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DISCRETIZATION CHAOS: VARIABLE STRUCTURE SYSTEMS WITH FINITE SWITCHING VALUES

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Discretization chaos in a class of variable structure control systems with finite switching values is discussed in this paper. It is shown that for the linear oscillatory systems with a certain class of sampling periods, discretized variable structure control enables periodic motions whose periods can be determined. The patterns of behaviours of the discretized system vary depending upon initial conditions as well as sampling periods. Simulation results are presented to show various behaviours.

1. Introduction

A variable Structure Control (VSC) has been studied extensively and received many applications (Utkin, 1992; Zinober, 1990). The main mechanism of VSC is the so-called sliding mode behaviour. The sliding mode is attained by designing the control laws that drive the system state to reach and remain at the intersection of a set of prescribed sliding surfaces. When in sliding, the system exhibits invariance properties, such as robustness to certain internal parameter variations and external disturbances. These invariance properties are maintained when the switching frequency is assumed to be infinite. This ensures that the system state resides on the prescribed switching surfaces that exhibit the desired dynamic characteristics.

However in the practical implementation of the VSC control, due to the physical limitation of switching equipment, the frequency of switching is fixed and not relatively high. Problems, such as severe chattering (or zigzagging), chaotic behaviours, etc., may appear (Yu, 1993; 1994).

A chaotic system is a nonlinear deterministic dynamical system whose behaviour is erratic and irregular and so sensitive to small changes in initial conditions that it is impossible to predict precisely the motion of the system (Grantham and Athalye, 1990). In other words, a chaotic system is a deterministic system exhibiting essentially random motion.

Discretization chaos is concerned with the deterministic system whose chaotic motion is caused solely by converting the continuous-time system to discrete-time system (Grantham and Athalye, 1990).

Discretization chaos in VSC systems with unbounded control magnitudes has been studied in (Yu, 1993; 1994). It has been shown that for a class of VSC systems,

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increasing sampling periods brings about the transition from the sliding mode to the pseudo-sliding mode, to irregular "bird" pattern behaviours, and further, to the instability.

The preliminary study of discretization chaos in another class of VSC systems with finite switching values has been undertaken in the paper (Yu, 1995) in which a two-dimensional oscillatory system was dealt with. To the best of the author's knowledge, this was the first time the discretization chaos in such VSC systems was studied. In this paper, we will extend the result in (Yu, 1995) to general controllable oscillatory linear systems. We will show that different sampling periods may lead the discretized VSC systems into periodic and chaotic behaviours.

The paper is organized as follows. In Section 2, the mechanism of VSC is briefly explained and the continuous–time VSC system concerned in this paper with finite switching values is analyzed in details. In Section 3, the periodic behaviours of the system discretized using commonly used zero-order-hold are studied. Periods when the system behaves periodically are determined. Simulation results are presented in Section 4 to confirm the theoretical investigation as well as various chaotic behaviours that are yet to be explored. Discussion and conclusions are included in Section 5.

2. Continuous–Time VSC System

This section aims to provide a brief introduction of the theory of VSC and details of the linear oscillatory system to be studied.

2.1. Mechanism of VSC

To demonstrate the mechanism of VSC, let us consider the following dynamical control system:

$$\dot{x} = f(x, u) \tag{1}$$

where $x \in \mathbb{R}^n$ is the system state and $u \in \mathbb{R}^1$ is a scalar control input to be determined. The control u is a closed-loop feedback which has the form $u = u(x)$. Once such a control is specified, the control system (1) becomes an autonomous continuous-time dynamical system $\dot{x} = f(x, u(x))$.

The prominent VSC is characterized by the control structure

$$u(x) = \begin{cases} u^+(x) & \text{if} \quad s(x) > 0 \\ u^-(x) & \text{if} \quad s(x) < 0 \end{cases} \tag{2}$$

where $u^+(x) \neq u^-(x)$ and $s(x) = 0$ is the switching manifold on which the discontinuity takes place. System (1) under control (2) is actually a differential equation with right-hand side discontinuity. The task is to design a control law u satisfying (2) such that the system trajectories reach and reside on the prescribed switching manifold $s(x) = 0$ for $t > t_0$, where t_0 is a particular moment. The prescribed switching manifold characterizes the desired dynamics to be achieved.

To ensure the sliding on $s(x) = 0$, the condition

$$s\,\dot{s} < 0 \tag{3}$$

is needed in the neighbourhood of $s(x) = 0$ for the design of u such that the state x crosses the switching manifold $s = 0$, for example, with $u^+(x)$ from $s > 0$ to $s < 0$, and recrosses back from $s < 0$ to $s > 0$ as soon as $u^-(x)$ takes place, and so forth, resulting in the value of u being altered between u^+ and u^-. The resulting motion is called a *sliding mode* because of the resulting chattering along $s = 0$.

Due to the discontinuity of the control on the switching manifold $s = 0$, the solution of the differential equation with right-hand side discontinuity should be redefined. Assuming $s = 0$ is a regular and smooth manifold, a function $x(t)$ is a *solution* to (1) with closed-loop feedback control $u = u(x)$ in the sense of Filippov (Zinober, 1990) if

$$\dot{x}(t) = \alpha f_0^+(x) + (1-\alpha) f_0^-(x) \tag{4}$$

where $f_0^+(x) = \langle \nabla s, f^+(x, u^+(x))\rangle$, $f_0^-(x) = \langle \nabla s, f^-(x, u^-(x))\rangle$, and a proper α $(0 \le \alpha \le 1)$ chosen such that $\dot{x}$ is orthogonal to the tangent of $s = 0$, so the solution remains on the manifold. Here $f^+(x, u^+(x))$ and $f^-(x, u^-(x))$ are limit values at $s = 0$ approaching from sides $s > 0$ and $s < 0$, respectively.

When in sliding mode, the system satisfies equations

$$s(x) = 0 \quad \text{and} \quad \dot{s}(x) = 0 \tag{5}$$

and exhibits invariance properties (Zinober, 1990) yielding motion which is independent of certain system parameters and disturbances.

The invariance properties are a very important aspect of control systems as the systems must perform well regardless of certain disturbances in their work environments. For further details on the theory of VSC, readers are referred to the books by Utkin (1992) and Zinober (1990).

2.2. Control System to Be Studied

In this paper, we consider the system in the controllable canonical form

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \vdots\; &\quad \vdots \\ \dot{x}_n &= -\sum_{i=1}^{n} a_i x_i + u \end{aligned} \tag{6}$$

$$y = x_1 \tag{7}$$

where $a_1, \cdots, a_n$ are the system constant parameters and u can only take finite switching values. The switching hyperplane for the system is chosen

$$s(x) = c_1 x_1 + c_2 x_2 + \ldots + c_{n-1} x_{n-1} + x_n = 0 \tag{8}$$

where $s(x)$ defines an asymptotically stable motion with properly chosen c_i, $i = 1, \cdots, n-1$. Equation $s(x) = 0$ prescribes the desired system dynamics to be tracked. One can easily see that $s = 0$ corresponds to the $(n-1)$-dimensional system

$$y^{(n-1)} + c_{n-1}y^{(n-2)} + \cdots + c_2\dot{y} + c_1 y = 0$$

So when the trajectory enters the sliding mode $s = 0$, the system dimension is reduced by one.

A natural candidate of VSC with finite switching values that forces the trajectory into the sliding mode $s = 0$ is

$$u = -\, b\ \mathrm{sgn}(s), \qquad b > 0 \tag{9}$$

where

$$\mathrm{sgn}(s) = \begin{cases} 1 & \text{for } s \geq 0 \\ -1 & \text{for } s < 0 \end{cases} \tag{10}$$

which is a bang-bang type of switching control. For inequality (3) one obtains

$$\begin{aligned} s\,\dot{s} &= s\Big[\sum_{i=1}^{n-1} c_i x_{i+1} + \dot{x}_n\Big] = s\Big[\sum_{i=1}^{n-1} c_i x_{i+1} - \sum_{i=1}^{n} a_i x_i + u\Big] \\ &= \sum_{i=1}^{n-1} c_i x_{i+1} s - \sum_{i=1}^{n} a_i x_i s - b|s| \end{aligned} \tag{11}$$

If the condition

$$-b < \sum_{i=1}^{n-1} c_i x_{i+1} - \sum_{i=1}^{n} a_i x_i < b \tag{12}$$

is satisfied, then $s\,\dot{s} < 0$ and the existence of the sliding mode $s = 0$ as well as the attractiveness towards it are guaranteed. Inequality (12) actually defines the region of attraction in the state space towards $s = 0$. Since $s = 0$ is asymptotically stable, the entire system is asymptotically stable and exhibits the desired dynamics $s(x) = 0$.

The formal solution of the system is

$$\tilde{x}(t) = \exp\Big(A(t - t_0)\Big)\tilde{x}(t_0) \tag{13}$$

with

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ -a_1 & -a_2 & -a_3 & \cdots & -a_n \end{bmatrix}$$

where $\tilde{x} = [x_1 + (b/a_1)\ \mathrm{sgn}(s), x_2, \ldots, x_n]^T$.

Definition 1. The linear system is called *oscillatory* if the eigenvalues of A are non-zero and imaginary. In this paper, the linear system dealt with is assumed to be oscillatory.

Figure 1 depicts the state space portrait of a two-dimensional oscillatory system ($a_1 > 0$, $a_2 = 0$) with the VSC. For the two-dimensional system, the system trajectory is governed by the switching between two oscillators whose equilibriums are at $(-b/a_1,\ 0)^T$ and $(b/a_1,\ 0)^T$. Two typical trajectories are shown in Fig. 1. The trajectory starting from x_a, which is inside the region of attraction, slides along $s = 0$ as soon as it hits the switching line. The other trajectory that starts from x_b, which is outside the region of attraction, exhibits oscillation before it enters the region of attraction. It then slides on the switching line as soon as it hits it.

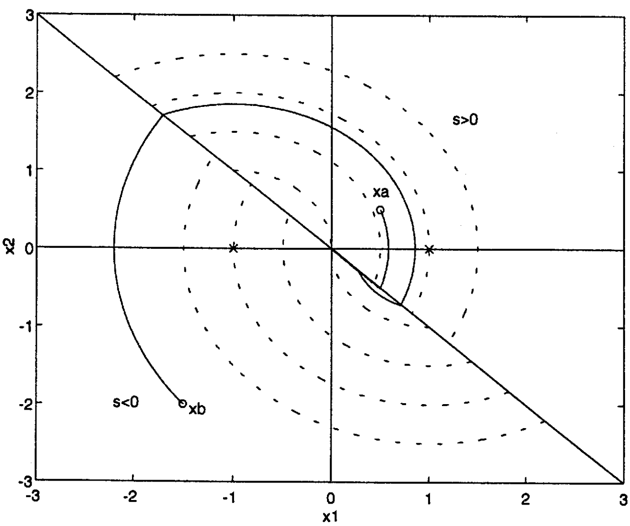

Fig. 1. Phase plane portrait and typical trajectories.

3. Discretization Chaos in the VSC System

3.1. Discretized System

A commonly used discretization scheme is the so-called zero-order-hold (Astrom and Wittenmark, 1984). The control is withheld in the half-closed half-open sampling interval $[kh, (k+1)h)$, where k is an integer and h is the sampling period. The discretized system can be derived as follows:

$$\tilde{x}(k+1) = \exp(Ah)\tilde{x}(k) \tag{14}$$

where $\tilde{x}(k) = [x_1(k) + (b/a_1)\ \mathrm{sgn}(s(k)), x_2(k), \ldots, x_n(k)]^T$. Equation (14) can also be written as

$$x(k+1) = \Phi x(k) + \Gamma(k, k+1) \tag{15}$$

where

$$\Phi = \exp(Ah)$$

$$\Gamma(k, k+1) = (b/a_1)\Big[\exp(Ah)e_1\text{sgn}(s(k)) - e_1\text{sgn}(s(k+1))\Big] \tag{16}$$

with $e_1 = [1, 0, \ldots, 0]^T$.

The fixed sampling period h allows overstepping of the desired equilibrium because the control value is withheld during the sampling interval. The system trajectory may never converge to the desired equilibrium at all, since the discretized system is obtained by switching between two linear systems whose equilibriums are different from the desired equilibrium. The problem of interest is the patterns of behaviours.

3.2. Chaotic Behaviours

Discretization chaos is concerned with the deterministic system whose chaotic behaviours are caused solely by converting the continuous-time system to a discrete-time one (Gratham and Athalye, 1990). That is, chaos occurs in a discrete-time analog of a continuous-time system for cases where the continuous-time system does not exhibit any chaos at all. The discretization chaos measures the deterioration of the system performance due to discretization. The terminology *sensitivity* to initial conditions in the continuous-time case should then be extended to the sensitivity to initial conditions as well as sampling periods in the discrete-time case. For example, trajectories starting from two neighbouring initial states may have completely different behaviours: one converges and the other diverges. Trajectories with two slightly different sampling periods may give rise to totally different behaviours.

The complete answer to the discretization chaos in system (7) is not available due to the complexity of the problem. However, for some class of systems, we can figure out conditions that enable periodic and chaotic behaviours. The results in (Yu, 1995) constitute an initial study of two-dimensional oscillatory systems. This paper deals with a more general case, but restricted by the following assumption.

Assumption 1. *It is assumed that system (7) is characterized by the properties:*

- *All the eigenvalues of A are nonzero and imaginary, i.e. with distinct frequencies $\omega_1, \omega_2, \ldots, \omega_{n/2}$ (actually $\pm w_i j$ is a pair of conjugate eigenvalues of A), where n is the rank of the system.*
- *There exists a frequency such that all other frequencies are its harmonics. Without loss of generality, assume ω_1 is the frequency and $\omega_i = l\omega_1$, $i \neq 1$ and $l > 0$ is an integer.*

In the following, we shall prove for certain classes of sampling periods that if a system behaves periodically, then we can compute its period. However, we do not have all answers to all cases. Accordingly, in Section 4 we demonstrate other behaviours including chaos through simulations.

Theorem 1. *For system (7), under Assumption 1, if the sampling period equals* $h = (2\pi)/(N\omega_1)$ *with even* N *and there exists a* $k_0 > 0$ *such that for* $k > k_0$, $s(k)s(k+1) < 0$, *i.e. the trajectory moves one and only one step on each side of the switching hyperplane* $s = 0$, *then the system exhibits periodic behaviours with period* n *for* $k > k_0$.

Proof. Without loss of generality, we assume $s(k_0) > 0$. Iterating (15) for $N = k - k_0 + 1$ steps yields

$$x(k+1) = \Phi^{k-k_0+1}x(k_0) + \sum_{i=k_0}^{k} \Phi^{i-k_0}\Gamma(k+k_0-i, k+k_0+1-i) \quad (17)$$

Since $s(k)s(k+1) < 0$ for $k > k_0$, we have

$$\begin{aligned}\Gamma(k, k+1) &= (b/a_1)\Big[\exp(Ah)e_1\mathrm{sgn}(s(k)) - e_1\mathrm{sgn}(s(k+1))\Big] \\ &= (b/a_1)\Big[\exp(Ah)e_1 + e_1\Big]\mathrm{sgn}(s(k)) = -\Gamma(k-1, k)\end{aligned}$$

and hence

$$\Gamma(k+k_0-i, k+k_0+1-i) = (-1)^{k-i}\Gamma(k_0, k_0+1) \quad (18)$$

Thus

$$\begin{aligned}x(k+1) &= \Phi^{k-k_0+1}x(k_0) + \sum_{i=k_0}^{k} \Phi^{i-k_0}(-1)^{k-i}\Gamma(k_0, k_0+1) \\ &= \Phi^{k-k_0+1}x(k_0) + (-1)^{k-k_0}\Gamma(k_0, k_0+1)\sum_{i=k_0}^{k} \Phi^{i-k_0}(-1)^{i-k_0} \\ &= \Phi^{k-k_0+1}x(k_0) + (-1)^{k-k_0}\Gamma(k_0, k_0+1)\sum_{i=0}^{k-k_0} \Phi^{i}(-1)^{i} \quad (19)\end{aligned}$$

since the entries of $\exp(ANh)$ contain the terms $\cos(\omega_i Nh)$ and $\sin(\omega_i Nh)$ for $i = 1, \ldots, n/2$ and $w_i = lw_i$. For $h = (2\pi/N\omega_1)$, $\cos(\omega_i Nh) = 1$ and $\sin(\omega_i Nh) = 0$ for $i = 1, \ldots, n/2$, which is the same as for the case $h = 0$, hence $\Phi^{k-k_0+1} = \Phi^N = I$ and eqn. (19) can be expressed as

$$x(k+1) = x(N+k_0) = x(k_0) + (-1)^{k-k_0}\Gamma(k_0, k_0+1)\sum_{i=0}^{k-k_0} \Phi^{i}(-1)^{i} \quad (20)$$

Since for an even integer m

$$A^m - B^m = (A-B)\sum_{i=0}^{m-1}(-1)^i A^{m-i-1}B^i$$

it follows that

$$\begin{aligned}\sum_{i=0}^{k-k_0} \Phi^i(-1)^i &= \sum_{j=0}^{k-k_0} \Phi^{k-k_0-j}(-1)^{k-k_0-j} \\ &= (-1)^{k-k_0} \sum_{j=0}^{k-k_0} \Phi^{k-k_0-j}(-1)^j \\ &= (-1)^{k-k_0} (\Phi - I)^{-1}(\Phi^{k-k_0+1} - I) \qquad (21)\end{aligned}$$

Again, since N is even and $\Phi^{k-k_0+1} = \Phi^N = I$, we see that (21) is zero. Hence eqn. (20) becomes

$$x(N + k_0) = x(k_0) \qquad (22)$$

which shows the period of the system trajectory is N. ∎

Corollary 1. *If the conditions of Theorem 1 are satisfied except that, for* $k > k_0$, *we have* $s(k_0)s(k_0+1) > 0$, $s(k_0+1)s(k_0+2) < 0$, $s(k_0+2)s(k_0+3) > 0$, $s(k_0+3)s(k_0+4) < 0$, $\cdots$, *i.e. on each side of the switching line* $s = 0$ *the system trajectory always moves two and only two steps before jumping to the opposite side of the switching line, then the system exhibits periodic behaviours with period* $2N$ *for* $k > k_0$.

Proof. Because on each side of the switching hyperplane, the system trajectory moves two and only two steps before jumping to the other side of the switching hyperplane, we can actually consider the case when the system trajectory moves with a "larger" step $2h$ which adds two steps together. From Theorem 1 one can conclude that the system exhibits the periodic behaviour with period n and the sampling period $h' = 2h$. Hence $h = (2\pi)/((2N)\omega)$ which indicates that the period of the "new" system trajectory is $2N$. ∎

Remark 1. From Corollary 1 one can see that if the conditions of Theorem 1 are satisfied and the system trajectory moves l steps and only l steps on each side of the switching hyperplane before jumping to the opposite side of the switching hyperplane, then the system exhibits a periodic behaviour with period lN.

Theorem 1 and Corollary 1 show that the period of the trajectory can be determined if the trajectory moves the same number of steps on each side of the switching hyperplane. However, there may be cases where the steps on each side are different. We have the following theorem to deal with it.

Theorem 2. *For system (7) under Assumption 1, if the sampling period is* $h = (2\pi)/(N\omega_1)$ *and there exists a* $k_0 > 0$ *such that, for* $k > k_0$, *the* p *steps when the trajectory moves on one side of* $s(k) = 0$ *are different from the* q *steps on the other side of* $s(k) = 0$, *then the system exhibits periodic behaviours with period* $n(p+q)$ *for* $k > k_0$.

Proof. Without loss of generality, we assume $s(k_0) > 0$. Iterating (15) for $N(p+q)$ steps starting from k_0 yields

$$x\Big(N(p+q)+k_0\Big)$$

$$= \Phi^{N(p+q)}x(k_0) + \sum_{i=k_0}^{N(p+q)-1+k_0} \Phi^{i-k_0}\Gamma\Big(N(p+q)+2k_0-i, N(p+q)+2k_0-i+1\Big) \quad (23)$$

Since we assume that the system trajectory moves p steps on one side of $s(k) = 0$ and q steps on the other side of $s(k) = 0$, it follows that, in the first term of (23), $\Phi^{N(p+q)} = I$ because it contains the terms $\cos(w_i N(p+q)h) = 1$ and $\sin(w_i N(p+q)h) = 0$ for $h = (2\pi)/(Nw_1)$ and $i = 1, \ldots, n/2$. In the second term of (23), we have

$$\sum_{i=k_0}^{N(p+q)-1+k_0} \Phi^{i-k_0}\Gamma\Big(N(p+q)+2k_0-i, N(p+q)+2k_0-i+1\Big) \quad (24)$$

From the index sequence

$$k_0, k_0+1, \ldots, k_0+p+q, k_0+p+q+1, \ldots, (p+q)N-1+k_0$$

one can see that the terms $\Gamma((N(p+q)+2k_0-i, N(p+q)+2k_0-i+1)$ corresponding to the subsequence

$$k_0+j, k_0+(p+q)+j, k_0+2(p+q)+j, \ldots, k_0+(N-1)(p+q)+j, \qquad j \in [1,(p+q)]$$

are the same, since the period of the pattern is $p+q$. The summation (24) can be rewritten as

$$\sum_{i=k_0}^{N(p+q)-1+k_0} \Phi^{i-k_0}\Gamma\Big(N(p+q)+2k_0-i, N(p+q)+2k_0-i+1\Big)$$

$$= \sum_{i=0}^{p+q-1}\left[\sum_{l=0}^{N-1} \Phi^{i+l(p+q)}\Gamma\Big(N(p+q)+k_0-\big(i+l(p+q)\big), N(p+q)+k_0 -\big(i+l(p+q)\big)+1\Big)\right]$$

For a given i, the term $\Gamma(N(p+q)+k_0-(i+l(p+q)), N(p+q)+k_0-(i+l(p+q))+1)$ is the same as for $l = 1, \ldots, N$ since the period of the pattern is $p+q$. Also, since $\Phi^{N(p+q)} = I$ and, for an integer m,

$$A^m - B^m = (A-B)\sum_{i=0}^{m-1} A^{m-i-1}B^i$$

we obtain

$$\sum_{l=0}^{N-1} \Phi^{i+l(p+q)} = \Phi^i \sum_{l=0}^{M-1} \Phi^{l(p+q)} = \Phi^i(\Phi^{(p+q)N} - I)(\Phi - I)^{-1} = 0$$

Therefore

$$x\Big(N(p+q)+k_0\Big) = x(k_0)$$

which shows that the period of the system trajectory is $N(p+q)$. ∎

Remark 2. The same argument applies to the case when the system trajectory moves Mp steps on one side of $s=0$ and Mq steps on the other side of $s=0$, where M is a positive integer. The period of the system trajectory is $MN(p+q)$.

Remark 3. Only two periodic behaviours have been explored in this section for a class of sampling periods. Other sampling periods may give rise to chaotic behaviours. However, even periodic behaviours do not mean the system is not chaotic. The trajectories starting from two slightly different initial states with the same sampling period may end up with different periods. Several typical behaviours will be demonstrated in Section 4. Investigations into reasons behind chaotic behaviours are under way.

4. Simulations

In simulations we choose the fourth-order system with $a_1 = 4$, $a_2 = 0$, $a_3 = 5$, $a_4 = 0$ such that it has four eigenvalues j, $-j$, $2j$, $-2j$ ($w_1 = 1$, $w_2 = 2$) where $j = \sqrt{-1}$. The second set of the conjugate eigenvalues $2j, -2j$ indicates that, as regards the frequency of oscillatory motion, $\omega_2 = 2\omega_1 = 2$. Also in the simulations, $b = 1$ is chosen and $c_1 = c_2 = c_3 = c_4 = 1$. For each simulation, 1000 iterations are made. In the figures, we project the four-dimensional trajectories onto a two-dimensional plane and a three-dimensional state space so that the motion can be visualized.

We set first $h = (2\pi)/10$. For the initial state $x(0) = (0.1, 0, 0, 0)^T$, the system trajectory moves one and only one step on each side of the switching hyperplane, the period of the trajectory is 10 (see Fig. 2), which confirms the result of Theorem 1. For the initial state $x(0) = (1, 0.5, 0.3, 0.2)^T$, the system trajectory moves two and only two steps on each side of the switching hyperplane, the period of the trajectory (see Fig. 3) is 20, which confirms Corollary 1. For the initial state $x(0) = (2, 1, 0.3, 0.2)$, the system trajectory moves again one and only one step on each side of the switching hyperplane so that the period of the trajectory is 10 (see Fig. 4). The sensitivity of the patterns of trajectory to initial conditions is observed.

We choose the second $h = (2\pi/7)$. For the initial state $x(0) = (1, 0.5, 0.3, 0.2)^T$, the system trajectory moves two and only two steps on each side of the switching hyperplane, showing that the irregularity disappears if the initial condition is changed. The period of the trajectory (see Fig. 5) is 14, which confirms Corollary 1. For the initial state $x(0) = (2, 1, 0.3, 0.2)^T$, the system trajectory moves two steps on the

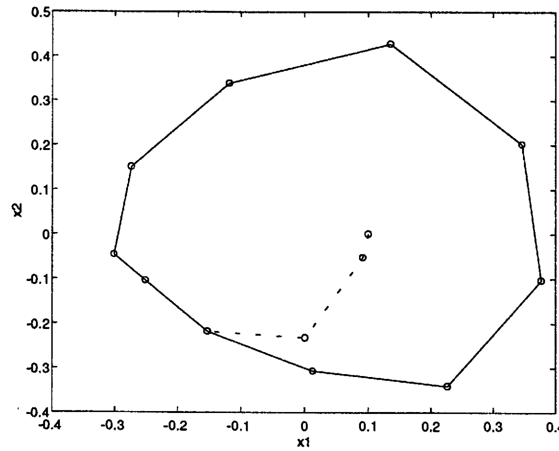

Fig. 2(a). 2D projection.

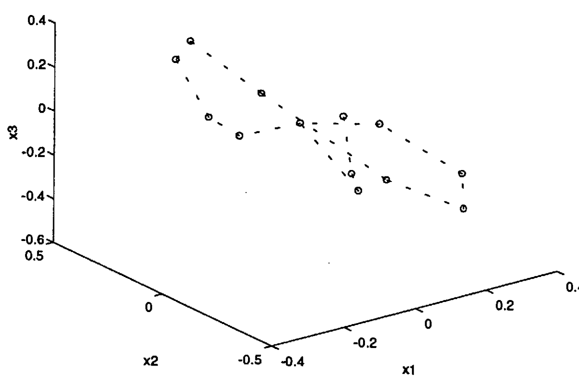

Fig. 2(b). 3D projection.

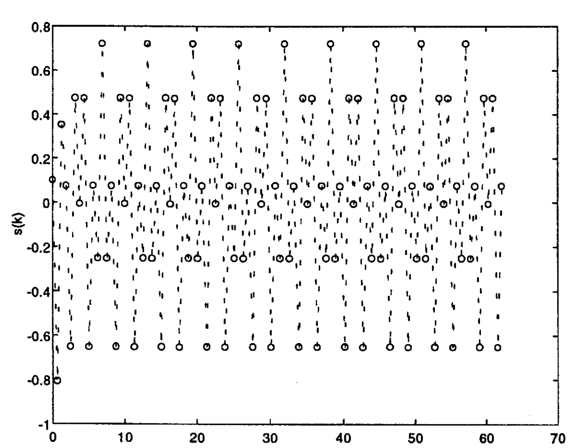

Fig. 2(c). Switching functions s versus t for 100 steps.

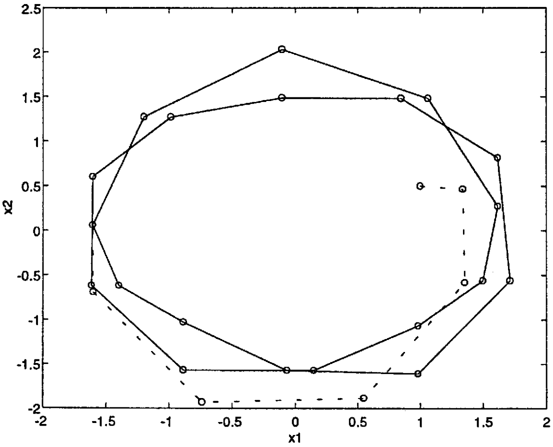

Fig. 3(a). 2D projection.

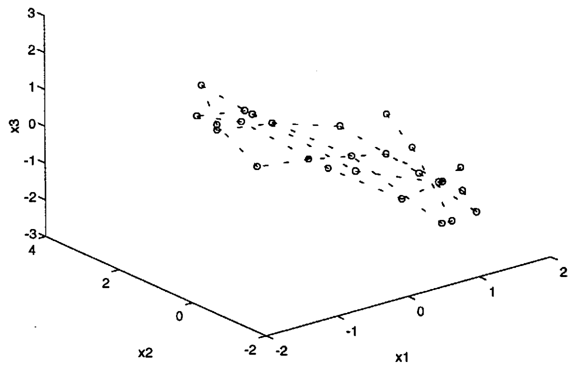

Fig. 3(b). 3D projection.

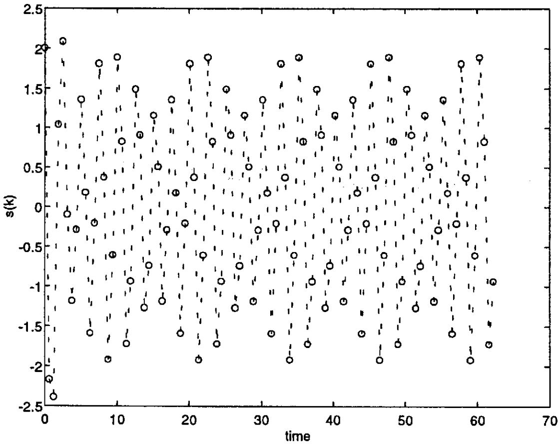

Fig. 3(c). Switching functions s versus t for 100 steps.

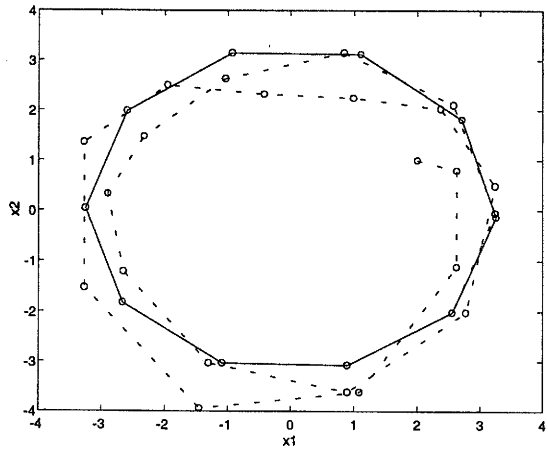

Fig. 4(a). 2D projection.

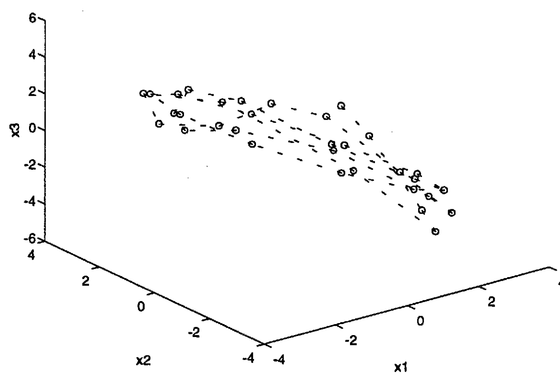

Fig. 4(b). 3D projection.

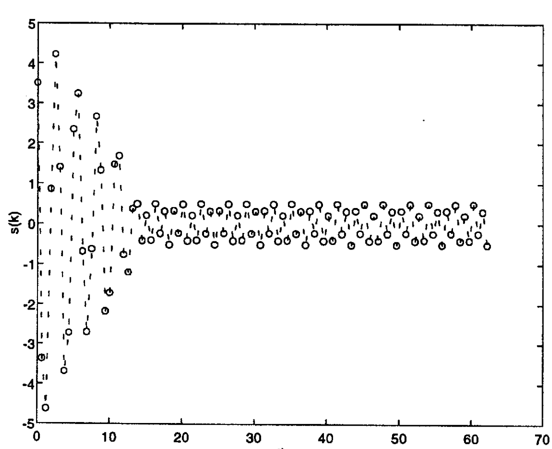

Fig. 4(c). Switching functions s versus t for 100 steps.

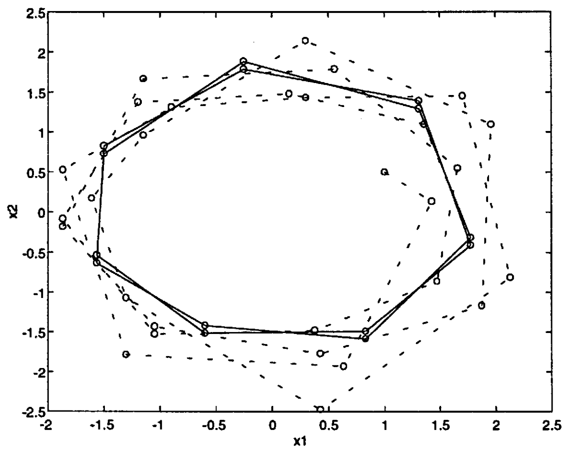

Fig. 5(a). 2D projection.

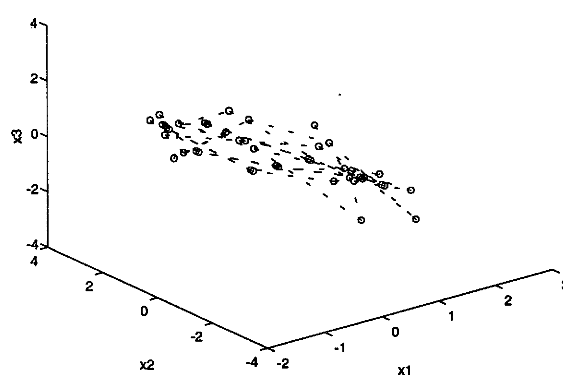

Fig. 5(b). 3D projection.

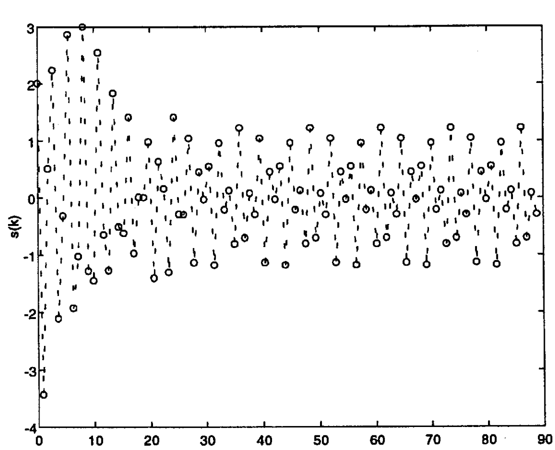

Fig. 5(c). Switching functions s versus t for 100 steps.

side $s > 0$ and one step on the side $s < 0$. From Fig. 6, the period is 21, which confirms Theorem 2. Again, this simulation provides another proof that different initial conditions may give rise to a totally different pattern of behaviours.

Further strange phenomena may be observed if we choose different n's and initial conditions. In Fig. 7, the chaotic motion is observed with the initial condition $x(0) = (0.1, 0, 0, 0)^T$ and $n = 5$. The system trajectory moves apparently in an irregular pattern on each side of $s = 0$ causing the trajectory to diverge. If another initial condition $x(0) = (10, 0.5, 0.3, 0.2)^T$ is chosen, the irregularity occurs as well, resulting in a strange pattern of divergence. Increasing iteration steps we observe that the trajectory goes to infinity.

The low sampling frequencies in the simulations above are chosen to demonstrate the periodic patterns of the system behaviours. If N is large, i.e. h is small, the period of the periodic trajectories will be large.

5. Discussion and Conclusion

The discretization chaos in the oscillatory systems using VSC with only finite switching values available has been demonstrated and studied. It has been shown that, for the oscillatory system, periods of the periodic behaviours for a certain class of sampling periods can be determined. Only two classes of movement patterns have been studied in this paper. Some behaviours are yet to be explored for other patterns and other classes of VSC systems.

To the author's knowledge, this is the first time discretization chaos in VSC systems with finite switching values is studied. More complex systems will be investigated in forthcoming publications.

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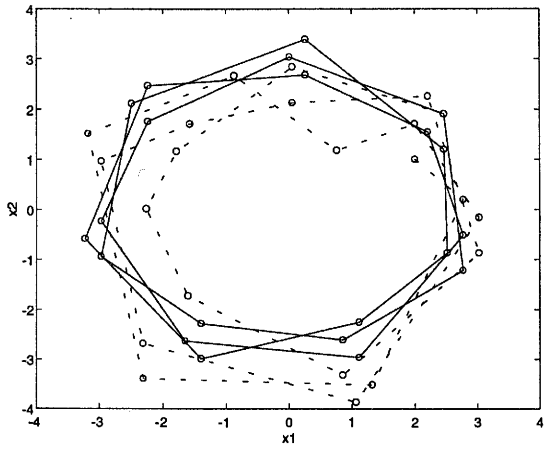

Fig. 6(a). 2D projection.

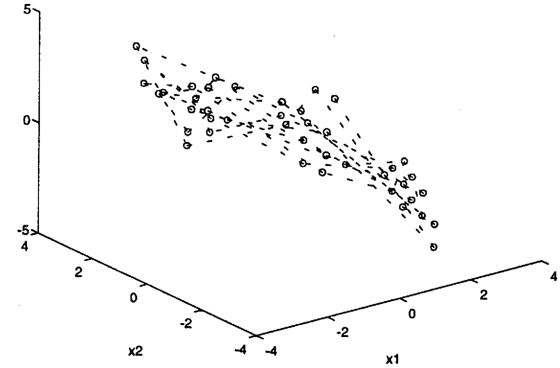

Fig. 6(b). 3D projection.

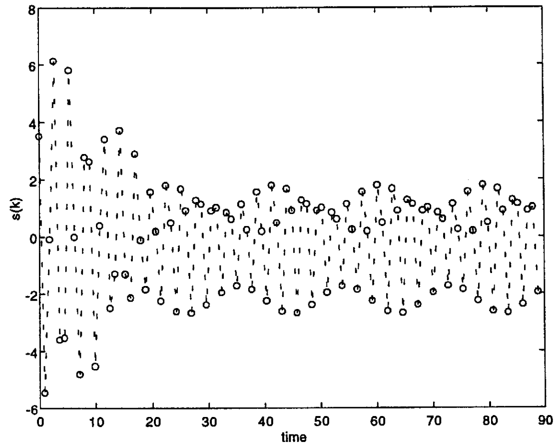

Fig. 6(c). Switching functions s versus t for 100 steps.

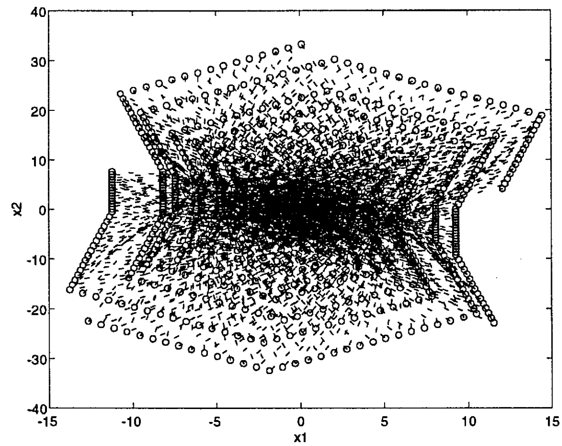

Fig. 7(a). 2D projection.

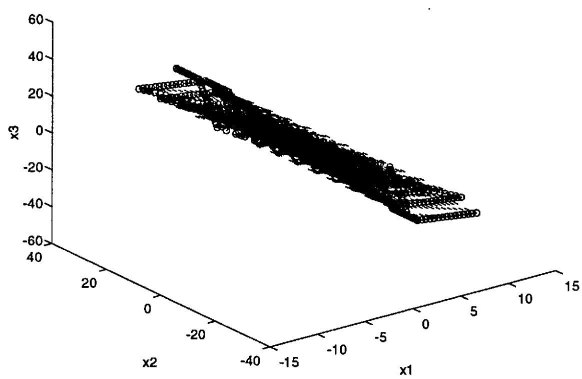

Fig. 7(b). 3D projection.

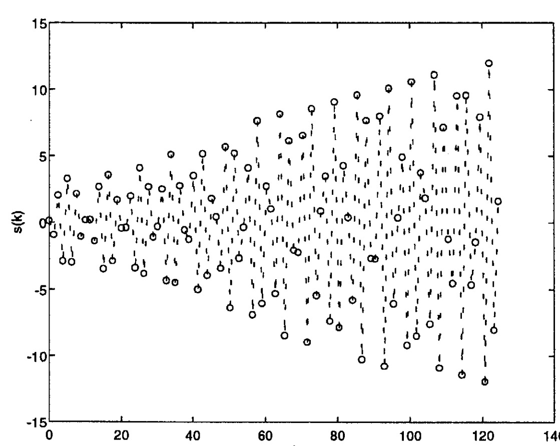

Fig. 7(c). Switching functions s versus t for 100 steps.

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