogeneous systems corresponding to the sequence (k_n, l_n). Next, replacing n by a positive real number t and assuming that k_t and l_t are positive real numbers, we obtain families of the non-homogeneous systems corresponding to the pair (k_t, l_t). For these families of systems there exist families of reliability functions.

It is easy to prove that the family of reliability of the non-homogeneous parallel-series systems is given by

$$\mathbb{R}_t(x) = \prod_{i=1}^{a}\left[1 - \left(F^{(i)}(x)\right)^{l_t}\right]^{q_i k_t}, \quad x \in (-\infty, \infty), \ t \in (0, \infty) \tag{1}$$

where

$$F^{(i)}(x) = \prod_{i=1}^{e_i}\left(F^{(i,j)}(x)\right)^{p_{ij}}, \qquad i = 1, 2, ..., a \tag{2}$$

We assume that the lifetime distributions do not necessarily have to be concentrated on the interval $\langle 0, \infty)$ and that a reliability function $R(x)$ is non-increasing, right-continuous, $R(-\infty) = 1$ and $R(+\infty) = 0$.

We shall investigate limit distributions of a standarized random variable $(X - b_t)/a_t$, where X is the lifetime of the non-homogeneous parallel-series system and $a_t = a(t) > 0$, $b_t = b(t) \in (-\infty, \infty)$ are some suitably chosen functions. Apart from that, since

$$P\left(\frac{X - b_t}{a_t} > x\right) = P(X > a_t x + b_t) = \mathbb{R}_t(a_t x + b_t)$$

we assume the following definition.

Definition 2. A reliability function $\mathbb{R}(x)$ is called an *asymptotic reliability function* of the non-homogeneous parallel-series system if there exist functions $a_t > 0$ and $b_t \in (-\infty, \infty)$ such that

$$\lim_{t \to \infty} \mathbb{R}_t(a_t x + b_t) = \mathbb{R}(x) \ \text{ for } \ x \in C_{\mathbb{R}}$$

where $C_{\mathbb{R}}$ is the set of continuity points of $\mathbb{R}(x)$. A pair (a_t, b_t) is called a norming functions pair.

In the whole paper, we assume the following notations: $x(t) \ll y(t)$, where $x(t)$ and $y(t)$ are positive functions, such that $x(t)$ is of order much less than $y(t)$; $x(t) \sim y(t)$, where $x(t)$ and $y(t)$ are positive or negative functions, such that $x(t)$ is of order $y(t)$; $x(t) \gg y(t)$, where $x(t)$ and $y(t)$ are positive functions, such that $x(t)$ is of much greater order than $y(t)$.

Lemma 1. *If*

(i) $\mathbb{R}(x) = \exp(-V(x))$ *is a reliability function,*

(ii) *the family* $\mathbb{R}_t(x)$ *is given by (1),*

(iii) $\lim_{t\to\infty} k_t = \infty$,

(iv) $a_t > 0$, $b_t \in (-\infty, \infty)$ *are some functions,*

then the assertion

$$\lim_{t\to\infty} \mathbb{R}_t(a_t x + b_t) = \mathbb{R}(x) \ \ for \ \ x \in C_{\mathbb{R}} \tag{3}$$

is equivalent to the assertion

$$\lim_{t\to\infty} k_t \sum_{i=1}^{a} q_i \left(F^{(i)}(a_t x + b_t)\right)^{l_t} = V(x) \ \ for \ \ x \in C_V \tag{4}$$

where C_V *is the set of continuity points of* $V(x)$ *and points such that* $V(x) = \infty$.

Proof. Let

$$\overline{\mathbb{R}}_t(x) = 1 - \prod_{i=1}^{a} \left[1 - \left(\widetilde{R}^{(i)}(x)\right)^{l_t}\right]^{q_i k_t}$$

for $x \in (-\infty, \infty)$, $t \in (0, \infty)$, be a reliability function family such that for $i = 1, 2, ..., a$

$$\widetilde{R}^{(i)}(x) = \prod_{i=1}^{e_i} \left(\widetilde{R}^{(i,j)}(x)\right)^{p_{ij}}, \qquad x \in (-\infty, \infty) \tag{5}$$

and

$$\widetilde{R}^{(i,j)}(x) = 1 - R^{(i,j)}(-x) = F^{(i,j)}(-x), \quad x \in C_{R^{(i,j)}} \tag{6}$$

Then, from Lemma 2 in (Kołowrocki, 1993b; 1994d), it follows that conditions (3) and

$$\lim_{t\to\infty} \overline{\mathbb{R}}_t(a_t x - b_t) = \overline{\mathbb{R}}(x) \ \text{ for } \ x \in C_{\overline{\mathbb{R}}} \tag{7}$$

where

$$\overline{\mathbb{R}}(x) = 1 - \exp(-\overline{V}(x))$$

are equivalent and

$$\mathbb{R}(x) = 1 - \overline{\mathbb{R}}(-x) \text{ for } x \in C_{\overline{\mathbb{R}}}$$

which means

$$V(x) = \overline{V}(-x) \text{ for } x \in C_{\overline{V}} \tag{8}$$

Moreover, by Lemma 1 (Kołowrocki, 1993b; 1994d), assertion (7) is equivalent to the assertion

$$\lim_{t\to\infty} k_t \sum_{i=1}^{a} q_i \left(\widetilde{R}^{(i)}(-a_t x - b_t)\right)^{l_t} = \overline{V}(-x) \text{ for } x \in C_{\overline{V}} \tag{9}$$

From eqns. (5) and (6) we have

$$\widetilde{R}^{(i)}(x) = \prod_{i=1}^{e_i} F^{(i,j)}(-x) = F^{(i)}(-x)$$

and then from (9) and (8), we get (4). This completes the proof. ■

3. Asymptotic Reliability Functions of a Homogeneous Parallel-Series System

A problem of the assignation of a closed class of the possible non-degenerate asymptotic reliability functions for the homogeneous parallel-series system is solved in (Kołowrocki, 1993a; 1994a; 1994c) and summarized in (Kołowrocki, 1994b). Before we formulate the main result of these papers, we define a function $A(t)$ for $t \in (0, \infty)$, such that

$$A(t) \sim \prod_{i=1}^{n} f_i\left(\rho(t)\right)$$

where $f_i(t)$ for $i = 1, 2, ..., n$, is the superposition of the function $\ln(t)$ taken i times, $\rho(t)$ is a positive function and n is such that

$$f_{n+1}\left(\rho(t)\right) \ll A \ln\left(\ln(t)\right), \quad A > 0$$

Theorem 1.

1° *If*

$$k_t = t, \quad l_t = c(\ln t)^{\rho(t)}, \quad t \in (0, \infty), \quad c > 0$$

where

$$\ln(\ln t) \ll |l_t - c \ln t| \quad and \quad \rho(t) \ll (\ln t)^{\lambda}$$

for every $\lambda > 0$ *and*

$$|\rho(\tau_\nu) - \rho(t) \lesssim \frac{\delta \ln \nu}{\ln t[\ln(\ln t)]}$$

for every natural $\nu \geq 2$, *where* $0 < \delta \neq 1$ *and* $\tau_\nu = \tau_\nu(t)$, $t \in (0, \infty)$, *is given by*

$$\frac{\tau_\nu}{t} = \nu^{\frac{1}{1-\rho(t)}}$$

or

$$s \ll |l_t - c\ln t| \lessapprox C\ln(\ln t), \quad s > 0, \ C > 0$$

and

$$|\rho(\tau_\nu) - \rho(t)| \sim \frac{\delta \ln \nu}{\ln t[\ln(\ln t)]}$$

for every natural $\nu \geq 2$, *where* $\delta > 0$ *and* $\tau_\nu = \tau_\nu(t)$, $t \in (0, \infty)$, *is given by*

$$\frac{\tau_\nu}{t} = \nu^{\frac{1}{[1-\rho(t)]\ln(\ln t)}}$$

or

$$\rho(t) \gtrapprox (\ln t)^\lambda, \qquad \lambda > 0$$

and

$$|\rho(\tau_\nu) - \rho(t)| \sim \frac{\delta \ln \nu}{\ln t[\ln(\ln t)]}$$

for every natural $\nu \geq 2$, *where* $\delta > 0$ *and* $\tau_\nu = \tau_\nu(t)$, $t \in (0, \infty)$, *is given by*

$$\frac{\tau_\nu}{t} = \nu^{\frac{1}{[1-\rho(t)]A(t)}}$$

then the only possible non-degenerate asymptotic reliability functions of the homogeneous parallel-series system are those of the following types

$$\mathbb{R}_1(x) = \begin{cases} \exp(-(-x)^{-\alpha}) & for \quad x < 0, \quad where \ \alpha > 0 \\ 0 & for \quad x \geq 0 \end{cases}$$

$$\mathbb{R}_2(x) = \begin{cases} 1 & for \quad x < 0 \\ \exp(-x^{-\alpha}) & for \quad x \geq 0, \quad where \ \alpha > 0 \end{cases}$$

and

$$\mathbb{R}_3(x) = \exp(-\exp x) \qquad for \quad x \in (-\infty, \infty)$$

2° *If*

$$k_t = t, \quad l_t - c\ln t \sim s, \quad s \in (-\infty, \infty), \quad c > 0, \quad t \in (0, \infty)$$

then the only possible non-degenerate asymptotic reliability functions of the homogeneous parallel-series system are those of the following types

$$\mathbb{R}_4(x) = \begin{cases} \exp\left(-\exp\left(-(-x)^\alpha - \frac{s}{c}\right)\right) & for \quad x < 0 \\ 0 & for \quad x \geq 0 \end{cases}$$

$$\mathbb{R}_5(x) = \begin{cases} 1 & for \quad x < 0 \\ \exp\left(-\exp\left(x^\alpha - \frac{s}{c}\right)\right) & for \quad x \geq 0 \end{cases}$$

$$\mathbb{R}_6(x) = \begin{cases} \exp\left(-\exp\left(-(-x)^\alpha - \frac{s}{c}\right)\right) & for \quad x < 0 \\ \exp\left(-\exp\left(\beta x^\alpha - \frac{s}{c}\right)\right) & for \quad x \geq 0 \end{cases}$$

where $\alpha > 0$, $\beta > 0$, *and*

$$\mathbb{R}_7(x) = \begin{cases} 1 & for \quad x < x_1 \\ \exp\left(-\exp(-\frac{s}{c})\right) & for \quad x_1 \leq x < x_2 \\ 0 & for \quad x \geq x_2 \end{cases}$$

where $x_1 < x_2$.

3° *If*

$$\lim_{t\to\infty} k_t = k, \qquad \lim_{t\to\infty} l_t = \infty$$

then the only possible non-degenerate asymptotic reliability functions of the homogeneous parallel-series system are those of the following types

$$\mathbb{R}_8(x) = \begin{cases} 1 & for \quad x \leq 0 \\ \left[1 - \exp(-x^{-\alpha})\right]^k & for \quad x > 0, \quad where \quad \alpha > 0 \end{cases}$$

$$\mathbb{R}_9(x) = \begin{cases} \left[1 - \exp\left(-(-x)^{-\alpha}\right)\right]^k & for \quad x < 0, \quad where \quad \alpha > 0 \\ 0 & for \quad x \geq 0 \end{cases}$$

and

$$\mathbb{R}_{10}(x) = \left[1 - \exp\left(-\exp(-x)\right)\right]^k \quad for \quad x \in (-\infty, \infty)$$

4. A Remark on Singular Limits

From (Kołowrocki, 1993b) it follows that the reliability functions family of the homogeneous parallel-series system may be convergent to some non-degenerate non-increasing functions which are not reliability functions. We call these functions singular limits. From (Kołowrocki, 1993b) we also know that, on the assumptions of Theorem 1, the only types of singular limits are as follows:

$$\mathbb{S}_1(x) = 0 \quad for \quad x \in (-\infty, \infty)$$

$$\mathbb{S}_2(x) = 1 \quad for \quad x \in (-\infty, \infty)$$

$$\mathbb{S}_3(x) = \exp(-1) \quad for \quad x \in (-\infty, \infty)$$

$$\mathbb{S}_4(x) = \begin{cases} 1 & for \quad x < 0 \\ \exp(-1) & for \quad x \geq 0 \end{cases}$$

$$\mathbf{S}_5(x) = \begin{cases} \exp(-1) & \text{for } x < 0 \\ 0 & \text{for } x \geq 0 \end{cases}$$

$$\mathbf{S}_6(x) = \begin{cases} 1 & \text{for } x \leq 0 \\ \exp\Big(- \exp(x^{-\alpha})\Big) & \text{for } x > 0, \text{ where } \alpha > 0 \end{cases}$$

$$\mathbf{S}_7(x) = \begin{cases} \exp\Big(- \exp\Big(- (-x)^{-\alpha}\Big)\Big) & \text{for } x < 0, \text{ where } \alpha > 0 \\ 0 & \text{for } x \leq 0 \end{cases}$$

$$\mathbf{S}_8(x) = \exp\Big(- \exp(\exp x)\Big) \qquad \text{for } x \in (-\infty, \infty)$$

and

$$\mathbf{S}_9(x) = \exp\Big(- \exp\Big(- \exp(-x)\Big)\Big) \qquad \text{for } x \in (-\infty, \infty)$$

5. Asymptotic Reliability Functions of a Non-homogeneous Parallel-Series System

In order to transfer the results concerning the homogeneous parallel-series system to the non-homogeneous system we shall use the following two practically important lemmas.

Lemma 2. *If*

(i) $\mathbb{R}(x) = \exp(-V(x))$ *is a non-degenerate reliability function,*

(ii) *the family* $\mathbb{R}_t(x)$ *is given by (1),*

(iii) $\lim_{t\to\infty} k_t = \infty$,

(iv) $a_t > 0,\ b_t \in (-\infty, \infty)$ *are some functions,*

(v) $F(x)$ *is one of the distribution functions given by (2) such that there exists* $T_1 > 0$ *such that for* $t > T_1$,

$$F^{(i)}(a_t x + b_t) \leq F(a_t x + b_t) \tag{10}$$

for all $x \in (-\infty, \infty)$ *and* $i = 1, 2, ..., a$ *and*

$$d(x) = \begin{cases} 0 & \text{for } x < x_0 \\ \lim\limits_{t\to\infty} \sum\limits_{i=1}^{a} q_i d_i(a_t x + b_t) & \text{for } x \geq x_0 \end{cases} \tag{11}$$

where

$$d_i(a_t x + b_t) = \left(\frac{F^{(i)}(a_t x + b_t)}{F(a_t x + b_t)} \right)^{l_t} \tag{12}$$

and $x_0 \in \langle -\infty, \infty)$ *is such a point that there exists* $T_2 > 0$ *such that for* $t > T_2$

$$F(a_tx+b_t)\begin{cases} =0 & for \quad x<x_0 \\ \neq 0 & for \quad x \geq x_0 \end{cases} \tag{13}$$

then the assertion

$$\lim_{t\to\infty} \mathbb{R}_t(a_tx+b_t)=\mathbb{R}(x) \quad for \quad x\in C_{\mathbb{R}} \tag{14}$$

is equivalent to the assertion

$$\lim_{t\to\infty} k_t\Big(F(a_tx+b_t)\Big)^{l_t} d(x)=V(x) \quad for \quad x\in C_V \tag{15}$$

where $V(x)$ *is a non-degenerate function. Moreover, the family* $[k_t(F(a_tx+b_t))^{l_t}]$ *is convergent to a non-degenerate function* $\overline{V}_0(x)$ *and*

$$\mathbb{R}(x)=\exp\Big(-d(x)V_0(x)\Big) \tag{16}$$

Proof. By Lemma 1, condition (14) holds if and only if

$$\lim_{t\to\infty} k_t\sum_{i=1}^{a} q_i\Big(F^{(i)}(a_tx+b_t)\Big)^{l_t}=V(x) \text{ for } x\in C_V$$

Hence

$$\lim_{t\to\infty} k_t\Big(F(a_tx+b_t)\Big)^{l_t}\sum_{i=1}^{a} q_i\left(\frac{F^{(i)}(a_tx+b_t)}{F(a_t+b_t)}\right)^{l_t}=V(x)$$

for $x\in C_V$ and $x\geq x_0$ and by (10), (13) and (12)

$$\mathbb{R}(x)=1 \quad \text{for} \quad x<x_0.$$

The last result and (11) mean that assertions (14) and (15) are equivalent. Moreover, since according to (10) and (11), for sufficiently large t

$$0\leq d(x)\leq 1 \quad \text{for} \quad x\in(-\infty,\infty) \tag{17}$$

then if the family $[k_t(F(a_tx+b_t))^{l_t}]$ is convergent to a degenerate function (Kołowrocki, 1994b), then, by (15) and (17), $V(x)$ is degenerate. But this is inconsistent with the assumption that $V(x)$ is non-degenerate. Therefore, this family is convergent to a non-degenerate function $V_0(x)$ and $\mathbb{R}(x)$ is given by (16). ■

Lemma 3. *If*

(i) $\mathbb{R}(x)=\exp(-V(x))$ *is a non-degenerate reliability function,*

(ii) *the family* $\mathbb{R}(x)$ *is given by (1),*

(iii) $\lim_{t\to\infty} k_t=k$, $\lim_{t\to\infty} l_t=\infty$,

(iv) $a_t>0$, $b_t\in(-\infty,\infty)$ *are some functions,*

(v) $F(x)$ *is one of the distribution functions given by (2) such that there exists* $T_1>0$ *such that for* $t>T_1$

$$F^{(i)}(a_tx+b_t)\leq F(a_tx+b_t) \tag{18}$$

for all $x \in (-\infty, \infty)$ *and* $i = 1, 2, ..., a$ *and*

$$d_i(x) = \begin{cases} 0 & for \quad x < x_0 \\ \lim\limits_{t\to\infty} d_i(a_t x + b_t) & for \quad x \geq x_0 \end{cases} \tag{19}$$

where

$$d_i(a_t x + b_t) = \left(\frac{F^{(i)}(a_t x + b_t)}{F(a_t x + b_t)}\right)^{l_t} \tag{20}$$

and $x_0 \in \langle -\infty, \infty)$ *is such a point that there exists* $T_2 > 0$ *such that for* $t > T_2$

$$F(a_t x + b_t) \begin{cases} = 0 & for \quad x < x_0 \\ \neq 0 & for \quad x \geq x_0. \end{cases} \tag{21}$$

then the assertion

$$\lim_{t\to\infty} \mathbb{R}_t(a_t x + b_t) = \mathbb{R}(x) \quad for \quad x \in C_{\mathbb{R}} \tag{22}$$

is equivalent to the assertion

$$\lim_{t\to\infty} \Big(F(a_t x + b_t)\Big)^{l_t} = \mathbb{F}_0(x), \quad x \in C_{\mathbb{F}_0} \tag{23}$$

where $\mathbb{F}_0(x)$ *is a non-degenerate distribution function. Moreover,*

$$\mathbb{R}(x) = \prod_{i=1}^{a} \Big[1 - d_i(x)\mathbb{F}_0(x)\Big]^{q_i k}, \quad x \in (-\infty, \infty) \tag{24}$$

Proof. By (1), we have

$$\mathbb{R}_t(a_t x + b_t) = \prod_{i=1}^{a} \Big[1 - \Big(F^{(i)}(a_t x + b_t)\Big)^{l_t}\Big]^{q_i k_t}$$

$$= \prod_{i=1}^{a} \left[1 - \left(\frac{F^{(i)}(a_t x + b_t)}{F(a_t x + b_t)}\right)^{l_t} \Big(F(a_t x + b_t)\Big)^{l_t}\right]^{q_i k_t} \tag{25}$$

for $x \geq x_0$, and by (18) and (21)

$$\mathbb{R}(x) = 1 \quad \text{for} \quad x < x_0$$

The last result and (19) mean that assertions (22) and (23) are equivalent. Moreover, since according to (18) and (19)

$$0 \leq d_i(x) \leq 1 \quad \text{for} \quad x \in (-\infty, \infty) \tag{26}$$

for all $i = 1, 2, ..., a$ and sufficiently large t, then if $\mathbb{F}_0(x)$ is degenerate (Kołowrocki, 1994b), then according to (25) and (26), $\mathbb{R}(x)$ is degenerate. But this is inconsistent with the assumption that $\mathbb{R}(x)$ is non-degenerate. Therefore, $\mathbb{F}(x)$ must be non-degenerate and $\mathbb{R}(x)$ is given by (24). ■

Theorem 2.

1° *If assumptions* 1° *of Theorem 1 on the pair* (k_t, l_t) *are satisfied and there exists a non-decreasing positive function* $d(x)$ *given by (11), then the only possible non-degenerate asymptotic reliability functions of the non-homogeneous parallel-series system are these of the following types*

$$\mathbb{R}'_1(x) = \begin{cases} \exp\Big(-d(x)(-x)^{-\alpha}\Big) & \text{for } x < 0, \quad \text{where } \alpha > 0 \\ 0 & \text{for } x \geq 0 \end{cases}$$

$$\mathbb{R}'_2(x) = \begin{cases} 1 & \text{for } x < 0 \\ \exp\Big(-d(x)x^{\alpha}\Big) & \text{for } x \geq 0, \quad \text{where } \alpha > 0 \end{cases}$$

and

$$\mathbb{R}_3(x) = \exp\Big(-d(x)\exp x\Big) \qquad \text{for } x \in (-\infty, \infty)$$

2° *If*

$$k_t = t, \quad l_t - c\ln t \sim s, \quad c > 0, \quad s \in (-\infty, \infty)$$

and there exists a non-decreasing positive function $d(x)$ *given by (11), then the only possible non-degenerate asymptotic reliability functions of the non-homogeneous parallel-series system are those of the following types*

$$\mathbb{R}'_4(x) = \begin{cases} \exp\Big(-d(x)\exp\Big(-(-x)^{\alpha} - \frac{s}{c}\Big)\Big) & \text{for } x < 0 \\ 0 & \text{for } x \geq 0 \end{cases}$$

$$\mathbb{R}'_5(x) = \begin{cases} 1 & \text{for } x < 0 \\ \exp\Big(-d(x)\exp\Big(x^{\alpha} - \frac{s}{c}\Big)\Big) & \text{for } x \geq 0 \end{cases}$$

$$\mathbb{R}'_6(x) = \begin{cases} \exp\Big(-d(x)\exp\Big(-(-x)^{\alpha} - \frac{s}{c}\Big)\Big) & \text{for } x < 0 \\ \exp\Big(-d(x)\exp\Big(\beta x^{\alpha} - \frac{s}{c}\Big)\Big) & \text{for } x \geq 0 \end{cases}$$

where $\alpha > 0$, $\beta > 0$, *and*

$$\mathbb{R}'_7(x) = \begin{cases} 1 & \text{for } x < x_1 \\ \exp\Big(-d(x)\exp(-\frac{s}{c})\Big) & \text{for } x_1 \leq x < x_2 \\ 0 & \text{for } x \geq x_2 \end{cases}$$

where $x_1 < x_2$.

3° *If*

$$\lim_{t\to\infty} k_t = k, \qquad \lim_{t\to\infty} l_t = \infty$$

and there exist non-decreasing functions $d_i(x)$ *given by (19), then the only possible non-degenerate asymptotic reliability functions of the non-homogeneous parallel-series system are those of the following types*

$$\mathbb{R}'_8(x) = \begin{cases} 1 & \textit{for } x \le 0 \\ \prod_{i=1}^{a} \Big[1 - d_i(x)\exp(-x^{-\alpha})\Big]^{q_i k} & \textit{for } x > 0, \quad \textit{where } \alpha > 0 \end{cases}$$

$$\mathbb{R}'_9(x) = \begin{cases} \prod_{i=1}^{a} \Big[1 - d_i(x)\exp\Big(-(-x)^{\alpha}\Big)\Big]^{q_i k} & \textit{for } x < 0, \quad \textit{where } \alpha > 0 \\ 0 & \textit{for } x \ge 0 \end{cases}$$

and

$$\mathbb{R}'_{10}(x) = \prod_{i=1}^{a} \Big[1 - d_i(x)\exp\Big(-\exp(-x)\Big)\Big]^{q_i k} \quad \textit{for} \quad x \in (-\infty, \infty)$$

Proof. We shall consider the following three cases.

Case 1. Since, by Lemma 2, the condition

$$\lim_{t\to\infty} \mathbb{R}_t(a_t x + b_t) = \mathbb{R}(x) \quad \text{for} \quad x \in C_{\mathbb{R}} \tag{27}$$

where $\mathbb{R}(x)$ is a non-degenerate reliability function given by

$$\mathbb{R}(x) = \exp\Big(-d(x)V(x)\Big) \tag{28}$$

is equivalent to the condition

$$\lim_{t\to\infty} k_t\Big(F(a_t x + b_t)\Big)^{l_t} = V(x), \quad x \in C_V \tag{29}$$

where $V(x)$ is a non-decreasing non-degenerate function, which is equivalent to the necessary condition of Lemma 1 given in (Kołowrocki, 1994b), then Theorem 1 is applicable for $\mathbb{R}(x)$ given by (28) in the sense that $V(x)$ is one of the types existing in the formulae for $\mathbb{R}_1(x)$, $\mathbb{R}_2(x)$ and $\mathbb{R}_3(x)$. Moreover, considering point 4, since all non-decreasing non-degenerate functions $V(x)$ existing in the formulae for singular limits have such properties that $V(-\infty)d(-\infty) \ne 0$ or $V(\infty)d(\infty) \ne \infty$ or $V(x)d(x)$ is a degenerate function, then $\mathbb{R}'_1(x)$, $\mathbb{R}'_2(x)$ and $\mathbb{R}'_3(x)$ are the only non-degenerate asymptotic reliability functions of the system.

Case 2. Since, by Lemma 2, condition (27), where $\mathbb{R}(x)$ is a non-degenerate reliability function given by (28) is equivalent to condition (29), where $V(x)$ is a non-decreasing non-degenerate function, which is equivalent to the necessary condition of Lemma 1 given in (Kołowrocki, 1994b), then Theorem 1 is applicable for $\mathbb{R}(x)$ given by (28) in the sense that $V(x)$ is one of the types existing in the formulae for $\mathbb{R}_5(x)$, $\mathbb{R}_6(x)$ and $\mathbb{R}_7(x)$. Moreover, considering point 4, since all fixed there non-increasing non-degenerate functions $V(x)$ have such properties that $V(-\infty)d(-\infty) \ne 0$ or $V(\infty)d(\infty) \ne \infty$ or $V(x)d(x)$ is a degenerate function, then $\mathbb{R}'_4(x)$, $\mathbb{R}'_5(x)$, $\mathbb{R}'_6(x)$ and $\mathbb{R}'_7(x)$ are the only non-degenerate asymptotic reliability functions of the system.

Case 3. Since, by Lemma 3,

$$\lim_{t\to\infty} \mathbb{R}_t(a_t x + b_t) = \prod_{i=1}^{a} \left[1 - d_i(x)\mathbb{F}_0(x)\right]^{q_i k}$$

and by Theorem 1 (point 3°, for $k = 1$), $\mathbb{F}_0(x)$ may only be equal to one of the non-degenerate distribution functions $\mathbb{F}_i(x) = \mathbb{R}_i(-x)$, $i = 1,2,3$, then the only non-degenerate asymptotic reliability functions of the system may only be the functions $\mathbb{R}'_8(x)$, $\mathbb{R}'_9(x)$ and $\mathbb{R}'_{10}(x)$. ■

6. An Example

Let the non-homogeneous parallel-series system be such that

$$R^{(i,j)}(x) = \begin{cases} 1, & x < -\frac{A_{ij}}{2} \\ \frac{1}{2} - \frac{x}{A_{ij}}, & -\frac{A_{ij}}{2} \le x < 0 \\ \frac{1}{2} - \frac{x}{2A_{ij}}, & 0 \le x < A_{ij} \\ 0, & x \ge A_{ij} \end{cases}$$

for $i = 1,2,...,a$, $j = 1,2,...,e_i$, and

$$A_i = \sum_{j=1}^{e_i} \frac{p_{ij}}{A_{ij}} \quad \text{for} \quad i = 1,2,...,a, \qquad A = \max_{1\le i\le a}\{A_i\}$$

If the pairs (k_t, l_t) and (a_t, b_t) satisfy the conditions

(i) $\qquad k_t = t, \qquad l_t = \dfrac{\ln t}{\ln 2}$

(ii) $\qquad a_t = \dfrac{1}{2Al_t}, \qquad b_t = 0$

then a reliability function

$$\mathbb{R}'_6(x) = \begin{cases} \exp\left(-\sum_{i=1}^{a} q_i \exp\left(\frac{A_i}{A}x\right)\right), & x \le 0 \\ \exp\left(-\sum_{i=1}^{a} q_i \exp\left(\frac{A_i}{2A}x\right)\right), & x > 0 \end{cases}$$

is the asymptotic reliability function of the system.

Justification. Since

$$a_t > 0 \quad \text{for} \quad t \in (-\infty, \infty) \quad \text{and} \quad \lim_{t\to\infty} a_t = 0$$

then for all i and j

$$F^{(i,j)}(a_t x) = \begin{cases} \dfrac{1}{2} + \dfrac{a_t x}{A_{i,j}}, & x \le 0 \\ \dfrac{1}{2} + \dfrac{a_t x}{2A_{i,j}}, & x > 0 \end{cases}$$

for sufficiently large t and according to (5)

$$\mathbb{R}_t(a_t x) = \prod_{i-1}^{a} \left[1 - \left(F^{(i)}(a_t x)\right)^{l_t}\right]^{q_i k_t}$$

for all $x \in (-\infty, \infty)$, where

$$F^{(i)}(a_t x) = \begin{cases} \prod_{j=1}^{e_i} \left[\dfrac{1}{2} + \dfrac{a_t x}{A_{ij}}\right]^{p_{ij}}, & x \le 0 \\ \prod_{j=1}^{e_i} \left[\dfrac{1}{2} + \dfrac{a_t x}{2A_{ij}}\right]^{p_{ij}}, & x > 0 \end{cases}$$

$$= \begin{cases} \dfrac{1}{2} \exp\left(2A_i a_t x - o(a_t)\right), & x \le 0 \\ \dfrac{1}{2} \exp\left(A_i a_t x - o(a_t)\right), & x > 0 \end{cases}$$

where $o(a_t) \ll a_t$.

Let $F(x)$ be one of the distribution functions $F^{(i)}(x)$ such that

$$F(x) = \begin{cases} 0, & x < -\dfrac{A}{2} \\ \prod_{j=1}^{e_i} \left[\dfrac{1}{2} + \dfrac{x}{A}\right]^{p_{ij}}, & -\dfrac{A}{2} \le x < 0 \\ \prod_{j=1}^{e_i} \left[\dfrac{1}{2} + \dfrac{x}{2A}\right]^{p_{ij}}, & 0 \le x < A \\ 1, & x \ge A \end{cases}$$

Hence

$$F(a_t x) = \begin{cases} \dfrac{1}{2} \exp\left(2A a_t x - o(a_t)\right), & x \le 0 \\ \dfrac{1}{2} \exp\left(A a_t x - o(a_t)\right), & x > 0 \end{cases}$$

and

$$\lim_{t \to \infty} \frac{F^{(i)}(a_t x)}{F(a_t x)} = \lim_{t \to \infty} \exp\left((A_i - A) a_t x - o(a_t)\right) \le 1$$

for $x \leq 0$ and

$$\lim_{t\to\infty} \frac{F^{(i)}(a_t x)}{F(a_t x)} = \lim_{t\to\infty} \exp\Big((A_i - A)a_t x - o(a_t)\Big) \leq 1$$

for $x > 0$, i.e.

$$F^{(i)}(a_t x) \leq F(a_t x), \qquad x \in (-\infty, \infty)$$

for sufficiently large t. Moreover

$$d_i(x) = \lim_{t\to\infty} \left(\frac{F^{(i)}(a_t x)}{F(a_t x)}\right)^{l_t} = \lim_{t\to\infty} \exp\left(\frac{A_i - A}{A}x - o\Big(\frac{1}{A}\Big)\right) = \exp\left(\frac{A_i - A}{A}x\right)$$

for $x \leq 0$ and

$$d_i(x) = \lim_{t\to\infty} \left(\frac{F^{(i)}(a_t x)}{F(a_t x)}\right)^{l_t} = \lim_{t\to\infty} \exp\left(\frac{A_i - A}{2A}x - o\Big(\frac{1}{2A}\Big)\right) = \exp\left(\frac{A_i - A}{2A}x\right)$$

for $x > 0$. Therefore

$$d(x) = \sum_{i=1}^{a} q_i d_i(x) = \begin{cases} \sum_{i=1}^{a} q_i \exp\left(\frac{A_i - A}{A}x\right), & x \leq 0 \\ \sum_{i=1}^{a} q_i \exp\left(\frac{A_i - A}{2A}x\right), & x > 0 \end{cases}$$

and further

$$\lim_{t\to\infty} k_t \Big(F(a_t x)\Big)^{l_t} d(x) = \begin{cases} d(x)\exp(x), & x \leq 0 \\ d(x)\exp\left(\frac{1}{2}x\right), & x > 0 \end{cases}$$

$$= \begin{cases} \sum_{i=1}^{a} q_i \exp\left(\frac{A_i}{A}x\right), & x \leq 0 \\ \sum_{i=1}^{a} q_i \exp\left(\frac{A_i}{2A}x\right), & x > 0 \end{cases}$$

Hence, by Lemma 2, $\mathbb{R}'_6(x)$ is the asymptotic reliability function of the system. ∎

7. Summary

In the paper, the closed class of asymptotic reliability functions for a homogeneous parallel-series system is given. It is a ten-element class and more extensive than the so far known three-element class of asymptotic distributions of minimum statistics of independent random variables with a common distribution function. This known result may be immediately obtained as a particular case of Theorem 1 given in this work. Similarly, as a particular case of this theorem we may obtain the well-known theorem on limit distributions of maximin statistics of independent random variables with the same distributions.

The result for the homogeneous system has been transferred to the non-homogeneous system. The fixed class of asymptotic reliability functions for non-homogeneous system is also ten-element but this class may be regarded as more extensive in the sense that ten types of asymptotic reliability functions, whose forms depend on some general properties of the reliability functions of particular components and on the frequencies of their appearance in the system, have been fixed.

All results have been obtained on the assumptions that lifetimes of the particular components were independent random variables and the pair (k_t, l_t) had a property such that at least k_t or l_t tended to infinity with some regularity of variation. This regularity can be diminished, because if the system has an asymptotic reliability function for the pairs (k_t, l_t) and (a_t, b_t) and it has the same asymptotic reliability function for other pairs (k_t', l_t') and (a_t', b_t'), then this system also has this asymptotic reliability function for the pairs $(\widetilde{k}_t, \widetilde{l}_t)$ and $(\widetilde{a}_t, \widetilde{b}_t)$, where $(\widetilde{k}_t, \widetilde{l}_t) = (k_t, l_t)$ and $(\widetilde{a}_t, \widetilde{b}_t) = (a_t, b_t)$ for some t and $(\widetilde{k}_t, \widetilde{l}_t) = (k_t, l_t')$ and $(\widetilde{a}_t, \widetilde{b}_t) = (a_t', b_t')$ for other t.

The example testifies that there exist systems which have asymptotic reliability functions from the fixed class.

From the practical point of view it is important that k_t and l_t should be natural numbers. In the case when k_t is not convergent, the return with k_t to the natural numbers is trivial because (see Lemma 1) if we replace k_t by its entire part, then $\mathbb{R}_t(x)$ has the same asymptotic reliability function. Whereas, l_t may be represented by $l_t = [l_t] + l_t - [l_t]$, where $[l_t]$ is the entire part of l_t, and since $(F(x))^{l_t - [l_t]}$ is again a distribution function, then according to (1), we may assume that the parallel subsystems have one component with the reliability function different from the remaining ones. In the case when k_t is convergent, k_t may be a natural number by the assumption and the return with l_t to the natural numbers is trivial because (see Lemma 3) if we replace l_t by its entire part, then $\mathbb{R}_t(x)$ has the same asymptotic reliability function.

Because of duality the result may be easily transferred to the series-parallel system.

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