

## UNIFORM DECAY RATES FOR SOLUTIONS TO A STRUCTURAL ACOUSTICS MODEL WITH NONLINEAR DISSIPATION

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In this work, the asymptotic behavior of solutions to a coupled hyperbolic/parabolic-like system is investigated. It is shown that with both components of the equation being subjected to nonlinear damping (boundary damping for the wave component, interior for the beam), a global uniform stability is attained for all (weak) solutions.

### 1. Introduction

#### 1.1. Statement of Problem

Let  $\Omega$  be a bounded open subset of  $\mathbb{R}^n$ ,  $n = 2$  or  $3$ , with Lipschitz boundary  $\partial\Omega = \Gamma_1 \cup \Gamma_0$ , and with both  $\Gamma_i$  being open and nonempty. We further specify that  $\Gamma_0$  be a simply connected segment of  $\partial\Omega$  and a smooth ( $C^2$ ) manifold. In this paper, we investigate the stability properties of functions  $\vec{z} = [z(t, x), z_t(t, x)]$  and  $\vec{v} = [v(t, x), v_t(t, x)]$  which solve the following system consisting of a coupled semilinear wave and “elastic” equation on finite time  $T$ :

$$\left\{ \begin{array}{ll} z_{tt} = \Delta z & \text{on } \Omega \times (0, T) \\ \frac{\partial z}{\partial \nu} = v_t - g_1(z_t) & \text{on } \Gamma_0 \times (0, T) \\ z = 0 & \text{on } \Gamma_1 \times (0, T) \\ \vec{z}(t = 0) = \vec{z}_0 \in H^1(\Omega) \times L^2(\Omega) \end{array} \right. \quad (1)$$

$$\left\{ \begin{array}{l} v_{tt} = -\mathcal{A}v - \mathcal{A}^{\frac{\eta}{2}} g_2\left(\mathcal{A}^{\frac{\eta}{2}} v_t\right) - z_t \text{ on } \Gamma_0 \times (0, T) \text{ (with fixed } \eta \in [0, 1]) \\ \vec{v}(t = 0) = \vec{v}_0 \in D\left(\mathcal{A}^{\frac{1}{2}}\right) \times L^2(\Gamma_0) \end{array} \right. \quad (2)$$

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where  $\mathring{A} : D(\mathring{A}) \subset L^2(\Gamma_0) \rightarrow L^2(\Gamma_0)$  is any positive definite self-adjoint operator, with its fractional powers therefore being well-defined. The  $g_i$  are functions on the real line which satisfy the following assumptions for  $i = 1, 2$ :

- (H1) (i)  $g_i(s)$  is continuous and monotone increasing;
- (ii)  $g_i(s)s > 0$  for  $s \neq 0$ ;
- (iii)  $m_i s \leq g_i(s) \leq M_i s$  for  $|s| > 1$  and  $0 < m_i \leq M_i$ .

In addition, we shall need the following geometrical assumptions:

- (H2) There exists a  $[C^2(\bar{\Omega})]^n$ -vector field  $h = [h_1(x), h_2(x), \dots, h_n(x)] \in [C^2(\bar{\Omega})]^n$  such that
  - (i)  $h \cdot \nu \leq 0$  on  $\Gamma_1$ , where  $\nu$  is the unit normal of  $\Gamma$  pointing toward the exterior of  $\Omega$ ;
  - (ii) The Jacobian matrix  $H(x)$  of  $h(x)$  is uniformly positive definite on  $\bar{\Omega}$ .

The coupled system is a nonlinear version of the “structural acoustics” model employed in (Banks *et al.*, 1991; Banks and Smith, 1995) to describe the active control of acoustic pressure in a chamber through the placement and implementation of piezoelectric ceramic patches on one of the (flexible) chamber walls. The modelling partial differential equation is a wave coupled to an elastic equation, with the coupling being accomplished through the velocity terms of the solution  $[z, v]$ . In the papers cited above, the abstract operator  $\mathring{A}$  is taken to be the biharmonic  $\Delta^2$  with appropriate homogeneous boundary conditions; and with  $\eta \equiv 1$ ,  $\mathring{A}^{1/2}$  is the Laplacian operator  $\Delta$ . The implemented control term for this particular “smart materials” application takes the form of a linear combination of derivatives of delta functions placed on the beam component (2) so as to mathematically describe the bending moments induced through the chamber wall  $\Gamma_0$  by the piezoelectric patches. In this paper, we deal strictly with the uncontrolled PDE model (1)–(2), and our objective here is to demonstrate that the “energy”  $E$  of the system, defined by

$$E(\vec{z}, \vec{v}, t) \equiv \frac{1}{2} \int_{\Omega} [|\nabla z(t)|^2 + |z_t(t)|^2] \, d\Omega + \frac{1}{2} \left\| \mathring{A}^{\frac{1}{2}} v(t) \right\|_{L^2(\Gamma_0)}^2 + \frac{1}{2} \left\| v_t(t) \right\|_{L^2(\Gamma_0)}^2 \tag{3}$$

decays uniformly as  $t \rightarrow \infty$ .

### 1.2. Preliminaries

Before dealing with the coupled system (1)–(2), we will consider its wave component as an equivalent abstraction, for which we will need the following background material:

- Let the operator  $A : D(A) \subset L^2(\Omega) \rightarrow L^2(\Omega)$  be defined by

$$Az = -\Delta z; \quad D(A) = \left\{ z \in H^2(\Omega) \ni z|_{\Gamma_1} = 0, \frac{\partial z}{\partial \nu} \Big|_{\Gamma_0} = 0 \right\} \tag{4}$$

Note that  $A$  is self-adjoint, positive definite, and hence the fractional powers of  $A$  are well defined.

- By (Grisvard, 1967), we have the following characterization:

$$D(A^{\frac{1}{2}}) = H^1_{\Gamma_1}(\Omega) = \left\{ z \in H^1(\Omega) \ni z = 0 \text{ on } \Gamma_1 \right\}$$

$$\text{with } \|z\|^2_{D(A^{\frac{1}{2}})} = \|A^{\frac{1}{2}}z\|^2_{L^2(\Omega)} = \int_{\Omega} |\nabla z|^2 d\Omega = \|z\|^2_{H^1_{\Gamma_1}(\Omega)} \forall z \in D(A^{\frac{1}{2}}) \quad (5)$$

where the last equality in (5) follows from Poincaré’s inequality.

- We define the map  $N$  by

$$\phi = N\psi \iff \begin{cases} \Delta\phi = 0 & \text{on } \Omega \\ \phi|_{\Gamma_1} = 0 & \text{on } \Gamma_1 \\ \frac{\partial\phi}{\partial\nu}\Big|_{\Gamma_0} = \psi & \text{on } \Gamma_0 \end{cases} \quad (6)$$

and elliptic theory will then yield that

$$N \in \mathcal{L}(L^2(\Gamma_1), D(A^{\frac{1}{2}})) \quad (7)$$

- Let  $\gamma : H^1(\Omega) \rightarrow H^{1/2}(\Gamma_0)$  be the restriction to  $\Gamma_0$  of the familiar Sobolev trace map; viz.

$$\forall z \in H^1(\Omega), \gamma(z) = \begin{cases} z|_{\Gamma_0} & \text{on } \Gamma_0 \\ 0 & \text{on } \Gamma_1 \end{cases} \quad (8)$$

It can then be shown easily that

$$N^*Az = \gamma(z) \quad \forall z \in D(A^{\frac{1}{2}}) \quad (9)$$

- We define the energy spaces

$$H_1 \equiv D(A^{\frac{1}{2}}) \times L^2(\Omega) \quad (10)$$

$$H_0 \equiv D(\mathring{A}^{\frac{1}{2}}) \times L^2(\Gamma_0) \quad (11)$$

With these definitions in hand, the system (1)–(2) is equivalent to the following abstract representation:

$$z_{tt} = -Az - ANg_1(N^*Az_t) + ANv_t \quad \text{on } (0, \infty) \times \Omega \quad (12)$$

$$v_{tt} = -\mathring{A}v - \mathring{A}^{\frac{\eta}{2}}g_2\left(\mathring{A}^{\frac{\eta}{2}}v_t\right) - N^*Az_t \quad \text{on } (0, \infty) \times \Gamma_0 \quad (13)$$

We will hence proceed to look for solutions  $[\vec{z}, \vec{v}]$  of (1)–(2) (and hence (12)–(13)) existing in  $H_1 \times H_0$  a.e. in time, and subsequently study their decay properties. Note that the pointwise defined terms  $N^*Az_t$  and  $g_2(\mathring{A}^{\eta/2}v_t)$ , for  $\eta > 0$ , are initially only formal representations, inasmuch as  $[z_t, v_t]$  are taken to live pointwise (in time) in  $L^2(\Omega) \times L^2(\Gamma_0)$  only.

1.3. Literature

The uniform stability for each of the individual wave and beam components of (1)–(2) has been well-established these past few years, and we spell out here a short (necessarily incomplete) list of results. Concerning the (linear) uncoupled elastic equation with  $\eta \in [\frac{1}{2}, 1]$ , we have the result of Chen and Triggiani (1989) that the associated abstract operator which models the PDE generates an analytic semigroup, which automatically provides for the exponential decay of the solution  $[v, v_t]$  of the second-order system

$$\begin{aligned} v_{tt} &= -\hat{A}v - \hat{A}^\eta v_t \text{ on } (0, \infty) \times \Gamma_1 \\ [v(0), v_t(0)] &= \bar{v}_0 \in H_0 \end{aligned} \tag{14}$$

For the wave equation with  $L^2(0, T; L^2(\Omega))$ -Neumann feedback control; viz.

$$\begin{aligned} z_{tt} &= \Delta z && \text{on } (0, \infty) \times \Omega \\ [z(0, x), z_t(0, x)] &= \bar{z}_0 && \text{on } \Omega \\ z(t, x) &= 0 && \text{on } (0, \infty) \times \Gamma_1 \\ \frac{\partial z(t, x)}{\partial \nu} &= -g_1(z_t(t, x)) && \text{on } (0, \infty) \times \Gamma_0 \end{aligned} \tag{15}$$

Chen (1981) proved the exponential stability of solutions (15), in the (linear) case that  $g_1(s) = s$ , under the geometrical conditions that  $\Omega$  be “star-shaped.” Lagnese (1983), and subsequently Triggiani (1989) through an alternate proof, showed the uniform stabilization of (15), again with  $g_1(s) = s$ , under the lessened constraint that there exists a  $[C^2(\bar{\Omega})]^n$ -vector field  $h(x)$  such that

- (h.i)  $h \cdot \nu \leq 0$  on  $\Gamma_1$ , where  $\nu$  denotes the unit-normal vector to  $\Gamma$ ;
- (h.ii)  $h$  is parallel to  $\nu$  on  $\Gamma_0$ ;
- (h.iii) The Jacobian matrix  $H(x)$  of  $h(x)$  is uniformly positive definite on  $\bar{\Omega}$ .

Also, Bardos *et al.* (1988) have derived stability results for linear wave equations with more general boundary conditions than those in (15), under the assumptions of geometric optics; however the techniques used in the proofs therein are not easily adaptable to our particular situation, based as they are on microlocal analysis and the propagation of singularities. In this work, we will pay particular attention to the result of Lasiecka and Triggiani (1992), who have shown the exponential decay of solutions of (15) without the constraint (h.ii). This result is proved by using the standard multipliers  $h \cdot \nabla z$  and  $z \operatorname{div} h$ , and invoking a (pseudodifferential) trace estimate which we state here for future reference:

**Lemma A.** (Lasiecka and Triggiani, 1992). *Let  $\epsilon > 0$  be arbitrarily small. Let  $z$  solve an arbitrary second-order hyperbolic equation on  $(0, T)$  with smooth space-dependent coefficients. Then with  $Q_T \equiv (0, T) \times \Omega$ ,*

$$\int_\epsilon^{T-\epsilon} \int_{\Gamma_0} \left( \frac{\partial z}{\partial \tau} \right)^2 d\Gamma_0 dt < C_{T,\epsilon} \left\{ \int_0^T \int_{\Gamma_0} \left\{ \left( \frac{\partial z}{\partial \nu} \right)^2 + z_t^2 \right\} d\Gamma_0 dt + \|z\|_{H^{\frac{1}{2}+\epsilon}(Q_T)}^2 \right\} \tag{16}$$

where the  $\epsilon$  on the left of (16) need not be the same as the  $\epsilon$  for the  $Q_T$ -norm on the right, and where  $\partial/\partial\tau$  denotes the tangential, and  $\partial/\partial\nu$  the co-normal derivative.

In addition, the works (Zuazua, 1990; You, 1990; Lasiecka and Tataru, 1993) all deal with boundary stabilization of the wave equation (15) with the full nonlinearity  $g_1$  in place, and varying geometrical conditions being assumed in each paper.

With regard to uniform stability for the linear version of the coupled model (1)–(2); i.e.  $g_1(s) = g_2(s) = s$ , we have the recent result in (Avalos, 1996) that with the geometrical condition (H2) in place, the coupled system decays exponentially for all initial data in  $[\vec{z}_0, \vec{v}_0] \in H_1 \times H_0$ . (The coupled system in this cited work is studied with  $\eta \equiv 1$ , but it is clear from the proof therein, although not explicitly stated, that exponential stability of the model is obtained for all  $\eta \in [0, 1]$ ). The strong stability of the linearization of (1)–(2) was ascertained in (Avalos and Lasiecka, 1998.)

The technical approach adopted in this paper for the proof of uniform stability of the nonlinear model (1)–(2) will essentially be that employed in (Avalos, 1996), an approach which in turn originated in (Lasiecka, 1989). Namely, we shall generate appropriate estimates for the energy functional  $\int_0^T E(\vec{z}, \vec{v}, t)dt$ , as opposed to the classical method of constructing a particular Lyapunov function for a generally nonlinear equation, and subsequently proving differential inequalities with respect to this Lyapunov function. Our initial estimate for the energy functional will be “tainted” with lower order terms, these having to be removed through a compactness/uniqueness argument. In attaining the preliminary estimate for our energy functional, the use of Lemma A above will be critical. Note that the assumptions in (H2) include no restrictions on the active portion of the boundary  $\Gamma_0$ . Indeed, it is owing to the control of the tangential derivative provided by Lemma A that one need not impose additional geometric assumptions upon the active portion of the boundary  $\Gamma_0$ ; e.g.  $h \cdot \nu \geq 0$  on  $\Gamma_0$ , or  $h$  being parallel to  $\nu$  on  $\Gamma_0$ .

### 1.4. Statement of Main Results

**Theorem 1.** (i) With the initial data  $[\vec{z}_0, \vec{v}_0] \in H_1 \times H_0$  and the parameter  $\eta \in [0, 1]$ , the system (1)–(2) has a unique solution  $[\vec{z}, \vec{v}] \in C([0, \infty); H_1 \times H_0)$ .

(ii) In addition, the velocity terms of the solution have the following regularity:

$$z_t|_{\Gamma_0} \in L^2_{loc}(0, \infty; L^2(\Gamma_0)) \tag{17}$$

$$v_t \in L^2_{loc}(0, \infty; D(\mathring{A}^{\frac{\eta}{2}})) \tag{18}$$

(consequently,  $g_1(N^*Az_t), g_2(\mathring{A}^{\eta/2}v_t) \in L^2_{loc}(0, \infty; L^2(\Gamma_0))$  by (H1)(iii)).

(iii) Furthermore, the solution  $[\vec{z}, \vec{v}]$  satisfies the following energy relation:

$$E(\vec{z}, \vec{v}, 0) = E(\vec{z}, \vec{v}, T) + \int_0^T \left[ \left( g_1(N^*Az_t), N^*Az_t \right)_{L^2(\Gamma_0)} + \left( g_2(\mathring{A}^{\frac{\eta}{2}}v_t), \mathring{A}^{\frac{\eta}{2}}v_t \right)_{L^2(\Gamma_0)} \right] dt \tag{19}$$

Before stating our stability result, we must first define some needed functions. Let  $f(x)$  be defined by

$$f(x) \equiv f_1(x) + f_2(x) \tag{20}$$

where the  $f_i$  are concave, strictly increasing functions, with  $f_i(0) = 0$ , and such that

$$f_i(sg_i(s)) \geq s^2 + g_i^2(s) \quad \text{for } |s| \leq 1 \tag{21}$$

note that such functions can always be constructed, given the hypotheses on the  $g_i$  in (H1) (Lasiecka and Tataru, 1993). With this function, we then define

$$\tilde{f}(x) \equiv f\left(\frac{x}{\text{meas}(\Sigma_{0T})}\right), \quad x \geq 0 \tag{22}$$

where  $\Sigma_{0T} \equiv (0, T) \times \Gamma_0$ . As  $\tilde{f}$  is monotone increasing, then  $cI + \tilde{f}$  is invertible for all  $c \geq 0$ . For  $K$  a positive constant, we then set

$$p(x) \equiv (cI + \tilde{f})^{-1}(Kx) \tag{23}$$

The function  $p$  is easily seen to be positive, continuous and strictly increasing with  $p(0) = 0$ . Finally, let

$$q(x) \equiv x - (I + p)^{-1}(x), \quad x > 0 \tag{24}$$

We can now proceed to state our stability result.

**Theorem 2.** *Assume that the hypotheses (H1)–(H2) are in place. Let  $[\vec{z}, \vec{v}]$  be the weak solution to the coupled system (1)–(2), assured by Theorem 1. With the energy  $E(\vec{z}, \vec{v}, t)$  as defined in (3), there exists a  $T_0 > 0$  such that*

$$E(\vec{z}, \vec{v}, t) \leq S\left(\frac{t}{T_0} - 1\right) \quad \text{for } t > T_0 \tag{25}$$

with  $\lim_{t \rightarrow \infty} S(t) = 0$ , where the contraction semigroup  $S(t)$  is the solution to the differential equation

$$\frac{d}{dt}S(t) + q(S(t)) = 0, \quad S(0) = E(\vec{z}, \vec{v}, 0) \tag{26}$$

(where  $q$  is as given in (24)). Here, the constant  $K$  (from definition (23)) will depend on  $E(\vec{z}, \vec{v}, 0)$  and time  $T_0$ , and the constant  $c$  (from definition (23)) is taken here to be  $c \equiv (m_1^{-1} + M_1 + m_2^{-1} + M_2)/\text{meas}(\Sigma_{0T_0})$ .

**Remark 1.** The assumptions (H2) on the vector field  $h$  will be satisfied if  $\Gamma_1$  is a sufficiently small portion of the boundary  $\partial\Omega$ ; viz. if  $\text{meas}(\Gamma_1) < (1/2) \text{meas}(\partial\Omega)$  (Bardos et al., 1988).

**Remark 2.** Concerning the boundary condition imposed upon the inactive portion  $\Gamma_1$ , the Dirichlet boundary condition  $z = 0$  on  $\Gamma_1 \times (0, \infty)$  in (1) can readily be replaced with  $\partial z / \partial \nu = 0$  on  $\Gamma_1 \times (0, \infty)$  without fundamentally changing the stability result for the system (1)–(2). Indeed, with  $\partial z / \partial \nu = 0$  on  $\Gamma_1 \times (0, \infty)$  the analysis will remain unchanged by simply taking the energy space  $H_1 \equiv (H^1(\Omega) / \mathbb{R}) \times (L^2(\Omega) / \mathbb{R})$ .

Since the heart of the matter here is the stability of the system (1)–(2), we relegate the proof of well-posedness (Theorem 1) to the Appendix below, and commence with the proof of uniform stability.

## 2. Proof of Theorem 2

Throughout, we will make use of the denotations  $Q_T \equiv (0, T) \times \Omega$ ,  $\Sigma_T \equiv (0, T) \times \Gamma$  and  $\Sigma_{iT} \equiv (0, T) \times \Gamma_i$ ,  $i = 0, 1$ , where  $T > 0$  is arbitrary. The proof of Theorem 2 proceeds through several steps.

**Proposition 1.** *There exists a constant  $C$ , independent of time  $0 < T < \infty$ , such that for all  $\epsilon > 0$  the solution component  $\vec{v}$  of (1)–(2) satisfies*

$$\begin{aligned} \int_{\epsilon}^{T-\epsilon} \left[ \left\| \mathring{A}^{\frac{1}{2}} v \right\|_{L^2(\Gamma_0)}^2 + \|v_t\|_{L^2(\Gamma_0)}^2 \right] dt &\leq C \left[ \left\| \mathring{A}^{\frac{1}{2}} v(T-\epsilon) \right\|_{L^2(\Gamma_0)}^2 + \|v_t(T-\epsilon)\|_{L^2(\Gamma_0)}^2 \right. \\ &\quad + \left\| \mathring{A}^{\frac{1}{2}} v(\epsilon) \right\|_{L^2(\Gamma_0)}^2 + \|v_t(\epsilon)\|_{L^2(\Gamma_0)}^2 \\ &\quad + \int_0^T \left( \|N^* A z_t\|_{L^2(\Gamma_0)}^2 + \left\| \mathring{A}^{\frac{\eta}{2}} v_t \right\|_{L^2(\Gamma_0)}^2 \right. \\ &\quad \left. \left. + \|g_2 \left( \mathring{A}^{\frac{\eta}{2}} v_t \right)\|_{L^2(\Gamma_0)}^2 \right) dt + \|v\|_{L^2(\Sigma_{0T})}^2 \right] \end{aligned}$$

*Proof.* As  $\mathring{A}$  is boundedly invertible, we easily have that

$$\int_0^T \|v_t\|_{L^2(\Gamma_0)}^2 dt \leq \left\| \mathring{A}^{-\frac{\eta}{2}} \right\| \int_0^T \left\| \mathring{A}^{\frac{\eta}{2}} v_t \right\|_{L^2(\Gamma_0)}^2 dt \tag{27}$$

Moreover, multiplying (13) by  $v$  and integrating from 0 to  $T$  yields

$$\begin{aligned} \int_0^T \left\| \mathring{A}^{\frac{1}{2}} v \right\|_{L^2(\Gamma_0)}^2 dt &= - \int_0^T \left( g_2 \left( \mathring{A}^{\frac{\eta}{2}} v_t \right), \mathring{A}^{\frac{\eta}{2}} v \right)_{L^2(\Gamma_0)} dt - \left[ (v_t, v)_{L^2(\Gamma_0)} \right]_0^T \\ &\quad + \int_0^T \left[ \|v_t\|_{L^2(\Gamma_0)}^2 - (N^* A z_t, v)_{L^2(\Gamma_0)} \right] dt \end{aligned} \tag{28}$$

(here we are implicitly invoking (17) and (18) so as to justify computations); using (27) and Cauchy-Schwarz on the right hand side of (28) will then give

$$\begin{aligned}
 \left(1 - \frac{\epsilon}{2}\right) \int_0^T \left\| \mathring{A}^{\frac{1}{2}} v \right\|_{L^2(\Gamma_0)}^2 dt &\leq C_\epsilon \left[ \int_0^T \left( \left\| N^* A z_t \right\|_{L^2(\Gamma_0)}^2 + \left\| \mathring{A}^{\frac{3}{2}} v_t \right\|_{L^2(\Gamma_0)}^2 \right) dt \right) \\
 &\quad + \int_0^T \left\| g_2 \left( \mathring{A}^{\frac{3}{2}} v_t \right) \right\|_{L^2(\Gamma_0)}^2 dt + \|v\|_{L^2(\Sigma_{0T})}^2 \\
 &\quad + \left\| \mathring{A}^{\frac{1}{2}} v(T) \right\|_{L^2(\Gamma_0)}^2 + \|v_t(T)\|_{L^2(\Gamma_0)}^2 \\
 &\quad + \left\| \mathring{A}^{\frac{1}{2}} v(0) \right\|_{L^2(\Gamma_0)}^2 + \|v_t(0)\|_{L^2(\Gamma_0)}^2 \Big] \tag{29}
 \end{aligned}$$

where  $C_\epsilon$  is independent of time. The result follows upon coupling (27) and (29), and replacing the interval  $(0, T)$  above with  $(\epsilon, T - \epsilon)$ . ■

We must likewise estimate the wave component  $\vec{z}$  of (1)–(2), for which we need the following:

**Lemma 1.** *Let  $\vec{w} \equiv [w, w_t] \in C([0, T]; H_1)$  satisfy the following wave equation:*

$$\begin{cases} w_{tt} = \Delta w & \text{on } Q_T \\ [w(t=0), w_t(t=0)] = \vec{w}_0 \in H_1 \\ w = 0 & \text{on } \Sigma_{1T} \end{cases} \tag{30}$$

with  $w_t|_{\Gamma_0}$  and  $(\partial w / \partial \nu)|_{\Gamma_0} \in L^2(0, T; L^2(\Gamma_0))$ . Then for every  $\epsilon > 0$ ,  $w$  satisfies the estimate

$$\begin{aligned}
 \int_\epsilon^{T-\epsilon} \left[ \left\| A^{\frac{1}{2}} w \right\|_{L^2(\Omega)}^2 + \|w_t\|_{L^2(\Omega)}^2 \right] dt &\leq C_T \left[ \int_{\Sigma_{0T}} \left[ \left( \frac{\partial w}{\partial \nu} \right)^2 + w_t^2 \right] d\Sigma_{0T} \right. \\
 &\quad + \|w\|_{H^{\frac{1}{2}+\epsilon}(Q_T)}^2 \Big] + C \left[ \left\| A^{\frac{1}{2}} w(T - \epsilon) \right\|_{L^2(\Omega)}^2 \right. \\
 &\quad + \|w_t(T - \epsilon)\|_{L^2(\Omega)}^2 + \left\| A^{\frac{1}{2}} w(\epsilon) \right\|_{L^2(\Omega)}^2 \\
 &\quad \left. + \|w_t(\epsilon)\|_{L^2(\Omega)}^2 \right] \tag{31}
 \end{aligned}$$

where the constant  $C_T$  depends upon time  $T$ , but constant  $C$  does not.



*Proof.* By Lemma 2.2 of (Lasićka and Tataru, 1993), there exists a sequence  $\{w^{(n)}\}_{n=1}^\infty \in C([0, T]; H^2(\Omega)) \cap C^1([0, T]; D(A^{1/2}))$  which satisfies

$$\begin{cases} w_{tt}^{(n)} = \Delta w^{(n)} \\ [w^{(n)}, w_t^{(n)}] \rightarrow [w, w_t] & \text{in } C([0, T]; H_1) \\ N^*Aw_t^{(n)} \rightarrow N^*Aw_t & \text{in } L^2(0, T; L^2(\Gamma_0)) \\ \frac{\partial w^{(n)}}{\partial \nu} \rightarrow \frac{\partial w}{\partial \nu} & \text{in } L^2(0, T; L^2(\Gamma_0)) \end{cases} \quad (32)$$

and it will thus suffice to prove the inequality (31) for a smooth solution  $w \in C([0, T]; H^2(\Omega)) \cap C^1([0, T]; D(A^{1/2}))$ . With the given vector field  $h(x)$  satisfying (H2)(i)–(ii), we have upon multiplying the wave equation in (30) by  $h \cdot \nabla w$  the standard identity (Triggiani, 1989, Appendix A):

$$\begin{aligned} \int_{Q_T} H \nabla w \cdot \nabla w \, dQ_T &= \int_{\Sigma_T} \frac{\partial w}{\partial \nu} h \cdot \nabla w \, d\Sigma_T + \frac{1}{2} \int_{\Sigma_T} w_t^2 h \cdot \nu \, d\Sigma_T \\ &\quad - \frac{1}{2} \int_{\Sigma_T} |\nabla w|^2 h \cdot \nu \, d\Sigma_T - \frac{1}{2} \int_{Q_T} \{w_t^2 - |\nabla w|^2\} \operatorname{div} h \, dQ_T \\ &\quad - \left[ (w_t, h \cdot \nabla w)_{L^2(\Omega)} \right]_0^T \end{aligned} \quad (33)$$

As  $[w, w_t] \in D(A^{1/2}) \times D(A^{1/2})$ , we then note that

$$\text{on } \Sigma_{1T}: w_t = 0; \quad \left| \frac{\partial w}{\partial \nu} \right| = |\nabla w|; \quad h \cdot \nabla w = h \cdot \nu \frac{\partial w}{\partial \nu}$$

and thus

$$\begin{aligned} \int_{\Sigma_{1T}} \frac{\partial w}{\partial \nu} h \cdot \nabla w \, d\Sigma_{1T} + \frac{1}{2} \int_{\Sigma_{1T}} w_t^2 h \cdot \nu \, d\Sigma_{1T} - \frac{1}{2} \int_{\Sigma_{1T}} |\nabla w|^2 h \cdot \nu \, d\Sigma_{1T} \\ = \frac{1}{2} \int_{\Sigma_{1T}} |\nabla w|^2 h \cdot \nu \, d\Sigma_{1T} \leq 0 \end{aligned} \quad (34)$$

after using the condition (H2)(i). Inserting (34) into (33) will therefore yield

$$\begin{aligned} \int_{Q_T} H \nabla w \cdot \nabla w \, dQ_T &\leq \int_{\Sigma_{0T}} \frac{\partial w}{\partial \nu} h \cdot \nabla w \, d\Sigma_{0T} + \frac{1}{2} \int_{\Sigma_{0T}} w_t^2 h \cdot \nu \, d\Sigma_{0T} \\ &\quad - \frac{1}{2} \int_{\Sigma_{0T}} |\nabla w|^2 h \cdot \nu \, d\Sigma_{0T} - \frac{1}{2} \int_{Q_T} \{w_t^2 - |\nabla w|^2\} \operatorname{div} h \, dQ_T \\ &\quad - \left[ (w_t, h \cdot \nabla w)_{L^2(\Omega)} \right]_0^T \end{aligned} \quad (35)$$

Hence, the implementation of condition (H2)(ii) and Cauchy-Schwarz on the estimate (35) will yield

$$\begin{aligned} \rho \int_{Q_T} |\nabla w|^2 dQ_T &\leq C \left[ \int_{\Sigma_{0T}} \left[ \left( \frac{\partial w}{\partial \nu} \right)^2 + w_t^2 \right] d\Sigma_{0T} \right. \\ &\quad + \int_{\Sigma_{0T}} |\nabla w|^2 d\Sigma_{0T} + \|\bar{w}(T)\|_{H_1}^2 + \|\bar{w}(0)\|_{H_1}^2 \\ &\quad \left. - \frac{1}{2} \int_{Q_T} \{w_t^2 - |\nabla w|^2\} \operatorname{div} h dQ_T \right] \end{aligned} \tag{36}$$

Now, to handle the last term on the right hand side of (36), we multiply the wave eqn. (30) by  $w \operatorname{div} h$  and integrate by parts to obtain

$$\begin{aligned} \int_{Q_T} \{w_t^2 - |\nabla w|^2\} \operatorname{div} h dQ_T &= \left[ (w_t, w \operatorname{div} h)_{L^2(\Omega)} \right]_0^T \\ &\quad + \int_{Q_T} w \nabla(\operatorname{div} h) \cdot \nabla w dQ_T \\ &\quad - \int_{\Sigma_{0T}} \frac{\partial w}{\partial \nu} w \operatorname{div} h d\Sigma_{0T} \end{aligned} \tag{37}$$

after using Green's Theorem and the identity  $\nabla(w \operatorname{div} h) \cdot \nabla w = w \nabla(\operatorname{div} h) \cdot \nabla w + |\nabla w|^2 \operatorname{div} h$ . We thus have upon majorizing the right hand side of (37) with the use of trace Theory and Poincaré's inequality,

$$\begin{aligned} \left| \int_{Q_T} \{w_t^2 - |\nabla w|^2\} \operatorname{div} h dQ_T \right| &\leq C_1 \left[ \int_{Q_T} w^2 dQ_T + \int_{\Sigma_{0T}} \left( \frac{\partial w}{\partial \nu} \right)^2 d\Sigma_{0T} + \|\bar{w}(T)\|_{H_1}^2 \right. \\ &\quad \left. + \|\bar{w}(0)\|_{H_1}^2 \right] + \frac{\epsilon}{4} \int_{Q_T} |\nabla w|^2 dQ_T \end{aligned} \tag{38}$$

where  $\epsilon > 0$  is arbitrarily small, and where the noncrucial dependence of  $C_1$  upon  $\epsilon$  has not been noted. Thus for  $\epsilon$  small enough, adding the inequalities (36) and (38) together yields

$$\begin{aligned} \left( \rho - \frac{\epsilon}{8} \right) \int_{Q_T} |\nabla w|^2 dQ_T &\leq C_1 \left[ \int_{\Sigma_{0T}} \left[ \left( \frac{\partial w}{\partial \nu} \right)^2 + w_t^2 \right] d\Sigma_{0T} + \int_{\Sigma_{0T}} |\nabla w|^2 d\Sigma_{0T} \right. \\ &\quad \left. + \|\bar{w}(T)\|_{H_1}^2 + \|\bar{w}(0)\|_{H_1}^2 + \int_{Q_T} w^2 dQ_T \right] \end{aligned} \tag{39}$$

Moreover, (38) and (39) together give:

$$\begin{aligned} \left(\rho - \frac{\epsilon}{8}\right) \int_{Q_T} w_t^2 dQ_T \leq C_1 \left[ \int_{\Sigma_{0T}} \left[ \left(\frac{\partial w}{\partial \nu}\right)^2 + w_t^2 \right] d\Sigma_{0T} + \int_{\Sigma_{0T}} |\nabla w|^2 d\Sigma_{0T} \right. \\ \left. + \|\vec{w}(T)\|_{H_1} + \|\vec{w}(0)\|_{H_1} + \int_{Q_T} w^2 dQ_T \right] \end{aligned} \quad (40)$$

(where the constants  $C_0$  and  $C_1$  above are not necessarily the same throughout). Replacing the interval  $(0, T)$  by  $(\epsilon, T - \epsilon)$  in the estimates (39) and (40), using the fact that on  $\Gamma$ ,  $|\nabla w|^2 = (\partial w / \partial \nu)^2 + (\partial w / \partial \tau)^2$ , and a subsequent application of Lemma A will give the desired estimate (31). The dependence on time of the constant  $C_T$  results then from the use of the microlocal estimate Lemma A. ■

Upon the combination of Proposition 1 and Lemma 1 (applied now to the solution component  $\vec{z}$  of (1)–(2)), we then have the preliminary estimate

$$\begin{aligned} \int_{\epsilon}^{T-\epsilon} E(\vec{z}, \vec{v}, t) dt \leq C_T \left[ \int_0^T \left( \|N^* Az_t\|_{L^2(\Gamma_0)}^2 + \|\mathring{A}^{\frac{1}{2}} v_t\|_{L^2(\Gamma_0)}^2 \right. \right. \\ \left. \left. + \|g_1(N^* Az_t)\|_{L^2(\Gamma_0)}^2 \right) dt + \int_0^T \|g_2(\mathring{A}^{\frac{1}{2}} v_t)\|_{L^2(\Gamma_0)}^2 dt \right. \\ \left. + \|z\|_{H^{\frac{1}{2}+\epsilon}(Q_T)}^2 + \|v\|_{L^2(\Sigma_{0T})}^2 \right] \\ + C \left[ E(\vec{z}, \vec{v}, T - \epsilon) + E(\vec{z}, \vec{v}, \epsilon) \right] \end{aligned} \quad (41)$$

where  $C_T$  depends upon  $T$ , but  $C$  does not. Applying the dissipativity property inherent in the relation (19), viz.  $\forall T \geq t_0 \geq 0$

$$\begin{aligned} E(\vec{z}, \vec{v}, t_0) = E(\vec{z}, \vec{v}, T) + \int_{t_0}^T \left[ \left( g_1(N^* Az_t), N^* Az_t \right)_{L^2(\Gamma_0)} \right. \\ \left. + \left( g_2(\mathring{A}^{\frac{1}{2}} v_t), \mathring{A}^{\frac{1}{2}} v_t \right)_{L^2(\Gamma_0)} \right] dt \end{aligned} \quad (42)$$

we can then couple (41) and (42) to obtain

**Proposition 2.** For time  $T$  large enough and  $\epsilon > 0$  sufficiently small, the following estimate holds for the solution  $[\vec{z}, \vec{v}]$  of (1)-(2):

$$\begin{aligned}
 E(\vec{z}, \vec{v}, T) \leq & \frac{C_T + 2C}{T - 2(C + \epsilon)} \left[ \int_0^T \left( \|N^*Az_t\|_{L^2(\Gamma_0)}^2 + \left\| \mathring{A}^{\frac{\eta}{2}}v_t \right\|_{L^2(\Gamma_0)}^2 \right. \right. \\
 & + \|g_1(N^*Az_t)\|_{L^2(\Gamma_0)}^2 \Big) dt + \int_0^T \|g_2(\mathring{A}^{\frac{\eta}{2}}v_t)\|_{L^2(\Gamma_0)}^2 dt \\
 & \left. + \|z\|_{H^{\frac{1}{2}+\epsilon}(Q_T)}^2 + \|v\|_{L^2(\Sigma_{0T})}^2 \right] \tag{43}
 \end{aligned}$$

Via a “nonlinear” compactness/uniqueness argument we now proceed to eliminate the lower order terms  $\|v\|_{L^2(\Sigma_{0T})}^2$  and  $\|z\|_{H^{1/2+\epsilon}(Q_T)}^2$ .

**Lemma 2.** With  $T$  sufficiently large, the inequality (43) implies that there exists a nonnegative constant  $C(E(\vec{z}, \vec{v}, 0))$  such that the solution  $[\vec{z}, \vec{v}]$  to (1)-(2) obeys the following inequality:

$$\begin{aligned}
 \|v\|_{C([0,T];L^2(\Gamma_0))}^2 + \|z\|_{H^{\frac{1}{2}+\epsilon}(Q_T)}^2 \leq & C(E(\vec{z}, \vec{v}, 0)) \left\{ \int_0^T \left[ \|z_t|_{\Gamma_0}\|_{L^2(\Gamma_0)}^2 \right. \right. \\
 & + \left\| \mathring{A}^{\frac{\eta}{2}}v_t \right\|_{L^2(\Gamma_0)}^2 + \|g_1(N^*Az_t)\|_{L^2(\Gamma_0)}^2 \\
 & \left. \left. + \|g_2(\mathring{A}^{\frac{\eta}{2}}v_t)\|_{L^2(\Gamma_0)}^2 \right] dt \right\} \tag{44}
 \end{aligned}$$

where the constant  $C(E(\vec{z}, \vec{v}, 0))$  remains bounded for bounded values of  $E(\vec{z}, \vec{v}, 0)$ .

*Proof.* If the lemma is false, there then exists a sequence of initial data  $\{\vec{z}_0^{(n)}, \vec{v}_0^{(n)}\}_{n=1}^\infty$  and a corresponding solution sequence  $\{z^{(n)}, v^{(n)}\}_{n=1}^\infty$  which satisfies for all  $n$ ,

$$\begin{cases} z_{tt}^{(n)} = \Delta z^{(n)} & \text{on } \Omega \times (0, T) \\ \frac{\partial z^{(n)}}{\partial \nu} = v_t^{(n)} - g_1(z_t^{(n)}) & \text{on } \Gamma_0 \times (0, T) \\ z^{(n)} = 0 & \text{on } \Gamma_1 \times (0, T) \\ \vec{z}^{(n)}(t=0) = \vec{z}_0^{(n)} \in H^1(\Omega) \times L^2(\Omega) \end{cases} \tag{45}$$

$$\begin{cases} v_{tt}^{(n)} = -\mathring{A}v^{(n)} - \mathring{A}^{\frac{\eta}{2}}g_2(\mathring{A}^{\frac{\eta}{2}}v_t^{(n)}) - z_t^{(n)} & \text{on } \Gamma_0 \times (0, T) \\ \vec{v}^{(n)}(t=0) = \vec{v}_0^{(n)} \in D(\mathring{A}^{\frac{1}{2}}) \times L^2(\Gamma_0) \end{cases} \tag{46}$$

with

$$\lim_{n \rightarrow \infty} \frac{\|v^{(n)}\|_{C([0,T];L^2(\Gamma_0))}^2 + \|z^{(n)}\|_{H^{\frac{1}{2}+\epsilon}(Q_T)}^2}{\int_0^T \left[ \|z_t^{(n)}\|_{L^2(\Gamma_0)}^2 + \|\mathring{A}^{\frac{\eta}{2}} v_t^{(n)}\|_{L^2(\Gamma_0)}^2 + \|g_1(N^*Az_t^{(n)})\|_{L^2(\Gamma_0)}^2 + \|g_2(\mathring{A}^{\frac{\eta}{2}}v_t^{(n)})\|_{L^2(\Gamma_0)}^2 \right] dt} = \infty \tag{47}$$

while the sequence of initial energy  $\{E(\overrightarrow{z^{(n)}}), \overrightarrow{v^{(n)}}, 0)\}_{n=1}^\infty$  is uniformly bounded in  $n$ . By the energy relation (19), the sequence  $\{E(\overrightarrow{z^{(n)}}), \overrightarrow{v^{(n)}}, t)\}_{n=1}^\infty$  is also bounded uniformly for  $0 \leq t \leq T$ , and consequently there exists a subsequence, still denoted by  $\{\overrightarrow{z^{(n)}}, \overrightarrow{v^{(n)}}\}_{n=1}^\infty$ , and  $[\overrightarrow{z}, \overrightarrow{v}] = [z, z_t, v, v_t]$  such that

$$\overrightarrow{z^{(n)}} \rightarrow \overrightarrow{z} \text{ in } L^\infty(0, T; H_1) \text{ weak star} \tag{48}$$

$$\overrightarrow{v^{(n)}} \rightarrow \overrightarrow{v} \text{ in } L^\infty(0, T; H_0) \text{ weak star} \tag{49}$$

We also have that  $z^{(n)} \rightarrow z$  weakly in  $H^1(Q_T)$ , and consequently by a classic compactness theorem (Lions and Magenes, 1972, p.99, Theorem 16.1) and Sobolev Trace Theory,

$$z^{(n)} \rightarrow z \text{ in } H^{\frac{1}{2}+\epsilon}(Q_T) \text{ strongly} \tag{50}$$

Moreover, we deduce from (49) and a compactness result of Simon's (1987, Corollary 4) that

$$v^{(n)} \rightarrow v \text{ in } C([0, T]; L^2(\Gamma_0)) \text{ strongly} \tag{51}$$

We now consider two possibilities for the limit  $[\overrightarrow{z}, \overrightarrow{v}]$ :

**Case I.**  $[\overrightarrow{z}, \overrightarrow{v}] \neq 0$ . Then with this assumption, the limit (47) implies that the sequences  $\{N^*Az_t^{(n)}\}$ ,  $\{\mathring{A}^{\eta/2}v_t^{(n)}\}$ ,  $\{g_1(N^*Az_t^{(n)})\}$  and  $\{g_2(\mathring{A}^{\eta/2}v_t^{(n)})\}$  each converge to 0 in  $L^2(\Sigma_{0T})$ . Upon passage to the limit in (45)–(46), we then have that  $[\overrightarrow{z}, \overrightarrow{v}]$  satisfies the system

$$\begin{cases} z_{tt} = \Delta z & \text{on } \Omega \times (0, T) \\ \frac{\partial z}{\partial \nu} = 0 & \text{on } \Gamma_0 \times (0, T) \\ z = 0 & \text{on } \Gamma_1 \times (0, T) \end{cases} \tag{52}$$

$$0 = \mathring{A}v \text{ on } \Gamma_0 \times (0, T) \tag{53}$$

As  $\mathring{A}$  is invertible, we immediately have that

$$v = 0 \tag{54}$$

Moreover, if we make the change of variable,  $w = z_t$ , then  $w$  solves

$$\begin{cases} w_{tt} = \Delta w & \text{on } \Omega \times (0, T) \\ \frac{\partial w}{\partial \nu} = w = 0 & \text{on } \Gamma_0 \times (0, T) \\ w = 0 & \text{on } \Gamma_1 \times (0, T) \end{cases} \tag{55}$$

Holmgren’s Uniqueness Theorem and (55) then implies that  $z_t = 0$  on  $Q_T$ , and consequently

$$z = 0 \tag{56}$$

after using the ellipticity of  $A$ . So  $[\vec{z}, \vec{v}] = 0$ , which contradicts our opening assumption.

**Case II.**  $[\vec{z}, \vec{v}] = 0$ . In this case, denoting

$$C_n \equiv \left( \|v^{(n)}\|_{C([0,T];L^2(\Gamma_0))}^2 + \|z^{(n)}\|_{H^{\frac{1}{2}+\epsilon}(Q_T)}^2 \right)^{\frac{1}{2}} \tag{57}$$

$$[\vec{z}^{(n)}, \vec{v}^{(n)}] \equiv \frac{1}{C_n} [z^{(n)}, v^{(n)}] \tag{58}$$

then

$$\|\vec{v}^{(n)}\|_{C([0,T];L^2(\Gamma_0))}^2 + \|\vec{z}^{(n)}\|_{H^{\frac{1}{2}+\epsilon}(Q_T)}^2 = 1 \text{ for every } n \tag{59}$$

and as  $[\vec{z}, \vec{v}] = 0$ , we have from (50) and (51) that  $\lim_{n \rightarrow \infty} C_n = 0$ . Also, one has *a fortiori* that  $[\vec{z}^{(n)}, \vec{v}^{(n)}]$  satisfies

$$\begin{cases} \vec{z}_{tt}^{(n)} = \Delta \vec{z}^{(n)} & \text{on } \Omega \times (0, T) \\ \frac{\partial \vec{z}^{(n)}}{\partial \nu} = \vec{v}_t^{(n)} - \frac{g_1(z_t^{(n)})}{C_n} & \text{on } \Gamma_0 \times (0, T) \\ \vec{z}^{(n)} = 0 & \text{on } \Gamma_1 \times (0, T) \\ \vec{z}^{(n)}(t=0) = \vec{z}_0^{(n)} \in H^1(\Omega) \times L^2(\Omega) \end{cases} \tag{60}$$

$$\begin{cases} \vec{v}_{tt}^{(n)} = -\mathbf{A}\vec{v}^{(n)} - \frac{\mathbf{A}^{\frac{\eta}{2}}g_2(\mathbf{A}^{\frac{\eta}{2}}v_t^{(n)})}{C_n} - \vec{z}_t^{(n)} & \text{on } \Gamma_0 \times (0, T) \\ \vec{v}^{(n)}(t=0) = \vec{v}_0^{(n)} \in D(\mathbf{A}^{\frac{1}{2}}) \times L^2(\Gamma_0) \end{cases} \tag{61}$$

with  $[\vec{z}_0^{(n)}, \vec{v}_0^{(n)}] = (1/C_n)[z_0^{(n)}, v_0^{(n)}]$ . In addition, (58) and (47) imply that

$$N^* \mathbf{A} \vec{z}_t^{(n)}, \mathbf{A}^{\frac{\eta}{2}} \vec{v}_t^{(n)} \rightarrow 0 \text{ in } L^2(0, T; L^2(\Gamma_0)) \text{ as } n \rightarrow \infty \tag{62}$$

Moreover, using the dissipative relation (42) (applied to  $[\vec{z}^{(n)}, \vec{v}^{(n)}]$ ), followed by the estimate (43), we have for all  $t \in (0, T]$ ,

$$\begin{aligned}
 E(\vec{z}^{(n)}, \vec{v}^{(n)}, t) dt \leq C_T & \left[ \int_0^T \|z_t^{(n)}|_{\Gamma_0}\|_{L^2(\Gamma_0)}^2 + \|\mathbf{A}^{\frac{\alpha}{2}} v_t^{(n)}\|_{L^2(\Gamma_0)}^2 \right. \\
 & + \|g_1(N^*Az_t^{(n)})\|_{L^2(\Gamma_0)}^2 + \|g_2(\mathbf{A}^{\frac{\alpha}{2}} v_t^{(n)})\|_{L^2(\Gamma_0)}^2 \\
 & \left. + \|v^{(n)}\|_{C([0,T];L^2(\Gamma_0))}^2 + \|z^{(n)}\|_{H^{\frac{1}{2}+\epsilon}(Q_T)}^2 \right] \quad (63)
 \end{aligned}$$

(where the constant  $C_T$  here is different from that in (43)). Dividing both sides of this inequality by  $C_n^2$  and subsequently considering (47), we then have that  $E(\vec{z}^{(n)}, \vec{v}^{(n)}, t)$  is uniformly bounded for  $0 \leq t \leq T$ . Thus there is a subsequence  $\{[\vec{z}^{(n)}, \vec{v}^{(n)}]\}$  and  $[\vec{z}, \vec{v}]$  such that

$$\begin{aligned}
 \vec{z}^{(n)} &\rightharpoonup \vec{z} && \text{in } L^\infty(0, T; H_1) && \text{weak star} \\
 \vec{v}^{(n)} &\rightharpoonup \vec{v} && \text{in } L^\infty(0, T; H_0) && \text{weak star} \\
 \vec{z}^{(n)} &\rightarrow \vec{z} && \text{in } H^{\frac{1}{2}+\epsilon}(Q_T) && \text{strongly} \\
 \vec{v}^{(n)} &\rightarrow \vec{v} && \text{in } C([0, T]; L^2(\Gamma_0)) && \text{strongly}
 \end{aligned}$$

The last two convergences above and (59) yield that

$$\|\vec{v}\|_{C([0,T];L^2(\Gamma_0))}^2 + \|\vec{z}\|_{H^{\frac{1}{2}+\epsilon}(Q_T)}^2 = 1 \quad (64)$$

But as  $g_1(N^*Az_t^{(n)})/C_n \rightarrow 0$  and  $g_2(\mathbf{A}^{\alpha/2} v_t^{(n)})/C_n \rightarrow 0$  in  $L^2(0, T; L^2(\Gamma_0))$ , by (47), we can then pass to the limit in (60)–(61), after recalling the convergences in (62). Subsequently invoking ellipticity and Holmgren’s Theorem, as was done in the final part of Case I, we arrive at  $\vec{z} = 0$  and  $\vec{v} = 0$ , a conclusion which contradicts (64). The proof of Lemma 2 is hence complete.

We thus have upon combining Proposition 2 and Lemma 2

**Proposition 3.** *For  $T > 0$  large enough, the solution  $[\vec{z}, \vec{v}]$  to (1)–(2) satisfies*

$$\begin{aligned}
 E(\vec{z}, \vec{v}, T) \leq C_T (E(\vec{z}, \vec{v}, 0)) & \int_0^T \left[ \|N^*Az_t\|_{L^2(\Gamma_0)}^2 + \|\mathbf{A}^{\frac{\alpha}{2}} v_t\|_{L^2(\Gamma_0)}^2 \right. \\
 & \left. + \|g_1(N^*Az_t)\|_{L^2(\Gamma_0)}^2 + \|g_2(\mathbf{A}^{\frac{\alpha}{2}} v_t)\|_{L^2(\Gamma_0)}^2 \right] dt \quad (65)
 \end{aligned}$$

where the constant  $C_T(E(\vec{z}, \vec{v}, 0))$  remains bounded for bounded values of  $E(\vec{z}, \vec{v}, 0)$ .

**2.1. Conclusion of Theorem 2**

Let

$$\Sigma_\alpha \equiv \{(t, x) \in \Sigma_{0T} \ni |v_t| > 1 \text{ a.e.}\}$$

$$\Sigma_\beta \equiv \Sigma_{0T} \setminus \Sigma_\alpha$$

Then using hypothesis (H1)(iii), we obtain

$$\int_{\Sigma_\alpha} \left( g_2^2 \left( \mathring{A}^{\frac{\eta}{2}} v_t \right) + \left( \mathring{A}^{\frac{\eta}{2}} v_t \right)^2 \right) d\Sigma_\alpha \leq (m_2^{-1} + M_2) \int_{\Sigma_\alpha} g_2 \left( \mathring{A}^{\frac{\eta}{2}} v_t \right) \mathring{A}^{\frac{\eta}{2}} v_t d\Sigma_\alpha \tag{66}$$

Moreover, from (21)

$$\int_{\Sigma_\beta} \left( g_2^2 \left( \mathring{A}^{\frac{\eta}{2}} v_t \right) + \left( \mathring{A}^{\frac{\eta}{2}} v_t \right)^2 \right) d\Sigma_\beta \leq \int_{\Sigma_\beta} f_2 \left( g_2 \left( \mathring{A}^{\frac{\eta}{2}} v_t \right) \mathring{A}^{\frac{\eta}{2}} v_t \right) d\Sigma_\beta \tag{67}$$

Then by Jensen’s inequality,

$$\begin{aligned} \int_{\Sigma_\beta} f_2 \left( g_2 \left( \mathring{A}^{\frac{\eta}{2}} v_t \right) \mathring{A}^{\frac{\eta}{2}} v_t \right) d\Sigma_\beta &\leq \text{meas}(\Sigma_{0T}) f_2 \left( \frac{1}{\text{meas}(\Sigma_{0T})} \int_{\Sigma_\beta} g_2 \left( \mathring{A}^{\frac{\eta}{2}} v_t \right) \mathring{A}^{\frac{\eta}{2}} v_t d\Sigma_\beta \right) \\ &= \text{meas}(\Sigma_{0T}) \tilde{f}_2 \left( \int_{\Sigma_\beta} g_2 \left( \mathring{A}^{\frac{\eta}{2}} v_t \right) \mathring{A}^{\frac{\eta}{2}} v_t d\Sigma_\beta \right) \end{aligned} \tag{68}$$

where  $\tilde{f}_2(s) \equiv f_2(s/\text{meas}(\Sigma_{0T}))$ . Thus,

$$\begin{aligned} \int_{\Sigma_{0T}} \left( g_2^2 \left( \mathring{A}^{\frac{\eta}{2}} v_t \right) + \left( \mathring{A}^{\frac{\eta}{2}} v_t \right)^2 \right) d\Sigma_{0T} &\leq (m_2^{-1} + M_2) \int_{\Sigma_{0T}} g_2 \left( \mathring{A}^{\frac{\eta}{2}} v_t \right) \mathring{A}^{\frac{\eta}{2}} v_t d\Sigma_{0T} \\ &\quad + \text{meas}(\Sigma_{0T}) \tilde{f}_2 \left( \int_{\Sigma_{0T}} g_2 \left( \mathring{A}^{\frac{\eta}{2}} v_t \right) \mathring{A}^{\frac{\eta}{2}} v_t d\Sigma_{0T} \right) \end{aligned} \tag{69}$$

In the very same way as was done above, we have

$$\begin{aligned} \int_{\Sigma_{0T}} \left( g_1^2 \left( N^* A z_t \right) + \left( N^* A z_t \right)^2 \right) d\Sigma_{0T} &\leq (m_1^{-1} + M_1) \int_{\Sigma_{0T}} g_1 \left( N^* A z_t \right) N^* A z_t d\Sigma_{0T} \\ &\quad + \text{meas}(\Sigma_{0T}) \tilde{f}_1 \left( \int_{\Sigma_{0T}} g_1 \left( N^* A z_t \right) N^* A z_t d\Sigma_{0T} \right) \end{aligned} \tag{70}$$



where  $\tilde{f}_1(s) \equiv f_1(s/\text{meas}(\Sigma_{0T}))$ . Splicing together (65) and (69)–(70), and further recalling the definition (22), we have

$$\begin{aligned}
 & E(\vec{z}, \vec{v}, T) \\
 & \leq C_T(E(\vec{z}, \vec{v}, 0)) \left[ M \int_{\Sigma_{0T}} \left[ g_1(N^*Az_t) N^*Az_t + g_2\left(\dot{A}^{\frac{\eta}{2}}v_t\right) \dot{A}^{\frac{\eta}{2}}v_t \right] d\Sigma_{0T} \right. \\
 & \left. + \text{meas}(\Sigma_{0T}) \tilde{f} \left( \int_{\Sigma_{0T}} \left[ g_1(N^*Az_t) N^*Az_t + g_2\left(\dot{A}^{\frac{\eta}{2}}v_t\right) \dot{A}^{\frac{\eta}{2}}v_t \right] d\Sigma_{0T} \right) \right] \quad (71)
 \end{aligned}$$

where  $M \equiv m_1^{-1} + M_1 + m_2^{-1} + M_2$ . Setting

$$K \equiv \frac{1}{C_T(E(\vec{z}, \vec{v}, 0)) \text{meas}(\Sigma_{0T})}, \quad c \equiv \frac{M}{\text{meas}(\Sigma_{0T})}$$

we then obtain

$$\begin{aligned}
 p[E(\vec{z}, \vec{v}, T)] & \leq \int_{\Sigma_{0T}} \left[ g_1(N^*Az_t) N^*Az_t + g_2\left(\dot{A}^{\frac{\eta}{2}}v_t\right) \dot{A}^{\frac{\eta}{2}}v_t \right] d\Sigma_{0T} \\
 & = E(\vec{z}, \vec{v}, 0) - E(\vec{z}, \vec{v}, T) \quad (72)
 \end{aligned}$$

where the function  $p$  is as defined in (23). To finish the proof of Theorem 2, we invoke the following result from (Lasiecka and Tataru, 1993):

**Lemma B.** *Let  $p$  be a positive, increasing function such that  $p(0) = 0$ . Since  $p$  is increasing, we can define an increasing function  $q$ ,  $q(x) \equiv x - (I + p)^{-1}(x)$ . Consider a sequence  $s_n$  of positive numbers which satisfies*

$$s_{m+1} + p(s_{m+1}) \leq s_m$$

Then  $s_m \leq S(m)$ , where  $S(t)$  is a solution to the differential equation

$$\frac{d}{dt}S(t) + q(S(t)) = 0, \quad S(0) = s_0$$

Moreover, if  $p(x) > 0$  for  $x > 0$ , then  $\lim_{n \rightarrow \infty} S(t) = 0$ .

With this result in mind, we replace  $T$  (resp.  $0$ ) in (72) with  $(n + 1)T$  (resp.  $nT$ ) to obtain

$$E(\vec{z}, \vec{v}, (n + 1)T) + p(E(\vec{z}, \vec{v}, (n + 1)T)) \leq E(\vec{z}, \vec{v}, nT) \tag{73}$$

for  $n = 0, 1, \dots$ . Applying Lemma B with  $s_m \equiv E(\vec{z}, \vec{v}, mT)$  thus results in

$$E(\vec{z}, \vec{v}, nT) \leq S(n), \quad n = 0, 1, \dots \tag{74}$$

Finally, using the dissipativity of  $E(\vec{z}, \vec{v}, t)$  inherent in the relation (42), we have for  $t = nT + \tau, 0 \leq \tau \leq T$ ,

$$\begin{aligned} E(\vec{z}, \vec{v}, t) &\leq E(\vec{z}, \vec{v}, nT) \leq S(n) = S\left(\frac{t - \tau}{T}\right) \\ &\leq S\left(\frac{t}{T} - 1\right) \quad \text{for } t > T \end{aligned} \tag{75}$$

where we have used the fact that  $S(\cdot)$  is dissipative.

The proof of Theorem 2 is now complete.

### 3. Appendix—Proof of Theorem 1

*Proof of Theorem 1(i).* For the proof of well-posedness for the system (1)–(2), we invoke here nonlinear semigroup theory (see (Barbu, 1976) for a general treatise). To wit, for  $[\vec{z}, \vec{v}] \equiv [z_1, z_2, v_1, v_2] \in H_1 \times H_0$ , we will define the operator  $\mathcal{A}: D(\mathcal{A}) \subset H_1 \times H_0 \rightarrow H_1 \times H_0$  as

$$\mathcal{A} \begin{bmatrix} \vec{z} \\ \vec{v} \end{bmatrix} = \begin{bmatrix} z_2 \\ -Az_1 - ANg_1(N^*Az_2) + ANv_2 \\ v_2 \\ -N^*Az_2 - \mathring{A}v_1 - \mathring{A}^{\frac{\eta}{2}}g_2(\mathring{A}^{\frac{\eta}{2}}v_2) \end{bmatrix} \tag{76}$$

with  $D(\mathcal{A}) = \{[\vec{z}, \vec{v}] \in [D(A^{1/2})]^2 \times [D(\mathring{A}^{1/2})]^2 \ni z_1 + Ng_1(N^*Az_2) - Nv_2 \in D(A)$  and  $\mathring{A}^{1/2}v_1 + \mathring{A}^{(\eta-1)/2}g_2(\mathring{A}^{\eta/2}v_2) \in D(\mathring{A}^{1/2})\}$ , and proceed to show that  $-\mathcal{A}$  is maximal monotone.

For the monotonicity: If  $[\vec{z}, \vec{v}]$  and  $[\vec{z}, \vec{v}] \in D(\mathcal{A})$ , then

$$\begin{aligned}
 & \left( \mathcal{A} \begin{bmatrix} \vec{z} \\ \vec{v} \end{bmatrix} - \mathcal{A} \begin{bmatrix} \vec{z} \\ \vec{v} \end{bmatrix}, \begin{bmatrix} \vec{z} \\ \vec{v} \end{bmatrix} - \begin{bmatrix} \vec{z} \\ \vec{v} \end{bmatrix} \right)_{H_1 \times H_0} \\
 &= \left( \begin{bmatrix} z_2 - \tilde{z}_2 \\ -Az_1 - ANg_1(N^*Az_2) + ANv_2 - (-A\tilde{z}_1 - ANg_1(N^*A\tilde{z}_2) + AN\tilde{v}_2) \\ -N^*Az_2 - \mathbf{A}v_1 - \mathbf{A}^{\frac{\eta}{2}}g_2(\mathbf{A}^{\frac{\eta}{2}}v_2) - (-N^*A\tilde{z}_2 - \mathbf{A}\tilde{v}_1 - \mathbf{A}^{\frac{\eta}{2}}g_2(\mathbf{A}^{\frac{\eta}{2}}\tilde{v}_2)) \end{bmatrix}, \begin{bmatrix} z_1 - \tilde{z}_1 \\ z_2 - \tilde{z}_2 \\ v_1 - \tilde{v}_1 \\ v_2 - \tilde{v}_2 \end{bmatrix} \right)_{H_1 \times H_0} \\
 &= \left( A^{\frac{1}{2}}(z_2 - \tilde{z}_2), A^{\frac{1}{2}}(z_1 - \tilde{z}_1) \right)_{L^2(\Omega)} + \left\langle AN(v_2 - \tilde{v}_2), (z_2 - \tilde{z}_2) \right\rangle_{[D(A^{\frac{1}{2}})]' \times D(A^{\frac{1}{2}})} \\
 &\quad - \left( A^{\frac{1}{2}}(z_1 - \tilde{z}_1), A^{\frac{1}{2}}(z_2 - \tilde{z}_2) \right)_{L^2(\Omega)} \\
 &\quad - \left\langle ANg_1(N^*Az_2) - ANg_1(N^*A\tilde{z}_2), z_2 - \tilde{z}_2 \right\rangle_{[D(A^{\frac{1}{2}})]' \times D(A^{\frac{1}{2}})} \\
 &\quad + \left( \mathbf{A}^{\frac{1}{2}}(v_2 - \tilde{v}_2), \mathbf{A}^{\frac{1}{2}}(v_1 - \tilde{v}_1) \right)_{L^2(\Gamma_0)} - \left( (N^*Az_2 - N^*A\tilde{z}_2), (v_2 - \tilde{v}_2) \right)_{L^2(\Gamma_0)} \\
 &\quad - \left( \mathbf{A}^{\frac{1}{2}}(v_1 - \tilde{v}_1), \mathbf{A}^{\frac{1}{2}}(v_2 - \tilde{v}_2) \right)_{L^2(\Gamma_0)} \\
 &\quad - \left\langle \mathbf{A}^{\frac{\eta}{2}}g_2(\mathbf{A}^{\frac{\eta}{2}}v_2) - \mathbf{A}^{\frac{\eta}{2}}g_2(\mathbf{A}^{\frac{\eta}{2}}\tilde{v}_2), v_2 - \tilde{v}_2 \right\rangle_{[D(\mathbf{A}^{\frac{1}{2}})]' \times D(\mathbf{A}^{\frac{1}{2}})} \\
 &= - \left( g_1(N^*Az_2) - g_1(N^*A\tilde{z}_2), N^*Az_2 - N^*A\tilde{z}_2 \right)_{L^2(\Gamma_0)} \\
 &\quad - \left( g_2(\mathbf{A}^{\frac{\eta}{2}}v_2) - g_2(\mathbf{A}^{\frac{\eta}{2}}\tilde{v}_2), \mathbf{A}^{\frac{\eta}{2}}v_2 - \mathbf{A}^{\frac{\eta}{2}}\tilde{v}_2 \right)_{L^2(\Gamma_0)} \leq 0
 \end{aligned}$$

after using the hypotheses (H1)–(H2) (note in particular that (H1)(iii) implies that  $g_1(N^*Az_2), g_2(\mathbf{A}^{\eta/2}v_2) \in L^2(\Gamma_0)$  for  $z_2 \in D(A^{1/2})$  and  $v_2 \in D(\mathbf{A}^{1/2})$ ). Thus  $-\mathcal{A}$  is monotone.

To show that  $-\mathcal{A}$  is maximal, we will verify the equivalent statement  $\mathcal{R}(I - \mathcal{A}) = H_1 \times H_0$ : For arbitrary  $[\vec{\phi}, \vec{\psi}] \in H_1 \times H_0$ , if there exists  $[\vec{z}, \vec{v}] \in D(\mathcal{A})$  such that

$$\begin{bmatrix} \vec{z} \\ \vec{v} \end{bmatrix} - \mathcal{A} \begin{bmatrix} \vec{z} \\ \vec{v} \end{bmatrix} = \begin{bmatrix} \vec{\phi} \\ \vec{\psi} \end{bmatrix} \tag{77}$$

then

$$\begin{aligned}
 & \begin{cases} z_1 - z_2 = \phi_1 \\ z_2 + Az_1 + ANg_1(N^*Az_2) - ANv_2 = \phi_2 \\ v_1 - v_2 = \psi_1 \\ v_2 + N^*Az_2 + \mathring{A}v_1 + \mathring{A}^{\frac{\eta}{2}}g_2\left(\mathring{A}^{\frac{\eta}{2}}v_2\right) = \psi_2 \end{cases} \\
 & \iff \\
 & \begin{cases} z_2 + Az_2 + ANg_1(N^*Az_2) - ANv_2 = \phi_2 - A\phi_1 \\ v_2 + N^*Az_2 + \mathring{A}v_2 + \mathring{A}^{\frac{\eta}{2}}g_2\left(\mathring{A}^{\frac{\eta}{2}}v_2\right) = \psi_2 - \mathring{A}\psi_1 \end{cases} \\
 & \iff \\
 & (\mathbf{F} + \mathbf{G}) \begin{bmatrix} z_2 \\ v_2 \end{bmatrix} = \begin{bmatrix} \phi_2 - A\phi_1 \\ \psi_2 - \mathring{A}\psi_1 \end{bmatrix} \tag{78}
 \end{aligned}$$

where  $\mathbf{F}, \mathbf{G}: D(A^{1/2}) \times D(\mathring{A}^{1/2}) \rightarrow [D(A^{1/2})]' \times [D(\mathring{A}^{1/2})]'$  are defined by

$$\mathbf{F} \equiv \begin{bmatrix} \mathbf{I} + A & -AN \\ N^*A & \mathbf{I} + \mathring{A} \end{bmatrix} \quad \text{and} \quad \mathbf{G} \begin{bmatrix} z \\ v \end{bmatrix} \equiv \begin{bmatrix} ANg_1(N^*Az) \\ \mathring{A}^{\frac{\eta}{2}}g_2\left(\mathring{A}^{\frac{\eta}{2}}v\right) \end{bmatrix} \tag{79}$$

In regard to the sum of these operators, we have the following:

**Proposition 4.**  $\mathcal{R}(\mathbf{F} + \mathbf{G}) = [D(A^{\frac{1}{2}})]' \times [D(\mathring{A}^{\frac{1}{2}})]'$ .

*Proof.* *A fortiori*,  $\mathbf{F} \in \mathcal{L}(D(A^{1/2}) \times D(\mathring{A}^{1/2}), [D(A^{1/2})]' \times [D(\mathring{A}^{1/2})]')$  is coercive, so if we can establish that  $\mathbf{G}$  is maximal monotone as a mapping of  $D(A^{1/2}) \times D(\mathring{A}^{1/2})$  into  $[D(A^{1/2})]' \times [D(\mathring{A}^{1/2})]'$ , the asserted result will promptly follow upon application of Corollary 1.3 of (Barbu, 1976, p.48). To this end, we first consider the component  $ANg_1(N^*Az)$  of  $\mathbf{G}$ : As  $g_1$  is monotone increasing, then  $g_1(\cdot) = \partial\Phi(\cdot)$  as a mapping from  $L^2(\Gamma_0)$  into itself, where  $\Phi$  is some proper, convex, lower semicontinuous functional on  $L^2(\Gamma_0)$ , and  $\partial\Phi$  denotes the subgradient of  $\Phi$  (Brezis, 1972, p.37). In particular then,

$$g_1(N^*A(\cdot)) = \partial\Phi(N^*A(\cdot))$$

as a mapping from  $D(A^{1/2})$  into  $L^2(\Gamma_0)$ . Since  $N^*A: D(A^{1/2}) \rightarrow H^{1/2}(\Gamma_0)$  is surjective, by (9) and Sobolev Trace theory, we then can invoke Lemma 2.1 of (Lasiecka, 1989) to obtain

$$\partial(\Phi N^*A(\cdot)) = ANg_1(N^*A(\cdot))$$

This above equality directly yields that  $ANg_1(N^*A(\cdot))$  is maximal monotone as a mapping of  $D(A^{1/2})$  into  $[D(A^{1/2})]'$ . By an almost identical line of reasoning,  $\mathring{A}^{\eta/2}g_2(\mathring{A}^{\eta/2}(\cdot))$  is maximal monotone as a mapping of  $D(\mathring{A}^{1/2})$  into  $[D(\mathring{A}^{1/2})]'$ ,

thereby giving that the entire structure  $\mathbf{G}$  is maximal monotone. The proof of Proposition 4 is concluded. ■

Proposition 1 thus provides the existence of  $[z_2, v_2] \in D(A^{1/2}) \times D(\mathring{A}^{1/2})$  which solves the operator equation (78), and setting

$$z_1 \equiv z_2 + \phi_1$$

$$v_1 \equiv v_2 + \psi_1$$

then *a fortiori*  $[\vec{z}, \vec{v}] = [z_1, z_2, v_1, v_2] \in D(\mathcal{A})$ , and using the equivalence between the equations (77) and (78), we deduce that  $\mathcal{R}(I - \mathcal{A}) = H_1 \times H_0$ . Consequently,  $-\mathcal{A}$  generates a nonlinear semigroup of contractions  $\{\mathcal{T}(t)\}_{t \geq 0}$  on  $\overline{D(\mathcal{A})} = H_1 \times H_0$ , and so for initial data  $[\vec{z}_0, \vec{v}_0] \in H_1 \times H_0$ ,  $[\vec{z}(t), \vec{v}(t)] \equiv \mathcal{T}(t) \left[ \begin{smallmatrix} \vec{z}_0 \\ \vec{v}_0 \end{smallmatrix} \right]$  will be the (weak) solution to (1)–(2), thereby proving Theorem 1(i).

We now endeavor to show the regularity posted in Theorem 1(ii), which is to be used in the computations performed in the proof of the stability result of Theorem 2; we start first with a preliminary result.

**Proposition 5.** *Suppose, in addition to the hypotheses (H1), that  $g_1$  and  $g_2$  are coercive; viz. for all  $s_1, s_2 \in \mathbb{R}$ , there exist positive constants  $\alpha_i, i = 1, 2$ , such that  $(g_i(s_1) - g_i(s_2))(s_1 - s_2) \geq \alpha_i |s_1 - s_2|^2$ . Then the following regularity properties hold true for the respective velocities of the solution  $[z, v]$  to (1)–(2):*

$$z_t|_{\Gamma_0} \in L^2(0, \infty; L^2(\Gamma_0)) \tag{80}$$

$$v_t \in L^2\left(0, \infty; D(\mathring{A}^{\frac{\eta}{2}})\right) \tag{81}$$

Consequently, because of (H1)(iii), we have also that  $g_1(N^*Az_t)$  and  $g_2(\mathring{A}^{\eta/2}v_t) \in L^2(0, \infty; L^2(\Gamma_0))$ .

*Proof.* If  $[\vec{z}_0, \vec{v}_0] \in D(\mathcal{A})$ , then by the semigroup property the corresponding solution  $[\vec{z}, \vec{v}]$  satisfies

$$\frac{d}{dt} \left[ \begin{smallmatrix} \vec{z} \\ \vec{v} \end{smallmatrix} \right] \in L^\infty(0, \infty; H_1 \times H_0) \tag{82}$$

with  $[\vec{z}(t), \vec{v}(t)] \in D(\mathcal{A})$  for  $t \geq 0$ . So as  $N^*A \in \mathcal{L}\left(D(A^{\frac{1}{2}}), L^2(\Gamma_0)\right)$ , we deduce that

$$z_t|_{\Gamma_0} \in L^\infty(0, \infty; L^2(\Gamma_0)) \tag{83}$$

(82) further implies that

$$\mathring{A}^{\frac{1}{2}}v_t \in L^\infty(0, \infty; L^2(\Gamma_0)) \tag{84}$$

Because of (H1)(iii), (83) and (84), we then have in turn

$$g_1(N^*Az_t) \in L^2(0, \infty; L^2(\Gamma_0)) \quad \text{and} \quad g_2\left(\mathring{A}^{\frac{\eta}{2}}v_t\right) \in L^\infty(0, \infty; L^2(\Gamma_0)) \tag{85}$$

Thus mindful of (83)–(85), we can multiply the equation (12) by  $z_t$ , (13) by  $v_t$ , integrate in time and space, subsequently integrate by parts and invoke the coercivity assumption on the  $g_i$  to obtain

$$E(\vec{z}, \vec{v}, T) + \int_0^T \left[ \alpha_1 \|N^*Az_t\|_{L^2(\Gamma_0)}^2 + \alpha_2 \left\| \mathbf{A}^{\frac{\eta}{2}}v_t \right\|_{L^2(\Gamma_0)}^2 \right] dt \leq E(\vec{z}, \vec{v}, 0) \tag{86}$$

where  $E(\vec{z}, \vec{v}, t)$  is as defined in (3). Since  $\overline{D(\mathbf{A})} = H_1 \times H_0$ , the inequality above can be extended to all  $[\vec{z}_0, \vec{v}_0] \in H_1 \times H_0$  so as to provide the regularity (80) and (81). Moreover, the assumption (H1)(iii), (80) and (81) will give  $g_1(N^*Az_t), g_2(\mathbf{A}^{\eta/2}v_t) \in L^2(0, \infty; L^2(\Gamma_0))$ . ■

*Proof of Theorem 1(ii).* Following a theme played in (Lasiecka and Tataru, 1993), we will approximate the solution  $[\vec{z}, \vec{v}]$  to (1)–(2), corresponding to fixed initial data  $[\vec{z}_0, \vec{v}_0] \in H_1 \times H_0$ , by the solution  $[\vec{z}^{(n)}, \vec{v}^{(n)}]$  to the following parametrized system:

$$z_{tt}^{(n)} = -Az^{(n)} - ANg_1\left(N^*Az_t^{(n)}\right) - \frac{1}{n}ANN^*Az_t^{(n)} + ANv_t^{(n)} \quad \text{on } (0, \infty) \times \Omega \tag{87}$$

$$v_{tt}^{(n)} = -\mathbf{A}v^{(n)} - \mathbf{A}^{\frac{\eta}{2}}g_2\left(\mathbf{A}^{\frac{\eta}{2}}v_t^{(n)}\right) - \frac{1}{n}\mathbf{A}^\eta v_t^{(n)} - NA^*z_t^{(n)} \quad \text{on } (0, \infty) \times \Gamma_0 \tag{88}$$

$$[\vec{z}(t=0), \vec{v}(t=0)] = [\vec{z}_0, \vec{v}_0] \tag{89}$$

As the functions  $g_i^{(n)}(s) \equiv g_i(s) + s/n, i = 1, 2$ , satisfy the assumptions (H1), there is indeed a solution  $[\vec{z}^{(n)}, \vec{v}^{(n)}] \in C([0, \infty); H_1 \times H_0)$  to (87), and as each  $g_i^{(n)}$  is coercive, Proposition 5 gives the additional regularity

$$z_t^{(n)}|_{\Gamma_0} \in L^2(0, \infty; L^2(\Gamma_0)) \tag{90}$$

$$v_t^{(n)} \in L^2(0, \infty; D(\mathbf{A}^{\frac{\eta}{2}})) \tag{91}$$

We also have, after the use of (90), (91) and the hypotheses (H1)(iii), that

$$g_1(N^*Az_t^{(n)}) \in L^2(0, \infty; L^2(\Gamma_0)) \tag{92}$$

$$g_2\left(\mathbf{A}^{\frac{\eta}{2}}v_t^{(n)}\right) \in L^2(0, \infty; L^2(\Gamma_0)) \tag{93}$$

We work now to obtain a bound on  $E(\vec{z}^{(n)}, \vec{v}^{(n)}, t)$ , for  $0 \leq t \leq T$  and fixed  $T > 0$ , which is uniform with respect to  $n$ . First of all, the inclusions (90), (91) and (93) will allow for the equality in (88) to be taken in  $L^2(0, T; [D(\mathbf{A}^{1/2})]')$ ; so taking the duality pairing of both sides of (88) with respect to  $v_t^{(n)}$  and subsequently integrating from 0 to  $t, 0 \leq t \leq T$ , followed by an integration by parts (this procedure

being entirely justified by the regularity posted in (90) and (91)) will then give the relation

$$\begin{aligned}
 & -2 \int_0^t \left( g_2 \left( \mathbf{A}^{\frac{n}{2}} v_t^{(n)} \right) + \frac{1}{n} \mathbf{A}^{\frac{n}{2}} v_t^{(n)}, \mathbf{A}^{\frac{n}{2}} v_t^{(n)} \right)_{L^2(\Gamma_0)} + \|\overline{v_0}\|_{H_0}^2 \\
 & \quad - \left\| \mathbf{A}^{\frac{1}{2}} v^{(n)}(t) \right\|_{L^2(\Gamma_0)}^2 - \left\| v_t^{(n)}(t) \right\|_{L^2(\Gamma_0)}^2 \, d\tau \\
 & = 2 \int_0^t \left( N^* A z_t^{(n)}, v_t^{(n)} \right)_{L^2(\Gamma_0)} \, d\tau \tag{94}
 \end{aligned}$$

The same sort of energy method cannot be directly applied to the wave component of  $E(\overline{z^{(n)}}, \overline{v^{(n)}}; t)$ , since  $\overline{z^{(n)}}$  need not be smooth enough to justify an integration by parts. However, since  $z_t^{(n)}|_{\Gamma_0} \in L^2(0, \infty; L^2(\Gamma_0))$  and  $g_1(N^* A z_t^{(n)}) \in L^2(0, \infty; L^2(\Gamma_0))$  for all  $n$ , Proposition 2.1 of (Lasiecka and Tataru, 1993, p.512)) does in fact allow the desired energy relation for all  $t > 0$

$$\begin{aligned}
 \|z_0\|_{H_1}^2 & = 2 \int_0^t \left[ \left( g_1 \left( N^* A z_t^{(n)} \right) + \frac{1}{n} N^* A z_t^{(n)}, N^* A z_t^{(n)} \right)_{L^2(\Gamma_0)} \right. \\
 & \quad \left. - \left( v_t^{(n)}, N^* A z_t^{(n)} \right)_{L^2(\Gamma_0)} \right] d\tau \\
 & \quad + \left\| A^{\frac{1}{2}} z^{(n)}(t) \right\|_{L^2(\Omega)}^2 + \left\| z_t^{(n)}(t) \right\|_{L^2(\Omega)}^2 \tag{95}
 \end{aligned}$$

Adding (94) and (95), we obtain for all  $n$  and  $t$ ,  $0 \leq t \leq T$ ,

$$\begin{aligned}
 E \left( \overline{z^{(n)}}, \overline{v^{(n)}}; 0 \right) & = E \left( \overline{z^{(n)}}, \overline{v^{(n)}}; t \right) \\
 & \quad + \int_0^t \left( g_1 \left( N^* A z_t^{(n)} \right) + \frac{1}{n} N^* A z_t^{(n)}, N^* A z_t^{(n)} \right)_{L^2(\Gamma_0)} \, d\tau \\
 & \quad + \int_0^t \left( g_2 \left( \mathbf{A}^{\frac{n}{2}} v_t^{(n)} \right) + \frac{1}{n} \mathbf{A}^{\frac{n}{2}} v_t^{(n)}, \mathbf{A}^{\frac{n}{2}} v_t^{(n)} \right)_{L^2(\Gamma_0)} \, d\tau \tag{96}
 \end{aligned}$$

The energy relation above immediately gives that for all  $n$  and  $t$ ,  $0 \leq t \leq T$ ,

$$E \left( \overline{z^{(n)}}, \overline{v^{(n)}}; t \right) \leq \|\overline{z_0}\|_{H_1}^2 + \|\overline{v_0}\|_{H_0}^2 \tag{97}$$

Moreover, the same relation (96) and (H1)(iii) yield that for all  $n$

$$\left\| N^* A z_t^{(n)} \right\|_{L^2(\Sigma_{0T})} + \left\| \mathbf{A}^{\frac{n}{2}} v_t^{(n)} \right\|_{L^2(\Sigma_{0T})} \leq C \left( \|\overline{z_0}\|_{H_1}, \|\overline{v_0}\|_{H_0} \right) \tag{98}$$

Thus,  $\{\{\overrightarrow{z^{(n)}}, \overrightarrow{v^{(n)}}\}\}_{n=1}^\infty$  is uniformly bounded in  $L^\infty(0, T; H_1 \times H_0)$  and  $\{[N^*Az_t^{(n)}, \mathring{A}^{\frac{\eta}{2}}v_t^{(n)}]\}_{n=1}^\infty$  in  $L^2(\Sigma_{0T})$ ; consequently there exists a subsequence, still denoted as  $\{\overrightarrow{z^{(n)}}, \overrightarrow{v^{(n)}}\}_{n=1}^\infty$ , and  $[\overrightarrow{z}, \overrightarrow{v}] \in L^\infty(0, T; H_1 \times H_0)$  such that

$$[z^{(n)}, z_t^{(n)}] \rightarrow [\tilde{z}, \tilde{z}_t] \text{ weakly in } L^\infty(0, T; H_1) \tag{99}$$

$$v^{(n)} \rightarrow \tilde{v} \text{ weakly in } L^\infty\left(0, T; D(\mathring{A}^{\frac{1}{2}})\right) \tag{100}$$

$$v_t^{(n)} \rightarrow \tilde{v}_t \text{ weakly in } L^2\left(0, T; D(\mathring{A}^{\frac{\eta}{2}})\right) \tag{101}$$

Moreover, by Simon’s compactness result (Simon, 1987)  $z^{(n)} \rightarrow \tilde{z}$  in  $L^\infty(0, T; H^{1-\epsilon}(\Omega))$  strongly  $\forall \epsilon > 0$ , and this convergence coupled with Sobolev Trace Theory gives us that

$$\begin{aligned} z^{(n)}|_{\Gamma_0} &\rightarrow \tilde{z}|_{\Gamma_0} \text{ strongly in } L^\infty(0, T; L^2(\Gamma_0)) \\ z_t^{(n)}|_{\Gamma_0} &\rightarrow \tilde{z}_t|_{\Gamma_0} \text{ weakly in } L^2(0, T; L^2(\Gamma_0)) \end{aligned} \tag{102}$$

In addition,  $g_1(N^*Az_t^{(n)})$  and  $g_2(\mathring{A}^{\eta/2}v_t^{(n)})$  are both uniformly bounded in  $L^2(0, T; L^2(\Gamma_0))$ , using the hypotheses (H1)(iii) and (98), and so there are subsequences  $\{g_1(N^*Az_t^{(n)})\}_{n=1}^\infty$ ,  $\{g_2(\mathring{A}^{\eta/2}v_t^{(n)})\}_{n=1}^\infty$  and  $\omega_1, \omega_2 \in L^2(0, T; L^2(\Gamma_0))$ , say, which satisfy

$$g_1(N^*Az_t^{(n)}) \rightarrow \omega_1 \text{ weakly in } L^2(0, T; L^2(\Gamma_0)) \tag{103}$$

$$g_2(\mathring{A}^{\frac{\eta}{2}}v_t^{(n)}) \rightarrow \omega_2 \text{ weakly in } L^2(0, T; L^2(\Gamma_0)) \tag{104}$$

Letting  $[\overrightarrow{z^{(m)}}, \overrightarrow{v^{(m)}}]$  and  $[\overrightarrow{z^{(n)}}, \overrightarrow{v^{(n)}}]$  be elements of the solution sequence, and using the energy relation (96), this time applied to the solution  $[\overrightarrow{z}, \overrightarrow{v}] \equiv [\overrightarrow{z^{(m)}} - \overrightarrow{z^{(n)}}], [\overrightarrow{v^{(m)}} - \overrightarrow{v^{(n)}}]$  to the system

$$\begin{aligned} \widehat{z}_{tt} &= -A\widehat{z} - AN\left(g_1(N^*Az_t^{(m)}) - g_1(N^*Az_t^{(n)})\right) \\ &\quad - ANN^*A\left(\frac{1}{m}z_t^{(m)} - \frac{1}{n}z_t^{(n)}\right) + AN\widehat{v}_t \end{aligned} \tag{105}$$

$$\begin{aligned} \widehat{v}_{tt} &= -\mathring{A}\widehat{v} - \mathring{A}^{\frac{\eta}{2}}\left(g_2(\mathring{A}^{\frac{\eta}{2}}v_t^{(m)}) - g_2(\mathring{A}^{\frac{\eta}{2}}v_t^{(n)})\right) \\ &\quad - \mathring{A}^\eta\left(\frac{1}{m}v_t^{(m)} - \frac{1}{n}v_t^{(n)}\right) - NA^*\widehat{z}_t \end{aligned} \tag{106}$$

$$[\overrightarrow{z}(t=0), \overrightarrow{v}(t=0)] = [0, 0]$$



we obtain

$$\begin{aligned}
 & \left\| \mathring{A}^{\frac{1}{2}} \left( z^{(m)}(t) - z^{(n)}(t) \right) \right\|_{L^2(\Omega)}^2 + \left\| z_t^{(m)}(t) - z_t^{(n)}(t) \right\|_{L^2(\Omega)}^2 \\
 & + \left\| \mathring{A}^{\frac{1}{2}} \left( v^{(m)}(t) - v^{(n)}(t) \right) \right\|_{L^2(\Gamma_0)}^2 + \left\| v_t^{(m)}(t) - v_t^{(n)}(t) \right\|_{L^2(\Gamma_0)}^2 \\
 & + \int_0^t \left( g_1 \left( N^* A z_t^{(m)} \right) - g_1 \left( N^* A z_t^{(n)} \right), N^* A \left( z_t^{(m)} - z_t^{(n)} \right) \right)_{L^2(\Gamma_0)} d\tau \\
 & + \int_0^t \left( g_2 \left( \mathring{A}^{\frac{3}{2}} v_t^{(m)} \right) - g_2 \left( \mathring{A}^{\frac{3}{2}} v_t^{(n)} \right), \mathring{A}^{\frac{3}{2}} v_t^{(m)} - \mathring{A}^{\frac{3}{2}} v_t^{(n)} \right)_{L^2(\Gamma_0)} d\tau \\
 & \leq \left( \frac{1}{m} + \frac{1}{n} \right) \left[ \left\| N^* A z_t^{(m)} \right\|_{L^2(\Sigma_{0T})}^2 + \left\| \mathring{A}^{\frac{3}{2}} v_t^{(m)} \right\|_{L^2(\Sigma_{0T})}^2 \right. \\
 & \quad \left. + \left\| N^* A z_t^{(n)} \right\|_{L^2(\Sigma_{0T})}^2 + \left\| \mathring{A}^{\frac{3}{2}} v_t^{(n)} \right\|_{L^2(\Sigma_{0T})}^2 \right] \tag{107}
 \end{aligned}$$

The estimate above thus implies that actually  $[z^{(n)}, v^{(n)}] \rightarrow [\vec{z}, \vec{v}]$  strongly in  $C([0, T]; H_1 \times H_0)$ , and of particular pertinence here,

$$\begin{aligned}
 & \lim_{n \rightarrow \infty} \int_0^T \left[ \left( g_1 \left( N^* A z_t^{(n)} \right), N^* A z_t^{(n)} \right)_{L^2(\Gamma_0)} + \int_0^T \left( g_2 \left( \mathring{A}^{\frac{3}{2}} v_t^{(n)} \right), \mathring{A}^{\frac{3}{2}} v_t^{(n)} \right)_{L^2(\Gamma_0)} \right] d\tau \\
 & = \int_0^T \left( \omega_1, N^* A \tilde{z}_t \right)_{L^2(\Gamma_0)} + \left( \omega_2, \mathring{A}^{\frac{3}{2}} \tilde{v}_t \right)_{L^2(\Gamma_0)} d\tau \tag{108}
 \end{aligned}$$

where  $\omega_1$  (resp.  $\omega_2$ ) is the weak limit posted in (103) (resp. 104). With the limits (99)–(104) and (108), and the fact that each  $g_i$  is monotonic increasing, we can then invoke Lemma 1.3 of (Barbu, 1976, p.42) to infer that

$$\omega_1 = g_1(N^* A \tilde{z}_t)$$

$$\omega_2 = g_2(\mathring{A}^{\frac{3}{2}} \tilde{v}_t)$$

With these convergences, along with those posted in (99)–(104), one can then pass to the limit in (87)–(88) so as to obtain the deduction that the weak limit  $[\vec{z}, \vec{v}] = [\vec{z}, \vec{v}]$ , where again  $[\vec{z}, \vec{v}]$  is the solution to (1)–(2). In particular then, we have by (102) and (101) the additional regularity

$$z_t|_{\Gamma_0} \in L^2(0, T; L^2(\Gamma_0)), \quad v_t \in L^2(0, T; D(\mathring{A}^{\frac{3}{2}}))$$

for arbitrary  $T > 0$ . The proof of Theorem 1(ii) is concluded.

*Proof of Theorem 1(iii).* The details here are identical to the derivation of the parametrized energy relations (94)–(96), and so will not be repeated.

The proof of Theorem 1 is now complete.

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