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APPROXIMATE RELATIVE CONTROLLABILITY OF RETARDED DYNAMICAL SYSTEMS

JERZY KLAMKA*

In this paper, linear abstract retarded dynamical systems defined in infinite-dimensional Hilbert spaces are considered. Using frequency-domain methods and spectral analysis for linear self-adjoint operators, the necessary and sufficient conditions for approximate relative controllability are formulated and proved. The method presented in the paper allows one to verify approximate relative controllability for abstract retarded dynamical systems by considering approximate controllability of simplified abstract dynamical systems without delays. Moreover, as an illustrative example, approximate relative controllability of a retarded distributed-parameter dynamical system is investigated. The presented results generalize to an infinite-dimensional class of retarded dynamical systems some controllability theorems which are known in the literature only for the finite-dimensional case.

1. Introduction

Controllability is one of the fundamental concepts in mathematical control theory (Bensoussan *et al.*, 1993; Klamka, 1992). Roughly speaking, controllability generally means that it is possible to steer a dynamical system from an arbitrary initial state to an arbitrary final state using a given set of admissible controls. In the literature, there are many different definitions of controllability which depend on the particular class of dynamical systems (Bensoussan *et al.*, 1993; Klamka, 1991; 1993b; Nakagiri, 1987; Nakagiri and Yamamoto, 1989; Narukawa, 1982; 1984; O'Brien, 1979; Park *et al.*, 1990; Triggiani, 1975a; 1976; 1978). For infinite-dimensional dynamical systems it is necessary to distinguish between the notions of approximate and exact controllability (Bensoussan *et al.*, 1993; Klamka, 1982; 1991; 1992; 1993a; 1993b; O'Brien, 1979; Triggiani, 1975a; 1976; 1978). This follows directly from the fact that in infinite-dimensional spaces there exist linear subspaces which are not closed. Moreover, for retarded dynamical systems there are two fundamental concepts of controllability, namely relative controllability and absolute controllability (Bensoussan *et al.*, 1993; Klamka, 1991; 1993b; Manitius, 1982; Nakagiri, 1987; Nakagiri and Yamamoto, 1989; Park *et al.*, 1990). Therefore, for retarded dynamical systems defined in infinite-dimensional state spaces, the following four kinds of controllability are considered: approximate relative controllability, exact relative controllability, approximate absolute controllability, and exact absolute controllability.

* Silesian Technical University, Institute of Automation, ul. Akademicka 16, 44–100 Gliwice, Poland, e-mail: jklamka@appia.gliwice.edu.pl

However, in the case of a finite number of scalar controls a dynamical system with infinite-dimensional state space cannot be exactly relatively or exactly absolutely controllable (Triggani, 1975b; 1977). On the other hand, the assumption of approximate controllability is essential in construction of an optimal control sequence in the minimum energy control problem for infinite-dimensional distributed-parameter systems without delays (Kobayashi, 1978). Therefore, combining the results given in (Klamka, 1991; Kobayashi, 1978), approximate relative controllability may be used in formulation and construction of the solution to the minimum-energy control problem for infinite-dimensional systems with delays. This is an open problem for future investigations.

Hence, the present paper is devoted to a study of approximate relative controllability for linear infinite-dimensional retarded dynamical systems. For such dynamical systems direct verification of approximate relative controllability is a rather difficult and complicated task. Therefore, using frequency-domain methods (Bensoussan *et al.*, 1993; Kobayashi, 1992; Nakagiri and Yamamoto, 1989) it is shown that approximate controllability of a linear retarded dynamical system can be checked by the approximate controllability condition for a suitably defined simplified infinite-dimensional dynamical system without delays. General results are then applied for approximate relative controllability investigations of distributed-parameter dynamical systems with one constant delay in the state variable.

The results presented in the paper extend controllability theorems given in (Bensoussan *et al.*, 1993; Klamka, 1982; 1991; Kobayashi, 1992; Nakagiri, 1987; Nakagiri and Yamamoto, 1989; O'Brien, 1979; Triggani, 1976; 1978) to a more general class of abstract retarded dynamical systems.

2. System Description and Basic Definitions

We begin with the basic notation and terminology used throughout the present paper. Let X be a separable Hilbert space. For a set $E \subset X$ the symbol $\mathrm{Cl}\, E$ denotes its closure. For a given real number $h > 0$ we denote by $L_2([-h,0],X)$ the separable Hilbert space of all strongly measurable and square integrable functions from $[-h,0]$ into X. Moreover, let us introduce the space $M_2([-h,0],X) = X \times (L_2([-h,0],X)$ (Bensoussan *et al.*, 1993; Klamka, 1991; Nakagiri, 1981, 1987, 1988; Nakagiri and Yamamoto, 1989) shortly denoted as M_2 which is a separable Hilbert space with standard scalar product

$$\langle g,f\rangle_{M_2} = \langle g^0,f^0\rangle_X + \langle g^1,f^1\rangle_{L_2} = \langle g^0,f^0\rangle_X + \int_{-h}^{0} \langle g^1(s),f^1(s)\rangle_X \,\mathrm{d}s$$

for $f = (f^0,f^1) \in M_2$ and $g = (g^0,g^1) \in M_2$.

Let $A_0 : X \supset D(A_0) \to X$ denote a linear, in general unbounded, self-adjoint, and positive-definite operator with domain $D(A_0)$ dense in X and compact resolvent $R(s; A_0)$ for all s in the resolvent set $\rho(A_0)$. Then the operator A_0 has the following properties (Chen and Russell, 1982; Klamka, 1991; Triggiani, 1976, 1978):

1) The operator A_0 has only the pure discrete-point spectrum $\sigma_p(A_0)$ consisting entirely of isolated real positive eigenvalues

$$0 < s_1 < s_2 < \ldots < s_i < \ldots, \qquad \lim_{i \to \infty} s_i = +\infty$$

Each eigenvalue s_i is of finite multiplicity $n_i < \infty$, $i = 1, 2, 3, \ldots$, equal to the dimensionality of the corresponding eigenmanifold.

2) The eigenvectors $x_{ik} \in D(A_0)$, for $i = 1, 2, 3, \ldots$ and $k = 1, 2, 3, \ldots, n_i$, form a complete orthonormal set in the separable Hilbert space X.

3) The operator A_0 has the spectral representation

$$A_0 x = \sum_{i=1}^{i=\infty} s_i \sum_{k=1}^{k=n_i} \langle x, x_{ik} \rangle_X \, x_{ik} \quad \text{for} \quad x \in D(A_0)$$

4) Fractional powers A_0, $0 < \alpha \leq 1$ of the operator A_0 can be defined as follows:

$$A_0^\alpha x = \sum_{i=1}^{i=\infty} s_i^\alpha \sum_{k=1}^{k=n_i} \langle x, x_{ik} \rangle_X \, x_{ik} \quad \text{for} \quad x \in D(A_0^\alpha)$$

where $D(A_0^\alpha) = \left\{ x \in X : \sum_{i=1}^{i=\infty} s_i^{2\alpha} \sum_{k=1}^{k=n_i} \left| \langle x, x_{ik} \rangle_X \right|^2 < \infty \right\}.$

5) The operators A_0^α, $0 < \alpha \leq 1$, are self-adjoint and positive-definite with domains dense in X and $-A_0^\alpha$ generates analytic semigroups on X. In particular, the operator $-A_0$ generates an analytic semigroup $T(t) : X \to X$ for $t \geq 0$.

We shall consider a linear abstract retarded dynamical control system described by the following functional differential equation (Nakagiri, 1981; 1986; 1987; 1988; Nakagiri and Yamamoto, 1989):

$$\dot{x}(t) = -A_0 x(t) + \sum_{r=1}^{r=p} c_r A_0^{\alpha_r} x(t - h_r) + \sum_{j=1}^{j=m} b_j u_j(t) \tag{1}$$

with the initial conditions

$$x(0) = g^0 \in X \quad \text{and} \quad x(t) = g^1(t) \in L_2\big([-h, 0], X\big) \tag{2}$$

where $0 < h_1 < h_2 < \ldots < h_r < \ldots < h_p$ are constant delays, $c_r \in \mathbb{R}$, $r = 1, 2, ..., p$, are given constants, $0 \leq \alpha_r < 1$, $r = 1, 2, ..., p$, are fractional powers of A_0, $b_j \in X$, $j = 1, 2, ..., m$.

The hereditary dynamical system (1) belongs to a special class of general dynamical systems with delays presented e.g. in the papers (Nakagiri; 1981; 1987; Nakagiri and Yamamoto, 1989). Frequently in applications, the operator A_0 is an unbounded differential operator and in this case the dynamical system (1) represents a hereditary distributed-parameter control system. Hereditary partial differential equations of the form (1) arise in population modelling, where x represents a population density which varies in space as well as in time, and the delay may result e.g. from the incubation period of a disease or the growth time of a food source (Memory, 1991).

It is generally assumed that the admissible controls $u_j(t)$ are elements of $L_2([0,\infty),\mathbb{R})$ for $j = 1,2,...,m$. It is well-known that the retarded system (1) with initial conditions (2) and control $u \in L_2([0,\infty),\mathbb{R}^m)$ has for $t > 0$ a unique mild solution $x(t;g,u) \in X$ (Nakagiri, 1981; Travis and Webb, 1974; 1976; Webb, 1976).

In the dynamical system (1), the space of control values is finite-dimensional and the control operator $B : \mathbb{R}^m \to X$ is given by

$$Bu = \sum_{j=1}^{j=m} b_j u_j(t) \tag{3}$$

The adjoint operator $B^* : X \to \mathbb{R}^m$ is defined as follows:

$$B^*x = \Big(\langle b_1, x\rangle_X, \langle b_2, x\rangle_X, \ldots, \langle b_j, x\rangle_X, \ldots, \langle b_m, x\rangle_X\Big) \tag{4}$$

For brevity, let us introduce the operator $\eta(t)$ defined as follows (Nakagiri, 1981; 1987; 1988; Nakagiri and Yamamoto, 1989):

$$\eta(t) = -\sum_{r=1}^{r=p} \chi_{(-\infty,-h_r]}(t) c_r A^{a_r} \tag{5}$$

where χ_E is the characteristic function of the interval E.

In what follows, we shall give short comments on spectral decomposition of the retarded dynamical system (1). A thorough analysis of this problem can be found in (Nakagiri, 1987; 1988; Webb, 1976).

First of all, for each $z \in \mathbb{C}$ we introduce the closed, densely-defined, linear operator

$$\Delta(z) = \Delta(z;A_0) = zI + A_0 - \sum_{r=1}^{r=p} c_r \exp(-zh_r) A_0^{a_r} \tag{6}$$

where I denotes the identity operator on X. By the retarded resolvent set $\rho(A_0,\eta)$ we mean the set of all values $z \in \mathbb{C}$ for which the operator $\Delta(z;A_0,\eta)$ has a bounded inverse with domain dense in X. In this case, $\Delta(z;A_0,\eta)^{-1}$ is the so-called retarded resolvent and is denoted by $R(z;A_0,\eta)$. The complement of $\rho(A_0,\eta)$ in the complex plane is called the retarded spectrum and is denoted by $\sigma(A_0,\eta)$. It is well-known that the retarded resolvent set $\rho(A_0,\eta)$ is open in $\mathbb{C}$ and the retarded resolvent $R(z;A_0,\eta)$ is an analytic function for $z \in \rho(A_0,\eta)$. Moreover, let us denote by

$\rho_0(A_0, \eta)$ the connected component of the resolvent set $\rho(A_0, \eta)$ which contains the right half-plane of the complex plane.

Let $x(t; g, 0)$ for $g \in M_2([-h, 0], X)$ denote the mild solution of the homogeneous dynamical system (1) with $u = 0$. Define the family of linear bounded operators $S(t) : M_2 \to M_2$ for $t \geq 0$ by

$$S(t)g = \Big(x(t; g, 0), x_t(s; g, 0)\Big) \qquad \text{for} \quad g \in M_2 \tag{7}$$

where $x_t(s; g, 0) = x(t + s; g, 0)$ for $s \in [-h, 0]$. Then $S(t)$ is a strongly continuous semigroup of bounded linear operators on M_2. Let A be the infinitesimal generator of the semigroup $S(t)$. Since A_0 has the compact resolvent, the spectrum $\sigma(A)$ is a pure discrete-point one consisting entirely of at most countable set of eigenvalues. In fact, we have

$$\sigma(A) = \bigcup_{i=1}^{i=\infty} \sigma_i \tag{8}$$

where

$$\sigma_i = \left\{ z \in \mathbb{C} : \Delta_i(z) = z + s_i - \sum_{r=1}^{r=p} c_r \exp(-z h_r) s_i^{a_r} = 0 \right\} \tag{9}$$

Now, we shall introduce various concepts of controllability for the retarded dynamical system (1). It is well-known that for retarded dynamical systems there exist two fundamental notions of controllability, namely related controllability and absolute controllability. In the present paper, we shall concentrate on the relative controllability. Since the dynamical system (1) is defined in the infinite-dimensional space X, it is necessary to distinguish between exact relative controllability and approximate relative controllability. However, since the control operator is finite-dimensional and therefore compact, the dynamical system (1) cannot be exactly relatively controllable for the infinite-dimensional space X (Triggiani, 1975b; 1977). Thus, in the sequel, we shall concentrate on approximate relative controllability. First of all, let R_t, $t > 0$, and R_∞ denote attainable sets given by

$$R_t = \Big\{ x(t; 0, u) \in X \;:\; u \in L_2\big([0, t], \mathbb{R}^m\big) \Big\} \quad \text{and} \quad R_\infty = \bigcup_{t>0} R_t \tag{10}$$

Definition 1. The dynamical system (1) is said to be *approximately relatively controllable in time* $t > 0$ if $\mathrm{Cl}\,(R_t) = X$.

Definition 2. The dynamical system (1) is said to be *approximately relatively controllable in finite time* if $\mathrm{Cl}\,(R_\infty) = X$.

Several others definitions of controllability for retarded dynamical systems can be found in the monographs (Bensoussan *et al.*, 1993; Klamka, 1991).

3. Approximate Relative Controllability

In this section, we shall formulate and prove criteria for approximate relative controllability in finite time of the retarded dynamical system (1). First of all, we shall introduce the following notation (Bensoussan *et al.*, 1993; Klamka, 1991; Triggiani, 1976):

$$B_i = \begin{bmatrix} \langle b_1, x_{i1}\rangle_X & \langle b_2, x_{i1}\rangle_X & \dots & \langle b_j, x_{i1}\rangle_X & \dots & \langle b_m, x_{i1}\rangle_X \\ \langle b_1, x_{i2}\rangle_X & \langle b_2, x_{i2}\rangle_X & \dots & \langle b_j, x_{i2}\rangle_X & \dots & \langle b_m, x_{i2}\rangle_X \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \langle b_1, x_{ik}\rangle_X & \langle b_2, x_{ik}\rangle_X & \dots & \langle b_j, x_{ik}\rangle_X & \dots & \langle b_m, x_{ik}\rangle_X \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \langle b_1, x_{in_i}\rangle_X & \langle b_2, x_{in_i}\rangle_X & \dots & \langle b_j, x_{in_i}\rangle_X & \dots & \langle b_m, x_{in_i}\rangle_X \end{bmatrix} \tag{11}$$

for $i = 1, 2, \dots$.

Now, let us recall a modified version of some necessary and sufficient condition for approximate relative controllability in finite time.

Lemma 1. (Nakagiri and Yamamoto, 1989) *The dynamical system (1) is approximately relatively controllable in finite time if and only if*

$$\bigcup_{z \in \rho_0(A_0,\eta)} \operatorname{Ker} B^* R(z; A_0, \eta) = \{0\} \tag{12}$$

Using methods similar to those given in (Nakagiri, 1986) we can prove the necessary and sufficient condition for approximate relative controllability in finite time.

Theorem 1. *The dynamical system (1) is approximately relatively controllable in finite time if and only if*

$$\operatorname{rank} B_i = n_i \quad \text{for each} \quad i = 1, 2, \dots \tag{13}$$

Proof. *(Necessity)* Suppose for the proof by contradiction that there exists at least one index $i_0 \geq 1$ such that

$$\operatorname{rank} B_{i_0} < n_{i_0} \tag{14}$$

Therefore, since the rows of (11) are linearly dependent, there exist real coefficients $\gamma_k, k = 1, 2, \dots, n_{i_0}$, $\sum_{k=1}^{k=n_{i_p}} \gamma_k^2 > 0$, such that

$$\sum_{k=1}^{k=n_{i0}} \gamma_k \langle b_j, x_{i_0 k}\rangle_X = \sum_{k=1}^{k=n_{i0}} \langle b_j, \gamma_k\, x_{i_0 k}\rangle_X = \langle b_j, \sum_{k=1}^{k=n_{i0}} \gamma_k\, x_{i_0 k}\rangle_X = \left\langle b_j, x^0\right\rangle_X = 0 \tag{15}$$

where $x^0 = \sum_{k=1}^{k=n_{i0}} \gamma_k x_{i_0 k}$ is a nonzero element. Therefore, by formulae (4), (6), (9) and (15) we deduce that there exist an eigenvalue $z_0 \in \sigma_{i_0}$ and a non-zero element $x^0 \in \operatorname{Ker} \Delta(z_0; A_0, \eta)$ such that

$$B^* x^0 = \Big(\langle b_1, x^0\rangle_X, \langle b_2, x^0\rangle_X, \ldots, \langle b_j, x^0\rangle_X, \ldots, \langle b_m, x^0\rangle_X\Big) = 0 \tag{16}$$

Let $z \in \rho(A_0, \eta)$. Since all the operators A_0^α for $0 \le \alpha \le 1$ are self-adjoint, by formula (6) the bounded operator $\Delta(z; A_0, \eta)$ is normal and, moreover, its inverse $\Delta(z; A_0, \eta)^{-1} = R(z; A_0, \eta)$ is also normal for all $z \in \rho(A_0, \eta)$. Furthermore, by formulae (6) and (9), for a given $z \in \rho(A_0, \eta)$ the eigenvalues of retarded resolvent $R(z; A_0, \eta)$ are equal to $\Delta_i(z)^{-1} \in \mathbb{C}$, $i = 1, 2, \ldots$. Therefore, for $x \in X$ we have

$$\begin{aligned}
R(z; A_0, \eta)x &= \Delta(z; A_0, \eta)^{-1} x = \Big(zI - A_0 - \sum_{r-1}^{r=p} c_r \exp(-zh_r) A_0^{\alpha_r}\Big)^{-1} x \\
&= \Big(zI - A_0 - \sum_{r=1}^{r=p} c_r \exp(-zh_r) A_0^{\alpha_r}\Big)^{-1} \sum_{i=1}^{i=\infty} \sum_{k=1}^{k=n_i} \langle x, x_{ik}\rangle_X \, x_{ik} \\
&= \sum_{i=1}^{i=\infty} \Big(z - s_i - \sum_{r=1}^{r=p} c_r \exp(-zh_r) A_0^{\alpha_r}\Big)^{-1} \sum_{k=1}^{k=n_i} \langle x, x_{ik}\rangle_X \, x_{ik} \\
&= \sum_{i=1}^{i=\infty} (\Delta_i(z))^{-1} \sum_{k=1}^{k=n_i} \langle x, x_{ik}\rangle_X \, x_{ik}
\end{aligned} \tag{17}$$

Therefore, from (5), (6) and (7) it follows directly that

$$B^* R(z; A_0, \eta) x^0 = B^* (\Delta i_0(z_0))^{-1} x^0 = (\Delta i_0(z_0))^{-1} B^* x^0 = 0 \tag{18}$$

for all $z \in \rho(A_0, \eta)$. This contradicts (2) and therefore, by Lemma 1, the dynamical system is not approximately relatively controllable in finite time. Hence the necessity follows.

(*Sufficiency*) Since the operator A_0 generates an analytic semigroup $T(t)$ for $t > 0$, condition (3) is the necessary and sufficient one for approximate controllability in any time interval for the dynamical system without delays (Klamka, 1991; 1993a; Triggiani, 1975a; 1976; 1978)

$$\dot{x}(t) = -A_0 x(t) + \sum_{j=1}^{j=m} b_j u_j(t) \tag{19}$$

Since attainable sets for the dynamical systems (1) and (19) are the same for $t \in [0, h_1]$, from Definitions 1 and 2 approximate relative controllability in finite time for the dynamical system (1) follows directly. Hence Theorem 1 follows. ■

Corollary 1. *Suppose that all eigenvalues s_i, $i = 1, 2, ...$, are simple, i.e. $n_i = 1$ for $i = 1, 2, \dots$. Then the dynamical system (1) is approximately relatively controllable in a finite time interval if and only if*

$$\sum_{j=1}^{j=m} \langle b_j, x_i \rangle_X^2 \neq 0 \qquad \text{for } i = 1, 2, ... \tag{20}$$

Proof. From Theorem 1 it follows immediately that, for the case when the multiplicities $n_i = 1$ for $i = 1, 2...$, the dynamical system (1) is approximately relatively controllable in finite time if and only if

$$B_i = \Big[\langle b_1, x_i \rangle_X \; \langle b_2, x_i \rangle_X \; ... \; \langle b_j, x_i \rangle_X \; ... \; \langle b_m, x_i \rangle_X \Big] \neq 0 \tag{21}$$

for i=1,2,... . Since the relations (20) and (21) are equivalent, Corollary 1 follows immediately. ■

Corollary 2. *The dynamical system (1) is approximately relatively controllable in finite time if and only if the dynamical system without delays*

$$\dot{x}(t) = A_0^{\beta} x(t) + \sum_{j=1}^{j=m} b_j u_j(t), \qquad 0 < \beta < \infty \tag{22}$$

is approximately controllable in finite time for some $\beta \in (0, \infty)$.

Proof. Comparing approximate controllability results given in (Bensoussan *et al.*, 1993; Klamka, 1991; Triggiani, 1976) with (13) it follows that the retarded dynamical system (1) is approximately relatively controllable in finite time if and only if the dynamical system without delays (22) is approximately controllable for $\beta = 1$. On the other hand, by (Narukawa, 1982) approximate controllability of the dynamical system (22) for $\beta = 1$ is equivalent to its approximate controllability for each $\beta \in (0, \infty)$. Hence Corollary 2 follows. ■

4. Example

Let us consider the retarded dynamical system with distributed parameters described by the following partial differential equation:

$$w_t(t, y) = -w_{yyyy}(t, y) + w_{yy}(t - h, y) + \sum_{j=1}^{j=m} b_j(y) u_j(y) \tag{23}$$

defined for $t > 0$, $y \in [0, L]$, with homogeneous boundary conditions

$$w(t, 0) = w(t, L) = w_{yy}(t, 0) = w_{yy}(t, L) = 0 \tag{24}$$

and with initial conditions

$$w(0,y) = g^0(y) \in L_2\big([0,L],\mathbb{R}\big) = X \quad \text{and} \quad w(t,y) = g^1(t,y) \in L_2\big([-h,0],X\big) \tag{25}$$

where $b_j(y) = b_j \in L_2([0,L],\mathbb{R}) = X$, $j = 1,2...,m$, are given functions, $u_j(t) \in L_2([0,\infty),\mathbb{R})$, $j = 1,2,...,m$, are scalar control functions, $h > 0$ is a constant delay.

The retarded linear partial differential equation (23) can be expressed in the abstract form (1) substituting $w(t,y) = x(t) \in X$ and using linear unbounded differential operator $A_0 : X \supset D(A_0) \to X$ defined as follows:

$$A_0 x = A_0 w(y) = w_{yyyy}(y) \tag{26}$$

$$D(A_0) = \Big\{x = w(y) \in H^4\big([0,L],\mathbb{R}\big) : w(0) = w(L) = w_{yy}(0) = w_{yy}(L) = 0\Big\} \tag{27}$$

where $H^4([0,L],\mathbb{R})$ denotes the fourth-order Sobolev space.

The unbounded linear differential operator A_0 has the following properties (Bensoussan *et al.*, 1993; Klamka, 1991; Triggiani, 1975a):

1. A_0 is a self-adjoint and positive-definite operator with domain $D(A_0)$ dense in the Hilbert space X.
2. There exists a compact inverse A_0^{-1} and, consequently, the resolvent $R(s;A_0)$ of A_0 is a compact operator for all $s \in \rho(A_0)$.
3. A_0 has a spectral representation

$$A_0 x = A_0 w(y) = \sum_{i=1}^{i=\infty} s_i \langle x, x_i \rangle_X x_i \quad \text{for} \quad x \in D(A_0)$$

   where $s_i > 0$ and $x_i(y) \in D(A_0), i = 1,2,...,$ are simple eigenvalues and eigenfunctions of A_0, respectively. Moreover,

$$s_i = \left(\frac{\pi i}{L}\right)^4, \; x_i(y) = \left(\frac{2}{L}\right)^{\frac{1}{2}} \sin\left(\frac{\pi i y}{L}\right) \quad \text{for} \quad y \in [0,L]$$

   and the set $\{x_i(y), i = 1,2,...\}$ forms a complete orthonormal system in X.
4. Fractional powers A_0^α, $0 < \alpha \le 1$, can be defined by

$$A_0^\alpha x = A_0^\alpha w(y) = \sum_{i=1}^{i=\infty} s_i^\alpha \langle x, x_i \rangle_X x_i \quad \text{for} \quad x \in D(A_0^\alpha) \quad \text{and} \quad 0 \le \alpha \le 1$$

which is also a densely defined, self-adjoint, and coercive operator domain in X. It should be noted that A_0 being a differential operator does not ensure at all that A_0^α is also a differential operator. However, for $\alpha = 1/2$, we have

$$\begin{aligned} A_0^{1/2} &= A_0^{1/2} w(y) = -w_{yy}(y) \\ D(A_0^{1/2}) &= \Big\{x = w(y) \in H^2\big([0,L],\mathbb{R}\big) : w(0) = w(L) = 0\Big\} \end{aligned} \tag{28}$$

Therefore the linear unbounded differential operator defined by (26) and (27) satisfies all the assumptions of Section 3 and hence eqn. (23) has the following abstract representation:

$$\dot{x}(t) = -A_0 x(t) - A_0^{1/2} x(t-h) + \sum_{j=1}^{j=m} b_j u_j(t) \tag{29}$$

Hence, using results presented in Section 3, it is possible to formulate the necessary and sufficient condition for approximate relative controllability of the abstract linear retarded dynamical system (23).

Theorem 2. *The dynamical system (23) is approximately relatively controllable in finite time if and only if*

$$\sum_{j=1}^{j=m} \left(\int_0^L \sqrt{\frac{2}{L}}\, b_j(y) \sin\left(\frac{\pi i y}{L}\right) \mathrm{d}y \right)^2 \neq 0 \quad \textit{for} \quad i = 1, 2, \ldots \tag{30}$$

Proof. Let us observe that the dynamical system (23) satisfies all the assumptions of Corollary 1. Therefore, taking into account the analytic formula for the eigenvectors $x_i(y) \in L_2([0, L], \mathbb{R})$, $i = 1, 2 \ldots$, and form of the inner product in the separable Hilbert space $L_2([0, L], \mathbb{R})$, we obtain inequalities (30) directly from relation (11). ■

5. Final Remarks

In recent years, we have witnessed considerable progress in the development of mathematical tools of population dynamics (Memory, 1991). The theory of linear differential operators and semigroups of bounded linear operators seems to be the most important. It can be applied to analysis of retarded differential equations. Retarded functional differential equations with delays in the state variables often arise in modelling population dynamics. Many population models containing time delays are described by retarded partial differential equations of the form (1) (see (Memory, 1991) for details). Such models are more realistic by allowing the population density to vary in both space and time.

In the present paper, controllability problems for linear abstract retarded dynamical systems have been considered. Using frequency-domain methods and spectral analysis of linear unbounded operators the necessary and sufficient conditions for approximate relative controllability have been formulated and proved. These conditions allow investigation of approximate relative controllability for abstract retarded dynamical systems by checking approximate controllability of abstract dynamical systems without delays.

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