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AN ALGEBRAIC APPROACH TO MODELLING AND PERFORMANCE OPTIMISATION OF A TRAFFIC ROUTE SYSTEM

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An algebraic approach to performance optimisation of collision-free traffic route systems is presented. The proposed models take into account traffic conditions in a wide area and react to the dynamic nature of the traffic flow. The adjustment of traffic-signal timings to minimise the total travel time through a city is considered. The approach is based on the $(\max, +)$ and $(\min, +)$ algebras which provide a framework to build an executable performance-oriented model for sequential and repetitive processes like a set of signalised intersections co-ordinating the traffic access to the routes through a sequence of signal timings. The concept of a quasi-rendez-vous synchronisation mechanism of processes is introduced. A computer example is provided in the final part of the paper to illustrate the effectiveness of the approach.

1. Introduction

The expanding transportation industry involves safety and congestion problems in urban areas. Signalised intersections in arterial networks are important components affecting the smooth traffic flow. In the present paper, the traffic route system (TRS) for a wide area in a city is considered as a class of discrete-event dynamic systems (DEDS). It consists of cyclic processes (the so-called traffic light processes, TLPs) such as signalised intersections which co-ordinate the traffic access to the routes through a sequence of signal timings (a signal timing plan, STP).

Many studies have been devoted in recent years to investigation of the performance of urban traffic systems, (Bretherton *et al.*, 1994; Guo and Hu, 1994). However, the problem has not been fully settled yet, in part because of interrupted traffic patterns in saturated flow conditions and signal policies related to turning vehicles. Simulations and queuing theory have been used to study activities at individual intersections. Since the activities at each intersection are asynchronous, such methods would be inappropriate to model a group of asynchronous activities. Signal models have been explored to find an optimum cycle time for traffic lights. These models

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have been developed mostly for isolated intersections which ignore the formation of platoons (D'Souza and Banaszak, 1995; McShane and Roess, 1990).

The capability of predicting the behaviour of TRSs (e.g. in order to estimate a throughway capacity, flexibility, and delay times) hinges on two types of knowledge. The first and most powerful one is the knowledge of the laws underlying a given phenomenon. When this knowledge is expressed in the form of equations which can, in principle, be solved, one can predict the future outcome of an experiment once the initial conditions are completely specified. The other method relies on discovering strong empirical regularities in the observed behaviour of the system, e.g. based on the data collected from simulation experiments. The approach proposed here follows the former. It aims at providing an algebraic model for performance evaluation of large TRSs. The goal is to find a state equation to calculate traffic routing characteristics without cumbersome and time-consuming simulations while providing a framework for the design of real-time control systems. The interest in such models is still growing in view of the increased congestion and accident rates.

In order to build an executable control-oriented model for TRS, the $(\max, +)$ and $(\min, +)$ algebras are used (Baccelli *et al.*, 1992; Obuchowicz and Banaszak, 1995). The model based on the $(\min, +)$ algebra rises a possibility to obtain the time required to travel along a given route (Obuchowicz *et al.*, 1995; Zaremba *et al.*, 1996). Thus, if a set of routes and signal timings are given, then this model can be used for creating a testing procedure which calculates the global time T_G of waiting for green signals. T_G is treated as a performance index because it influences the traffic fume and capacity. The $(\min, +)$ algebra model, however, is not a useful tool to determine a procedure for the adjustment of traffic signal timings minimising T_G. This is because the traffic signal timings cannot be calculated immediately in this framework.

If an urban traffic system can be considered as a discrete-event dynamic system, then a $(\max, +)$ algebra model can be constructed. This model is based on the concept of a "green line", i.e. a route such that the vehicles moving along it do not wait for green lights at all passed intersections. It forces a quasi-rendez-vous synchronisation of the TLPs corresponding to the intersections passed by the "green line." Two TLPs are said to have a quasi-rendez-vous if every vehicle moving along the route connecting the corresponding intersections does not have to wait for green lights. Thus the problem of TLPs synchronisation becomes similar to the well-known problem of the rendez-vous synchronisation of a cyclic process, where the $(\max, +)$ algebra was successfully applied (Braker, 1993; Obuchowicz and Banaszak, 1995).

The paper is organised as follows. In Section 2 the main problem is formulated. The respective formalism of $(\min, +)$ and $(\max, +)$ algebras is introduced in Section 3. A $(\min, +)$ algebraic model of vehicles routing is proposed in Section 4. Section 5 presents the main results based on a $(\max, +)$ algebraic model of traffic signal synchronisation, as well as an illustrative example of a TRS. In the last section we summarise the paper and indicate some directions of future research.

2. Problem Statement

2.1. Smooth-Flow Traffic Problem

Let us consider the traffic route system for a wide urban area, i.e. a regular, mesh-like arterial network with signalised intersections. Each pair of neighbouring intersections in the network is specified by the mean travel time required to drive from one position in the city to another. This means that owing to a given speed limit the relevant distance between two neighbouring intersections can easily be calculated. The traffic signal timings at the intersections are limited by the period of changing lights. The minimal periods are limited by both the time required to cross the street by pedestrians and the velocities of vehicles passing along one direction from one intersection to another. On the other hand, the maximal periods result from the traffic flow along a particular route.

Given a set of traffic routes, the problem under consideration consists in determining the signal-timing plans (the light timings) and their relative phases which minimise the overall time which vehicles have to spend at the signalised intersections while waiting for the green light and following the presumed routes.

Of course, the constraints regarding the minimal and maximal periods of signal timings may contradict one another and, accordingly, no solution can be found. Therefore, in what follows we restrict our attention to the case where no limits are specified for the maximal period of light timings. In this way, a solution to the problem, i.e. a timetable of the signal timings, follows from the period of an arterial network which is caused by the cyclic nature of distributed TLPs.

The problem specification can be represented in terms of a marked-graph formalism. Since the marked-graph representation can be transformed to a state-graph specification (which is, in turn, equivalent to a state equation), a timetable of signal timings can be easily calculated within the framework of a $(\max, +)$ algebra.

2.2. Illustrative Example

A city traffic route system (TRS), shown in Fig. 1, is given. It consists of two four-directional intersections I and II and of a three-directional intersection III. Based on the analysis of traffic streams in the TRS, i.e. the average flow per hour, important routes with significant streams can be selected (routes 1, 2 and 3 in Fig. 1). If T_G is defined as the overall waiting time (for green lights) for vehicles moving along these routes, then the TRS with $T_G = 0$ (routes 1, 2 and 3 are "green routes") is treated as the most efficient system. Thus the problem reduces to finding a signal-timing plan (STP) (which co-ordinates the traffic access to the routes through a sequence of signal timings), its schedule and related traffic signal phases between directly-connected intersections, while minimising the overall waiting time T_G (for green lights) for the vehicles moving along the selected routes.

For a given intersection, there are many possibilities of choosing a signal scheduling (SS) which guarantee an admissible, i.e. collision-free, flow through this intersec-

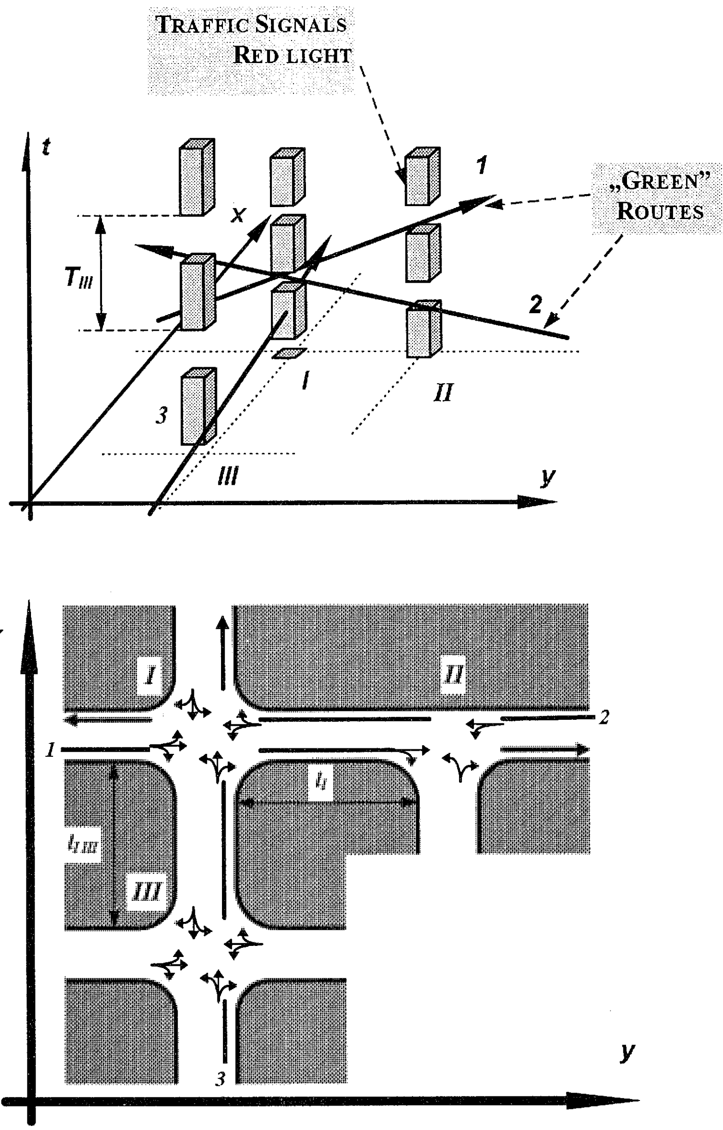


Fig. 1. Street layout and signalised intersection location: I, II, III—signalised intersection location; 1, 2, 3—routes with significant streams ("green routes").

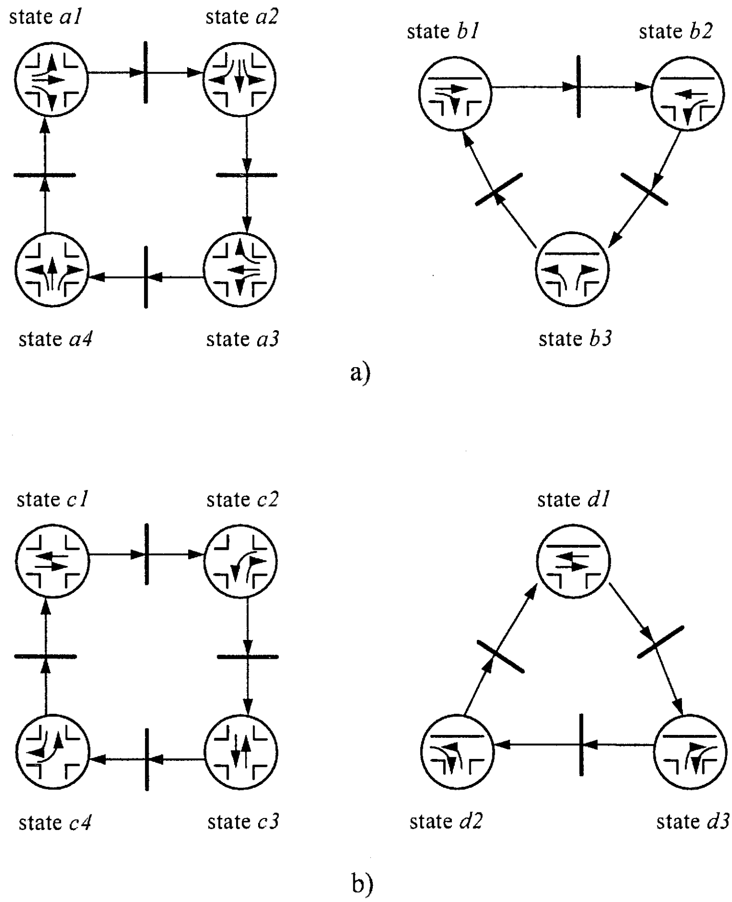


Fig. 2. Two possibilities of choosing SS which guarantee a collision-free flow through the intersection: (a) SS-A and (b) SS-B, for four-directional and three-directional intersections. The directions of movement for vehicles having green lights are marked with arrows in circles.

tion. Two examples presented in Fig. 2. The SS-A (Fig. 2(a)) permits the vehicles arriving from only one direction to cross the intersection. In the SS-B (Fig. 2(b)) vehicles can cross the intersection simultaneously from the i-th to the j-th direction and from the j-th to the i-th direction. If an SS is chosen, then the minimal time interval g_{ki} during which the k-th intersection has to be in the i-th state (e.g. this is the time required to pass the intersection by a given number of vehicles or the time needed to pass the crosswalk by pedestrians) can be determined.

3. Formalism

In this section the $(\max, +)$ and $(\min, +)$ algebras are introduced. Some graph-theoretical interpretations of the equations written in this formalism are presented. A new operation called modulo is also defined.

3.1. (Max, +) Algebra

Definition 1. The $(\max, +)$ algebraic structure $(\mathbb{R}_{\max}, \oplus, \otimes)$ is defined as follows:

- $\mathbb{R}_{\max} = \mathbb{R} \cup \{-\infty\}$, where $\mathbb{R}$ is the field of real numbers;
- $\forall a, b \in \mathbb{R}_{\max} : a \oplus b = \max(a, b)$;
- $\forall a, b \in \mathbb{R}_{\max} : a \otimes b = a + b$ and $\forall a \in \mathbb{R}_{\max} : a \otimes (-\infty) = (-\infty) \otimes a = (-\infty)$.

Remark 1. The $(\max, +)$ algebra exhibits the following properties:

- associativity of addition: $\forall a, b, c \in \mathbb{R}_{\max} : (a \oplus b) \oplus c = a \oplus (b \oplus c)$,
- commutativity of addition: $\forall a, b \in \mathbb{R}_{\max} : a \oplus b = b \oplus a$,
- associativity of multiplication: $\forall a, b, c \in \mathbb{R}_{\max} : (a \otimes b) \otimes c = a \otimes (b \otimes c)$,
- right and left distributivity of multiplication over addition:
  $\forall a, b, c \in \mathbb{R}_{\max} : (a \oplus b) \otimes c = (a \otimes b) \oplus (b \otimes c)$
  $\forall a, b, c \in \mathbb{R}_{\max} : a \otimes (b \oplus c) = (a \otimes b) \oplus (a \otimes c)$
- existence and absorbing of a zero element:
  $\exists \varepsilon \in \mathbb{R}_{\max}\ \forall a \in \mathbb{R}_{\max} : a \oplus \varepsilon = a \quad (\varepsilon = -\infty)$
  $\forall a \in \mathbb{R}_{\max} : a \otimes \varepsilon = \varepsilon$
- existence of an identity element: $\exists e \in \mathbb{R}_{\max}\ \forall a \in \mathbb{R}_{\max} : a \otimes e = a \quad (e = 0)$,
- any element of the $\mathbb{R}_{\max}$ space has no inverse with respect to the operation $\oplus$, because the equation $a \oplus x = \varepsilon$ has a solution only for $a = \varepsilon$ and $x = \varepsilon$.

Definition 2. The *parallel composition* (sum) $\oplus$ of matrices $\boldsymbol{A} = (A_{ij})$ and $\boldsymbol{B} = (B_{ij})$ of the same size is defined by the equation

$$(\boldsymbol{A} \oplus \boldsymbol{B})_{ij} = A_{ij} \oplus B_{ij} \tag{1}$$

Definition 3. The *series composition* (product) $\otimes$ of an $m \times n$ matrix $\boldsymbol{A} = (A_{ij})$ and an $n \times p$ matrix $\boldsymbol{B} = (B_{ij})$ is defined by

$$(\boldsymbol{A} \otimes \boldsymbol{B})_{ij} = \bigoplus_{k=1}^{n} (A_{ik} \otimes B_{kj}) \tag{2}$$

Definition 4. The *spectral equation* in the $(\max, +)$-algebra language has the form

$$\boldsymbol{A} \otimes \boldsymbol{x} = \lambda \otimes \boldsymbol{x} \tag{3}$$

Proposition 1. (Baccelli *et al.*, 1992) *Every square matrix has at least one eigenvalue. An* irreducible square matrix *has a unique eigenvalue. (A matrix $\boldsymbol{A}$ is called irreducible if there exists no permutation matrix $\boldsymbol{P}$ such that $\boldsymbol{P}^T \boldsymbol{A}\boldsymbol{P}$ is upper triangular, where $\boldsymbol{P}^T$ is the transpose of $\boldsymbol{P}$.)*

3.2. Correspondence of Graphs and Matrices

Definition 5. A *digraph* $G(\boldsymbol{A})$ corresponding to an $n \times n$ matrix $\boldsymbol{A}$ is the pair (ϑ, ζ), where ϑ is the set of the vertices of $G(\boldsymbol{A})$ and ζ is the set of the arcs of $G(\boldsymbol{A})$, such that the vertices are labelled from 1 to n, and

$$\forall i, j \in \vartheta : \ (j, i) \notin \zeta \Longleftrightarrow A_{ij} = \varepsilon \tag{4}$$

The value A_{ij} is the weight assigned to the arc (j, i). We can also define a matrix corresponding to the digraph $G(\vartheta, \zeta)$.

Remark 2. If a digraph $G(\vartheta, \zeta)$ is strongly-connected, i.e. if there exists a path from every node i to any node j, then the corresponding matrix is irreducible.

Definition 6. The *average weight* of a path $\rho = (i_1 \to i_2 \to \cdots \to i_{l-1} \to i_l)$ is defined as $\hat{w}_\rho = w_\rho / l_\rho$, where w_ρ is the *weight* of ρ (the sum of the weights of individual arcs), l_ρ is the *length* of ρ (the number of arcs in the path).

Definition 7. The *circuit mean* is the average weight of a circuit.

Proposition 2. (Baccielli *et al.*, 1992) *For a strongly-connected graph the* maximum circuit mean, *taken over all circuits, is equal to the eigenvalue of the corresponding matrix.*

3.3. (Min,+) Algebra

Definition 8. The $(\min, +)$ *algebraic structure* $(\mathbb{R}_{\min}, \vee, \bullet)$ is defined as follows:

- $\mathbb{R}_{\min} = \mathbb{R} \cup \{\infty\}$;
- $\forall a, b \in \mathbb{R}_{\min} : \ a \vee b = \min(a, b)$;
- $\forall a, b \in \mathbb{R}_{\min} : \ a \bullet b = a + b$ and $\forall a \in \mathbb{R}_{\min} : \ a \bullet \infty = \infty \bullet a = \infty$.

Proposition 3. *The* $(\min, +)$ *algebra is* isomorphic *to the* $(\max, +)$ *algebra.*

Proof. The proof is immediate if we select the required isomorphism in the form

$$\phi\colon \mathbb{R}_{\min} \longrightarrow \mathbb{R}_{\max}, \quad \phi(a) = -a \tag{5}$$

■

Remark 3. A consequence of the above-mentioned isomorphism is the inheritance of all properties of the $(\max, +)$ algebra by the $(\min, +)$ algebra, and conversely. In particular, the $(\min, +)$ algebra exhibits the properties included in Remark 1 if the element $\tau = \infty$ is treated as a zero element. Similarly, Proposition 1 is true if the *parallel* (sum) and *series compositions* (product) of the matrices, and the *spectral equation* are defined analogously to Definitions 2, 3 and 4, respectively, i.e. they take the forms

$$(\boldsymbol{A} \vee \boldsymbol{B})_{ij} = A_{ij} \vee B_{ij} \tag{6}$$

$$(\boldsymbol{A} \bullet \boldsymbol{B})_{ij} = \bigvee_{k=1}^{n} (A_{ik} \bullet B_{kj}) \tag{7}$$

$$\boldsymbol{A} \bullet \boldsymbol{x} = \lambda \bullet \boldsymbol{x} \tag{8}$$

One can also define a correspondence between a digraph $G(\vartheta, \zeta)$ and a matrix $\boldsymbol{A}$ in the same way as in Definition 5, by introducing the condition

$$\forall i, j \in \vartheta : \ (i, j) \notin \zeta \Longleftrightarrow A_{ij} = \tau \tag{9}$$

in lieu of (4).

Lemma 1. *If $\boldsymbol{A}$ is a matrix with non-negative elements and the diagonal elements equal to e, then*

$$\exists k_0 \ \forall k \geq k_0 : \quad \boldsymbol{A}_{\bullet}^{k+1} = \boldsymbol{A}_{\bullet}^{k} \tag{10}$$

where $\boldsymbol{A}_{\bullet}^{k} = \boldsymbol{A} \bullet \boldsymbol{A}_{\bullet}^{k-1}, \quad \boldsymbol{A}_{\bullet}^{0} = \boldsymbol{E}_{\bullet}, \quad (\boldsymbol{E}_{\bullet})_{ij} = \begin{cases} e & \text{for } i = j \\ \tau & \text{for } i \neq j \end{cases}.$

Proof. If $\boldsymbol{A}$ has non-negative elements, then it is associated with a digraph with non-negative weights. The A_{ij} can be understood as the weight of a minimal path of unit length between vertices j and i. Thus $(\boldsymbol{A}_{\bullet}^{k})_{ij}$ specifies the weight of the shortest path of the length equal to k between vertices j and i. The resulting shortest path $i \to j$ cannot contain any cycles, because the path of the same arcs, apart from those of a cycle, has the weight being less than or equal to that of the path containing the cycle. For a finite n, each path which does not contain a cycle can have at most $n-1$ arcs. Thus there exists an upper limit for k_0. ■

3.4. Operator Modulo

Definition 9. The *modulo* operator $\Diamond$ is defined as follows:

$$\begin{aligned} &\forall a, t \in [e, \tau) \ \exists b \in [e, t) : \ a \Diamond t = b \Longleftrightarrow \exists k \in \mathbb{N} \ \ a = b \bullet t_{\bullet}^{k} \\ &\forall a \in [e, \tau) : \ a \Diamond \tau = a; \quad \forall a, \in [e, \tau) : \ \tau \Diamond a = \tau \end{aligned} \tag{11}$$

where $\mathbb{N}$ denotes the set of natural numbers. If $\boldsymbol{A} = (A_{ij})$ is an $m \times n$ matrix with non-negative entries and $\boldsymbol{b} = (b_i)$ is an m-element vector with non-negative elements, then $\boldsymbol{B} = \boldsymbol{A} \Diamond \boldsymbol{b}$ is an $m \times n$ matrix defined by

$$B_{ij} = A_{ij} \Diamond b_i \tag{12}$$

It is easy to see that b is the remainder if a is divided by an integer number t (k times).

4. Vehicle Routing

In this section two problems are considered: calculation of the time required to drive along a given route in the TRS and finding the shortest route (in the sense of time) between two points. A model based on the $(\min, +)$ algebra is used to solve these problems.

4.1. A System Without Signalised Intersections

The formalism of the $(\min, +)$ algebra is a natural language for the problem of seeking the minimal path between the vertices of a given graph. The digraph can model e.g. a street network and the minimal path can reflect the shortest (in the sense of the travel time) path between given points in the considered city, under the condition that directly passing through the intersections is guaranteed. Let $\boldsymbol{A} = [A_{ij}]_{i,j=0}^{n+1}$ be an $(n+2) \times (n+2)$ matrix with $A_{ii} = e$ and $A_{ij} = \tau$ for $i \neq j$ if there is no direct connection between the intersections i and j, and $A_{ij} = t_{ij}$ otherwise, t_{ij} being the travel time between two directly-connected intersections j and i. The starting and destination points are labelled with $i = 0$ and $i = n+1$, respectively.

Lemma 2. *If y is the minimum travel time from a starting point to a destination point, then*

$$\exists k_0 : \quad y = x_{n+1}(k_0) = \left(\boldsymbol{A}_{\bullet}^{k_0} \bullet \boldsymbol{x}(0)\right)_{n+1} \tag{13}$$

where $\boldsymbol{x}(0) = [x_0 \ \tau \ \cdots \ \tau]^{\mathrm{T}}$ is an $(n+2)$-dimensional vector with the indices of elements $i = 0, 1, \dots, n+1$ and x_0 stands for the starting time of travel.

Proof. The i-th element of the vector $\boldsymbol{x}(1) = \boldsymbol{A} \bullet \boldsymbol{x}(0)$ represents the earliest time of the arrival at the i-th intersection along a road which is at most single-branched, $i = 1, \dots, n$, and at the destination point for $i = n+1$. If the element is equal to τ, such a one-branched road does not exist. Therefore the vector

$$\boldsymbol{x}(k) = \boldsymbol{A} \bullet \boldsymbol{x}(k-1) = \boldsymbol{A}_{\bullet}^{k} \bullet \boldsymbol{x}(0) \tag{14}$$

consists of minimal arriving times at the corresponding intersections along at most k-branched roads. Similarly,

$$\boldsymbol{x}(k+1) = \boldsymbol{A}_{\bullet}^{k+1} \bullet \boldsymbol{x}(0) \tag{15}$$

The matrix $\boldsymbol{A}$ is associated with a graph with non-negative weights which are equal to the time lengths between the intersections. Thus, based on Lemma 1, we obtain

$$\exists k_0 \ \forall k \geq k_0 : \ \boldsymbol{x}(k) = \boldsymbol{x}(k+1) \tag{16}$$

This result justifies that it is pointless to calculate the states later than $\boldsymbol{x}(k_0)$ and the proof is completed. ■

It is easy to see that the above algorithm is equivalent to the standard shortest-path algorithm.

Example 1. Let us consider the street network presented in Fig. 3. In order to simplify the calculations, we will assume that the average travel speed of the vehicle between the intersections is equal to unity. The matrix of the system is in the form

$$A = \begin{bmatrix} e & 5 & \tau & \tau & \tau & \tau \\ 5 & e & 10 & 15 & \tau & \tau \\ \tau & 10 & e & \tau & 15 & \tau \\ \tau & 15 & \tau & e & 20 & \tau \\ \tau & \tau & 15 & 20 & e & 5 \\ \tau & \tau & \tau & \tau & 5 & e \end{bmatrix}$$

Clearly, the longest paths without a cycle in this system have four edges.

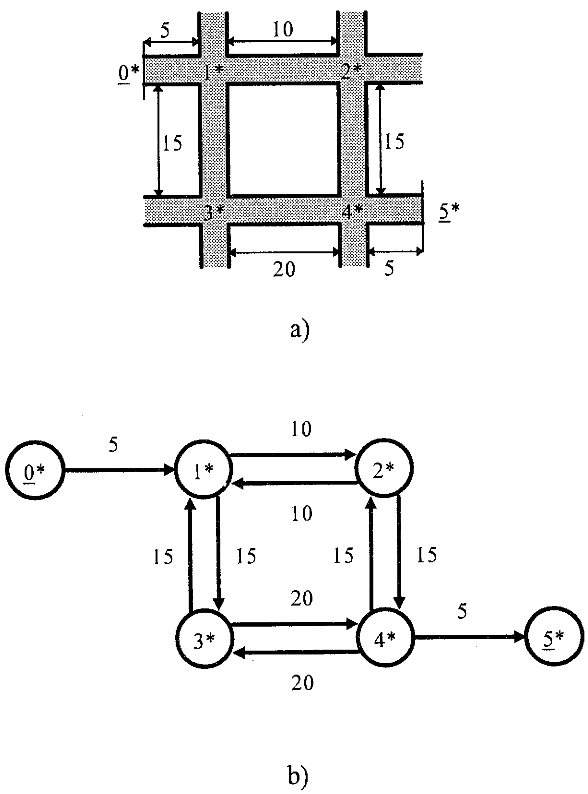


Fig. 3. (a) The arrangement of four intersections (1^* to 4^*) with given starting ($\underline{0}^*$) and destination ($\underline{5}^*$) points; distances between the intersections and points $\underline{0}^*$ and $\underline{5}^*$, in distance units, are also given; (b) the graph-theoretical representation of the system.

Thus $y = x_5(4) = (A_\bullet^4 \bullet x(0))_5 = 35$, where

$$A_\bullet^4 = \begin{bmatrix} e & 5 & 15 & 20 & 30 & 35 \\ 5 & e & 10 & 15 & 25 & 30 \\ 15 & 10 & e & 25 & 15 & 20 \\ 20 & 15 & 25 & e & 20 & 25 \\ 30 & 25 & 15 & 20 & e & 5 \\ 35 & 30 & 20 & 25 & 5 & e \end{bmatrix}$$

and the initial state $\boldsymbol{x}(0)$ is defined as in Lemma 2 with starting time $x_0 = e$.

Analysing the travel times between the individual points of the system, we easily observe that the route $\underline{0}^* \to 1^* \to 2^* \to 4^* \to \underline{5}^*$ leads to the minimum travel time.

♦

4.2. A System With Signalised Intersections

The necessary condition in accepting the thesis of Lemma 2 is the certainty that the vehicle will never be forced to wait to pass through the intersections in its route. This assumption is not very realistic with intersections where traffic lights exist. Let us define the following traffic light parameters:

- $\boldsymbol{T} = [T_i]_{i=0}^{n+1}$ as the traffic-light change period vector of a given intersection, where $T_0 = T_{n+1} = \tau$ and T_i is the time required by the traffic lights to change at the i-th intersection, $i = 1, \dots, n$;

- $\boldsymbol{R} = [R_{ij}]_{i,j=0}^{n}$ as an $(n+2) \times (n+2)$ matrix, $R_{ij} = r_{ij}$ for off-diagonal elements, if a direct connection exists between the i-th and the j-th point; $R_{ij} = e$ otherwise, r_{ij} being the time durations of the red light at the i-th intersection when coming from the j-th point of the system, $i = 1, \dots, n$, $j = 0, \dots, n+1$;

- $\boldsymbol{P} = [P_{ij}]_{i,j=0}^{n}$ as an $(n+2) \times (n+2)$ matrix, $P_{ij} = p_{ij}$ for off-diagonal elements if a direct connection exists between the i-th and the j-th point; $P_{ij} = e$ otherwise, p_{ij} being the traffic light phase at the i-th intersection at a fixed moment $t_0 = e$, counted from the last preceding lighting of the red light for the entrance onto that intersection from the j-th point of the network, $i = 1, \dots, n$, $j = 0, \dots, n+1$;

The yellow light is not considered in our model.

Example 2. In Fig. 3 a street network composed of four intersections is shown. The time distance matrix is represented by the matrix $\boldsymbol{A}$ of Example 1. Let the parameters of the intersections have the following values:

$$\text{period} \qquad\qquad \text{time of red light} \qquad\qquad \text{traffic light phases at } t_0 = e$$

$$\boldsymbol{T} = \begin{bmatrix} \tau \\ 8 \\ 20 \\ 16 \\ 12 \\ \tau \end{bmatrix}, \quad \boldsymbol{R} = \begin{bmatrix} e & e & e & e & e & e \\ 6 & e & 6 & 6 & e & e \\ e & 15 & e & e & 15 & e \\ e & 12 & e & e & 12 & e \\ e & e & 9 & 9 & e & e \\ e & e & e & e & e & e \end{bmatrix}, \quad \boldsymbol{P} = \begin{bmatrix} e & e & e & e & e & e \\ 5 & e & 1 & 3 & e & e \\ e & 2 & e & e & 7 & e \\ e & 6 & e & e & 10 & e \\ e & e & e & 3 & e & e \\ e & e & e & e & e & e \end{bmatrix}$$

♦

4.2.1. The Travel Time Along a Given Route

In this part we answer the question how much time a vehicle has to travel along a given route. Thus, in the matrices $\boldsymbol{A}$, $\boldsymbol{R}$ and $\boldsymbol{P}$ and in the vector $\boldsymbol{T}$, we take into account only those parameters of the system which concern the given route.

Theorem 1. *The travel time along a given route is given by*

$$y = x_{n+1}(n+1) \tag{17}$$

where x_{n+1} *is the* $(n+1)$-th *component of the state vector derived from the equations*

$$\begin{cases} \boldsymbol{x}(k) = \boldsymbol{B}(k-1) \bullet \boldsymbol{x}(k-1) \\ \boldsymbol{B}(k-1) = \boldsymbol{C} + \boldsymbol{Z}(k-1) \\ \boldsymbol{C} = \boldsymbol{A} + \boldsymbol{R} \\ \boldsymbol{Z}(k-1) = -\Big\{ \Big[\big(\boldsymbol{A} + \boldsymbol{P} + \mathbf{e} \bullet \boldsymbol{x}^T(k-1)\big) \Diamond \boldsymbol{T} \Big] \vee \boldsymbol{R} \Big\} \end{cases} \tag{18}$$

and $\boldsymbol{x}(0)$ *is defined in Lemma 2,* $\mathbf{e}$ *being the* $(n+2)$*-dimensional vector of all elements equal to* e.

Proof. Passing through one intersection. Let us assume that the travel route passes over only one intersection (Fig. 4). Let l_{10} be the distance between the starting point and the intersection, and v_{10} be the average travel speed along this part of the route. Moreover, we denote by l_{21} and v_{21} the distance between the intersection and the destination point, and the average travel speed along this part of the route, respectively. Thus the travel times along these parts of the route are

$$t_{i,i-1} = \frac{l_{i,i-1}}{v_{i,i-1}} \qquad \text{for } i = 1, 2 \tag{19}$$

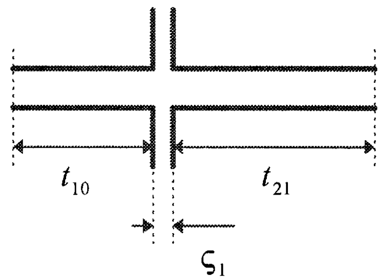


Fig. 4. Travel route through one intersection.

Let us assume that the travel time through the intersection is negligible, i.e. $\varsigma_1 = e$. Let the starting time of travel at the starting point be the reference time point $x_0 = e$ (Fig. 5). T_1 is the light changing period of the intersection and r_{10} is the time of the red light when the vehicle gets to the intersection (Fig. 5).

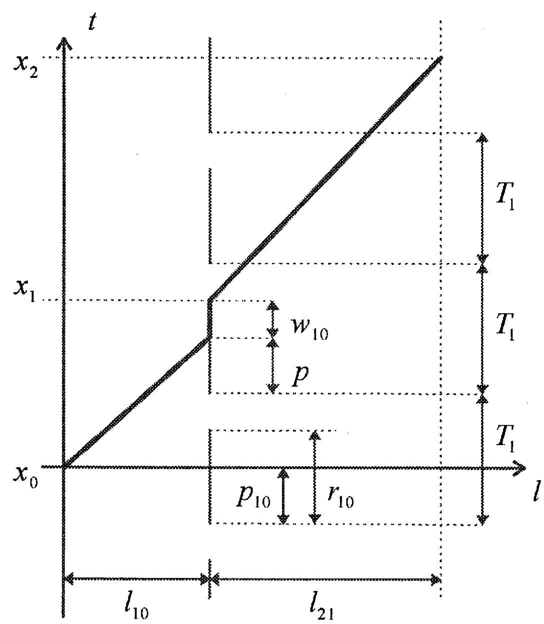


Fig. 5. Graphic interpretation of travelling along a route with one intersection.

If p is the time elapsed since the last red light, we say that the state of the traffic lights when reaching the intersection is in phase p. Thus the moment of the red-light lighting has zero phase and the moment of green light lighting has r_{10} phase. If the reference time point $t_0 = 0$ is laid at a fixed moment of red-light lighting, then the phase p at any moment t can be calculated from the formula

$$p = t \Diamond T_1 \tag{20}$$

However, as has been stated above, the reference time point is $t_0 = x_0 = e$. Then the traffic-light phase upon reaching the intersection, p_{10}, does not necessarily have to be equal to e. At that time

$$p = \big(p_{10} + (t \Diamond T_1)\big) \Diamond T_1 = (p_{10} + t) \Diamond T_1 \tag{21}$$

In particular, at the moment at which the vehicle reaches the intersection after travelling the distance l_{10}, the traffic light is equal to:

$$p = (p_{10} + x_0 + t_{10}) \Diamond T_1 = (p_{10} + t_{10}) \Diamond T_1 \tag{22}$$

If $p \geq r_{10}$, i.e. the vehicle encounters a green light, then it passes through the intersection directly, otherwise it must wait for $r_{10} - p$ units of time. Therefore the waiting time at the intersection is equal to

$$w_{10} = r_{10} - (p \vee r_{10}) \tag{23}$$

and the time of leaving the intersection is given by

$$x_1 = (t_{10} + w_{10}) \bullet x_0 = \big(c_{10} + z_{10}(x_0)\big) \bullet x_0 \tag{24}$$

where

$$c_{10} = t_{10} + r_{10}, \quad z_{10}(x_0) = -\big((t_{10} + p_{10} + x_0) \Diamond T_1\big) \vee r_{10} \tag{25}$$

The total travel time is equal to

$$y = t_{21} \bullet x_1 \tag{26}$$

According to the definitions of the matrices $\boldsymbol{A}$, $\boldsymbol{R}$, $\boldsymbol{P}$ and $\boldsymbol{T}$, the parameters describing the system composed of only one intersection, $n = 1$, are as follows:

$$\boldsymbol{A} = \begin{bmatrix} e & \tau & \tau \\ t_{10} & e & \tau \\ \tau & t_{21} & e \end{bmatrix}, \quad \boldsymbol{T} = \begin{bmatrix} \tau \\ T_1 \\ \tau \end{bmatrix}$$
$$\boldsymbol{R} = \begin{bmatrix} e & e & e \\ r_{10} & e & e \\ e & e & e \end{bmatrix}, \quad \boldsymbol{P} = \begin{bmatrix} e & e & e \\ p_{10} & e & e \\ e & e & e \end{bmatrix} \tag{27}$$

Moreover, we have the initial state vector in the form

$$x(0) = \begin{bmatrix} x_0 = e \\ x_1 = \tau \\ x_2 = \tau \end{bmatrix}$$

By analogy to (24) we define the matrices

$$C = A + R, \quad Z(0) = -\Big[\big(A + P + e \bullet x^T(0)\big)\Diamond T\Big] \vee R \tag{28}$$

where e is an $(n+2)$-dimensional vector consisting of the elements equal to e. Then

$$\begin{aligned} x(1) &= \big(C + Z(0)\big) \bullet x(0) \\ &= \left(\begin{bmatrix} e & \tau & \tau \\ t_{10} + r_{10} & e & \tau \\ \tau & t_{21} & e \end{bmatrix} + \begin{bmatrix} e & e & e \\ -\big[t_{10} + p_{10})\Diamond T_1\big] \vee r_{10} & e & e \\ e & e & e \end{bmatrix}\right) \otimes \begin{bmatrix} e \\ \tau \\ \tau \end{bmatrix} \\ &= \begin{bmatrix} e \\ t_{10} + r_{10} - \big[(t_{10} + p_{10})\Diamond T_1\big] \vee r_{10} \\ \tau \end{bmatrix} \end{aligned} \tag{29}$$

By calculating $Z(1)$ it can be shown, in accordance with (18), that

$$x(2) = \big(C + Z(1)\big) \bullet x(1) = \begin{bmatrix} e \\ t_{10} + r_{10} - \big[(t_{10} + p_{10})\Diamond T_1\big] \vee r_{10} \\ t_{10} + r_{10} - \big[(t_{10} + p_{10})\Diamond T_1\big] \vee r_{10} + t_{21} \end{bmatrix} \tag{30}$$

Subsequent iterations do not change the state vector. The individual elements of this vector represent the values of the starting times, leaving the intersection and reaching destination point. These values are in accordance with (23) and (25), and the rule of calculating subsequent state vectors is in accordance with the thesis of Theorem 1.

Passing through n intersections. Let us consider a more general case, i.e. passing through n intersections. The methodology of solving the problem does not change: it consists in repeating one intersection route n times. The time of leaving the i-th intersection is the starting time for the route with the $(i+1)$-th intersection. ∎

Example 3. Consider a route through three intersections illustrated in Fig. 6. Assume (the units are irrelevant) that $l_{10} = 720$, $l_{21} = 500$, $l_{32} = 600$, $l_{43} = 350$, $v_{10} = 60$, $v_{21} = 50$, $v_{32} = 30$, $v_{43} = 35$, $r_{10} = 13$, $r_{21} = 23$, $r_{32} = 13$, $p_{10} = 8$, $p_{21} = 25$, $p_{32} = 10$, $T_1 = 18$, $T_2 = 30$, $T_3 = 15$.

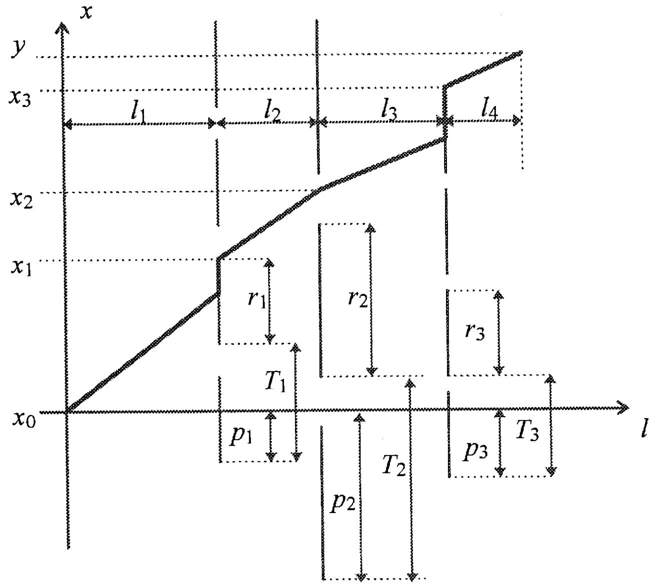


Fig. 6. Graphic interpretation of travelling along a route with three intersections.

Therefore $t_{10} = 12,\ t_{21} = 10,\ t_{32} = 20,\ t_{43} = 10,$

$$\boldsymbol{T} = \begin{bmatrix} \tau \\ 18 \\ 30 \\ 15 \\ \tau \end{bmatrix}, \qquad \boldsymbol{A} = \begin{bmatrix} 0 & \tau & \tau & \tau & \tau \\ 12 & 0 & \tau & \tau & \tau \\ \tau & 10 & 0 & \tau & \tau \\ \tau & \tau & 20 & 0 & \tau \\ \tau & \tau & \tau & 10 & 0 \end{bmatrix}$$

$$\boldsymbol{R} = \begin{bmatrix} e & e & e & e & e \\ 13 & e & e & e & e \\ e & 23 & e & e & e \\ e & e & 13 & e & e \\ e & e & e & e & e \end{bmatrix}, \qquad \boldsymbol{P} = \begin{bmatrix} e & e & e & e & e \\ 8 & e & e & e & e \\ e & 25 & e & e & e \\ e & e & 10 & e & e \\ e & e & e & e & e \end{bmatrix}$$

and hence

$$\boldsymbol{C} = \boldsymbol{A} + \boldsymbol{R} = \begin{bmatrix} e & \tau & \tau & \tau & \tau \\ 25 & e & \tau & \tau & \tau \\ \tau & 33 & e & \tau & \tau \\ \tau & \tau & 33 & e & \tau \\ \tau & \tau & \tau & 10 & e \end{bmatrix}$$

The subsequent iterations give

$$x(0) = \begin{bmatrix} e \\ \tau \\ \tau \\ \tau \\ \tau \end{bmatrix}, \quad B(0) = \begin{bmatrix} e & \tau & \tau & \tau & \tau \\ 23 & e & \tau & \tau & \tau \\ \tau & 10 & e & \tau & \tau \\ \tau & \tau & 20 & e & \tau \\ \tau & \tau & \tau & 10 & e \end{bmatrix}$$

$$x(1) = \begin{bmatrix} e \\ 23 \\ \tau \\ \tau \\ \tau \end{bmatrix}, \quad B(1) = \begin{bmatrix} e & \tau & \tau & \tau & \tau \\ 23 & e & \tau & \tau & \tau \\ \tau & 10 & e & \tau & \tau \\ \tau & \tau & 20 & e & \tau \\ \tau & \tau & \tau & 10 & e \end{bmatrix}$$

$$x(2) = \begin{bmatrix} e \\ 23 \\ 33 \\ \tau \\ \tau \end{bmatrix}, \quad B(2) = \begin{bmatrix} e & \tau & \tau & \tau & \tau \\ 23 & e & \tau & \tau & \tau \\ \tau & 10 & e & \tau & \tau \\ \tau & \tau & 30 & e & \tau \\ \tau & \tau & \tau & 10 & e \end{bmatrix}$$

$$x(3) = \begin{bmatrix} e \\ 23 \\ 33 \\ 63 \\ \tau \end{bmatrix}, \quad B(3) = \begin{bmatrix} e & \tau & \tau & \tau & \tau \\ 23 & e & \tau & \tau & \tau \\ \tau & 10 & e & \tau & \tau \\ \tau & \tau & 30 & e & \tau \\ \tau & \tau & \tau & 10 & e \end{bmatrix}$$

$$x(4) = \begin{bmatrix} e \\ 23 \\ 33 \\ 63 \\ 73 \end{bmatrix}$$

whence $y = x_4(4) = 73$. ♦

4.2.2. The Shortest Travel Time Between Given Two Points in the System

Let us consider a street system in which there exist many possible travel routes between given points in the system.

Theorem 2. *We have*

$$\exists k_0: \ y = x_{n+1}(k_0) = \left(\left(\overset{k_0-1}{\underset{j=0}{\bullet}} B(j) \right) \bullet x(0) \right)_{n+1} \tag{31}$$

where y is the shortest possible travel time between given points in the city, $\boldsymbol{B}(j)$ and $\boldsymbol{x}(0)$ being defined by (18) and (13), respectively.

Proof. The proof is similar to that in the classical problem of finding the shortest path in a graph (Lemma 2). The only thing left to do is to prove that $\boldsymbol{B}(j)$ is always non-negative. On the basis of (18) we get

$$\boldsymbol{Z}(j) \geq -\boldsymbol{R} \tag{32}$$

$$\boldsymbol{B}(j) = \boldsymbol{A} + \boldsymbol{R} + \boldsymbol{Z}(j) \geq \boldsymbol{A} + \boldsymbol{R} - \boldsymbol{R} = \boldsymbol{A} \tag{33}$$

and $\boldsymbol{A}$ the as the matrix of the time distances between the intersections in the city network, is non-negative. Therefore the matrix $\boldsymbol{B}(j)$ for any k is always non-negative. ■

Example 4. Let us consider all the intersections of the city street network in Example 1 (Fig. 3). The matrices $\boldsymbol{P}$ and $\boldsymbol{R}$ and the vector $\boldsymbol{T}$ are given in Example 2. Let the initial state be the same as in Example 1. Subsequent iterations produce

$$\boldsymbol{x}(0) = \begin{bmatrix} \boldsymbol{e} \\ \tau \\ \tau \\ \tau \\ \tau \\ \tau \end{bmatrix}, \quad \boldsymbol{B}(0) = \begin{bmatrix} e & 5 & \tau & \tau & \tau & \tau \\ 9 & e & 10 & 15 & \tau & \tau \\ \tau & 10 & e & \tau & 15 & \tau \\ \tau & 15 & \tau & e & 20 & \tau \\ \tau & \tau & 15 & 20 & e & 5 \\ \tau & \tau & \tau & \tau & 5 & e \end{bmatrix}$$

$$\boldsymbol{x}(1) = \begin{bmatrix} e \\ \boldsymbol{9} \\ \tau \\ \tau \\ \tau \\ \tau \end{bmatrix}, \quad \boldsymbol{B}(1) = \begin{bmatrix} e & 5 & \tau & \tau & \tau & \tau \\ 9 & e & 10 & 15 & \tau & \tau \\ \tau & 24 & e & \tau & 15 & \tau \\ \tau & 15 & \tau & e & 20 & \tau \\ \tau & \tau & 15 & 20 & e & 5 \\ \tau & \tau & \tau & \tau & 5 & e \end{bmatrix}$$

$$x(2) = \begin{bmatrix} e \\ 9 \\ 33 \\ \mathbf{24} \\ \tau \\ \tau \end{bmatrix}, \qquad B(2) = \begin{bmatrix} e & 5 & \tau & \tau & \tau & \tau \\ 9 & e & 12 & 19 & \tau & \tau \\ \tau & 24 & e & \tau & 15 & \tau \\ \tau & 15 & \tau & e & 20 & \tau \\ \tau & \tau & 24 & 20 & e & 5 \\ \tau & \tau & \tau & \tau & 5 & e \end{bmatrix}$$

$$x(3) = \begin{bmatrix} e \\ 9 \\ 33 \\ 24 \\ \mathbf{44} \\ \tau \end{bmatrix}, \qquad B(3) = \begin{bmatrix} e & 5 & \tau & \tau & \tau & \tau \\ 9 & e & 12 & 19 & \tau & \tau \\ \tau & 24 & e & \tau & 24 & \tau \\ \tau & 15 & \tau & e & 20 & \tau \\ \tau & \tau & 24 & 20 & e & 5 \\ \tau & \tau & \tau & \tau & 5 & e \end{bmatrix}$$

$$x(4) = \begin{bmatrix} e \\ 9 \\ 33 \\ 24 \\ 44 \\ \mathbf{49} \end{bmatrix}$$

whence $y = x_5(4) = 49$.

Comparing two alternative routes shown in Fig. 7, we see that the route with the shortest travel time is $\underline{0}^* \to 1^* \to 3^* \to 4^* \to \underline{5}^*$, unlike Example 1 (Fig. 3), where the shortest route is also the fastest route (passing through intersection 2^*). Introducing traffic-lights results in a longer route (through intersection 3^*) becoming faster. ♦

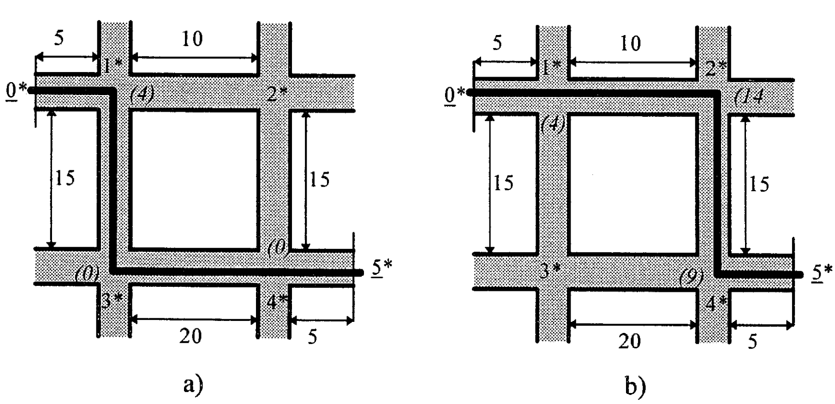


Fig. 7. Intersection arrangement of Fig. 3 with the fastest route (a) and an alternative route (b). The numbers in brackets denote the waiting times at each intersection.

Theorem 2 gives the shortest possible travel time between given starting and destination points in a city street network with traffic-light driven intersections. It does not clearly indicate, however, which route is optimal. To answer this question, Dijkstra's algorithm can be employed (Dijkstra, 1959; Dreyfus, 1969) in which an accessory index matrix is used. Its element (i, j) contains the index of the point in the network from which the j-th point is reached after at most i steps. The number of columns in this matrix is equal to the number of points in the system, and the number of rows is the maximum number of iterations which does not exceed $n-1$ (see the proof of Lemma 1). After completion of the algorithm, it is sufficient to read the suitable sequence of subsequent indices from the destination point to the starting point.

Lemma 3. *The problem of finding the shortest route in a street network with signalised intersections is of complexity class P.*

Proof. The complexity of Dijkstra's algorithm, when looking for the shortest path in the graph, is of order N^3, where N is the number of nodes (Dreyfus, 1969). This refers to the situation where each subsequent state of the system is calculated according to (14) with $\boldsymbol{A}$ composed of constant elements. In the situation of a street network with signalised intersections, the state equation is similar to (18), although $\boldsymbol{B}(k)$ changes during each iteration, in accordance with (18). Based on (18) it can be shown that the number of operations needed to calculate $\boldsymbol{B}(k)$ of size $N \times N$ during each iteration is equal to mN^2, where m is a constant independent of N. Thus the complexity of the problem of finding the shortest route in a street system with signalised intersection is $O(N^6)$. ■

Therefore the procedure aimed at determining a route with the shortest travel time can be seen as a modification of the standard Dijkstra algorithm in which the system matrix $\boldsymbol{B}(k)$ depends on the previous state vector.

5. Allocation of Signal Timings

In the section a $(\max, +)$ algebraic model of a TRS is introduced and an illustrative example is presented.

5.1. The Time Marked Graph of the TRS

If each of the time moments of changing signals is treated as an event, then the considered TRS is a particular case of the discrete-event dynamic system (DEDS). If a signal scheduling (SS) is chosen and the "green lines" are determined, then there exists an order of events. It results from two facts. The events of SS corresponding to the same intersection follow each other (see Fig. 2) and this order can be represented by a cyclic traffic-light process (TLP). The quasi-rendez-vous synchronisation of TLPs corresponding to a given "green line" is required. Let x_I^g and x_I^r be respectively the time moments (events) at which the green and red signals start at intersection I for vehicles moving along the "green" route. Similarly, x_{II}^g and x_{II}^r are these moments at intersection II. The quasi-rendez-vous synchronisation of the neighbouring TLPs forces two relations between them (see Fig. 8):

$$\begin{cases} x_I^g + t_{I,II}^g \geq x_{II}^g \Longrightarrow x_I^g \geq x_{II}^g - t_{I,II}^g \\ x_{II}^r \geq x_I^r + t_{I,II}^r \end{cases} \tag{34}$$

where $t_{I,II}$ is the average travel time between the intersections.

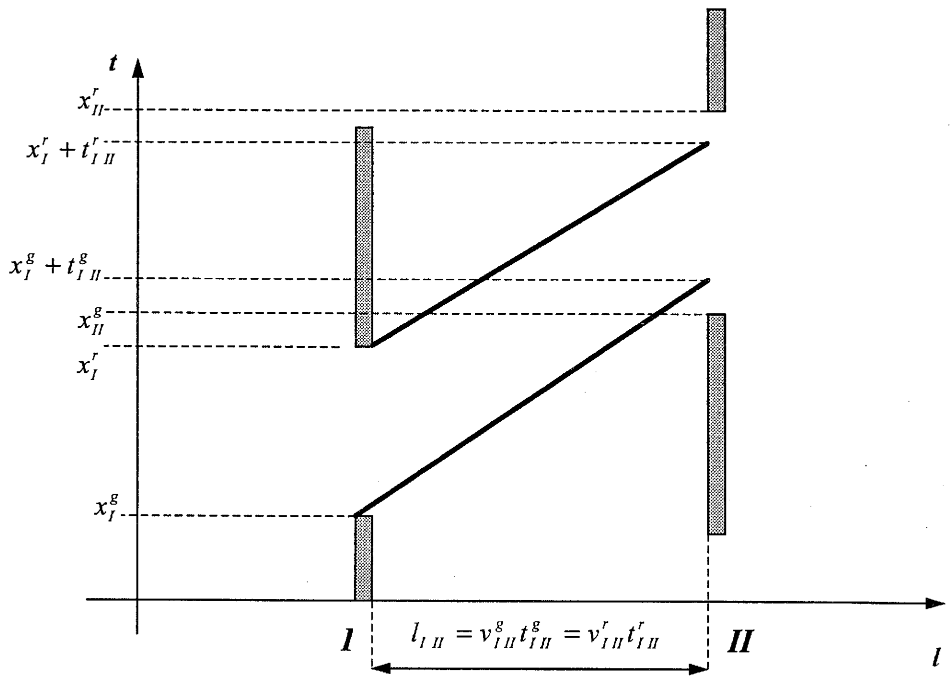
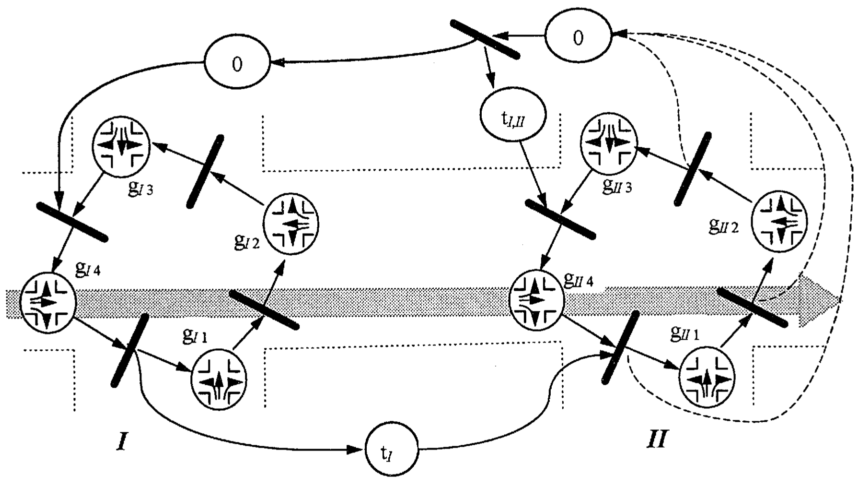
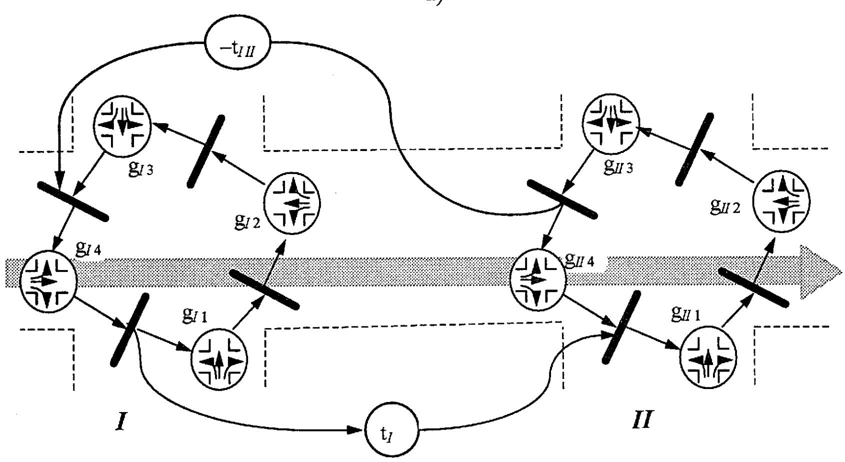


Fig. 8. An illustration of relation (34).

The first relation of (34) means that the vehicle which started from intersection I at time x_I^g cannot reach intersection II before the time x_{II}^g (it cannot be forced to wait for the green signal). The other means that the red signal at intersection II cannot start before the last vehicle arriving from intersection I reaches intersection II. In general, the identity $t_{I,II}^g = t_{I,II}^r$ is not necessary (these two time intervals can be different when a road narrows from two lanes into one), but we assume that $t_{I,II}^g = t_{I,II}^r = t_{I,II}$ for simplification. The relations (34) allow us to connect the TLPs corresponding to the neighbouring intersections in order to get a time market graph (Fig. 9).



a)



b)

Fig. 9. (a) The TMG of the TRS consisting of two intersections (I and II) and one "green" route. Three dashed arcs describe an alternative dependence of events. (b) A modification of the preceding TMG, which represents only a stationary realisation of the system. This graph contains a non-realistic connection with negative time, but it avoids alternative arcs.

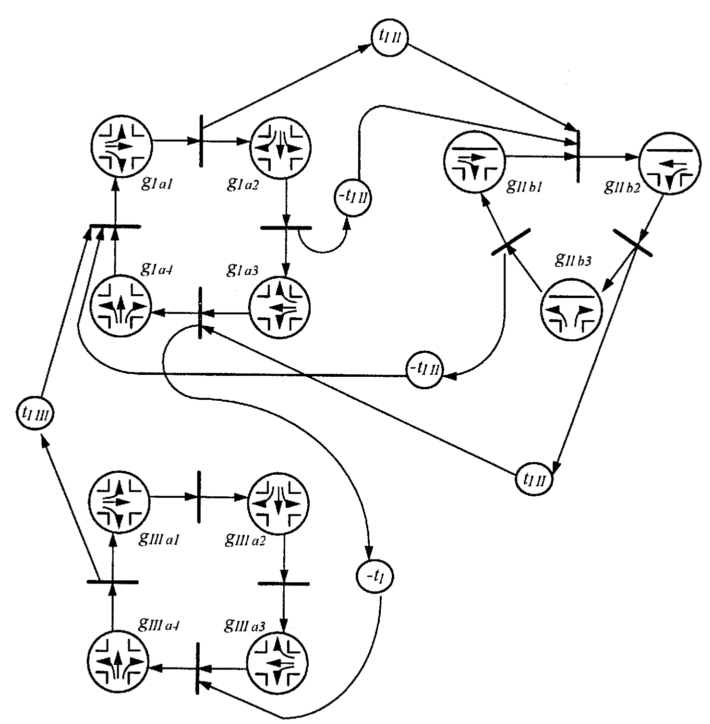
The first relation of (34) shows that x_I^g depends on x_{II}^g and some negative time parameter $-t_{I,II}$. It can be substituted by the relations

$$x_I^g \geq x_{II}^g - t_{I,II} \longrightarrow \begin{cases} x_I^g \geq x \\ x_{II}^g = x + t_{I,II} \end{cases} \tag{35}$$

where the event x precedes x^g_{II} by the time $t_{I,II}$. Unfortunately, it is difficult to say which event of the TLP corresponding to intersection II precedes x in a real steady-state realisation (see Fig. 9(a)). Thus one can create a set of alternative TMGs which differ from one another in only one arc of alternative arcs (dashed in Fig. 9(a)).

In order to obtain the steady state of the system, each TMG from this set has to be examined. The number of considered TMGs increases exponentially against the number of "green" routes. This disadvantage can be overcome in the case when the arcs labelled with negative weights are introduced to the graph instead the sets of alternative arcs (Fig. 9(b)). Such a modified graph describes the unique steady state of the system, but it does not have any classical interpretation of TMGs. It turns out, however, that there are no obstacles to apply the (max, +) algebraic formalism to create a state equation based on a modified TMG and to obtain all the desired quantities, such as the TRS period and the traffic-signal timetable.

Example 5. Let the TRS configuration shown in Fig. 1 with two types of SS (see SS-A and SS-B in Fig. 2) be considered. If 1, 2 and 3 are "green" routes, then the graphs of the state order for both types of SS are presented in Figs. 10 and 11. If



Notation:

g_{ki} – minimal time during which the k-th intersection has to be in the i-th state;
t_{km} – average travel time between the k-th and m-th intersection ($k, m = I, II, III$).

Fig. 10. The modified TMG for TRS presented in Fig. 1 and SS-A (Fig. 2(a)).

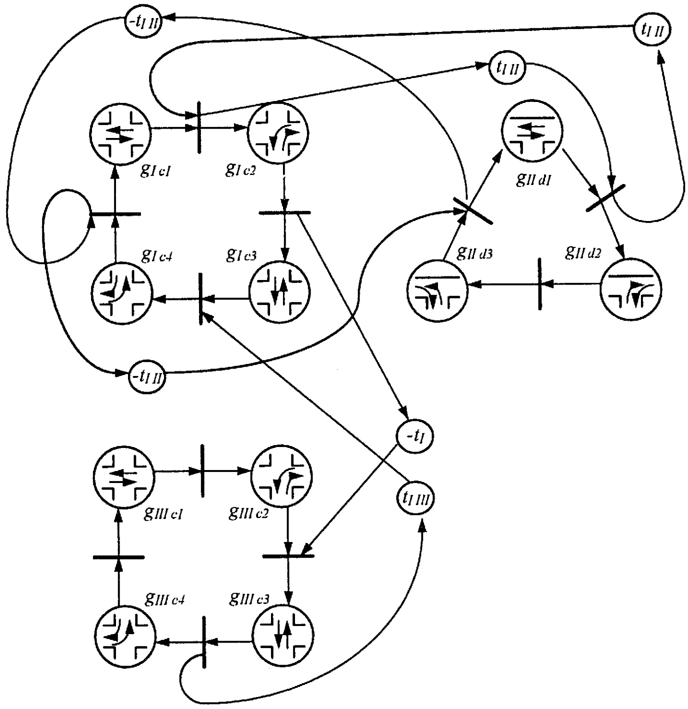


Fig. 11. The modified TMG for TRS presented in Fig. 1 and SS-B (Fig. 2(b)).

there exists an arc between vertices i and j, then, according to the selected initial state, the starting time moment x_j of the j-th traffic state has to satisfy the condition:

$$x_j(k) \geq x_i(k - q_{ij}) + w_{ij}, \quad q_{ij} \in \{0, 1, \dots, m\} \tag{36}$$

where w_{ij} is the weight of the (i,j) arc and m is a natural number.

Unfortunately, any realisable algorithm of determining the number q_{ij} of previous iterations associated with arc (i,j) is unknown, apart from the full inspection of all possible sets $\{q_{ij}\}_{i,j=1}^{n}$, in order to obtain a sensible system realisation. In most practical situations, a trial-and-error method is employed. ♦

5.2. (Max, +) Algebraic Model of a Traffic Route System

If $x(k)$ is the vector of the starting moments (events) for traffic states in the k-th interaction, then the order of system events (36) can be modelled within the (max, +) algebra framework as follows:

$$\begin{cases} \boldsymbol{x}(k) = \bigotimes_{q=0}^{m} \left(\boldsymbol{A}_q \otimes \boldsymbol{x}(k-q)\right) \\ \boldsymbol{x}(0) = \boldsymbol{x}_0, \qquad \boldsymbol{x}(-1) = \boldsymbol{x}_{-1}, \quad \dots, \quad \boldsymbol{x}(-m) = \boldsymbol{x}_{-m} \end{cases} \tag{37}$$

where $\{\boldsymbol{A}_q | q = 0, m\}$ is the set of some $(\max, +)$-algebra matrices such that $\bigoplus_{q=0}^{m} \boldsymbol{A}_q$ is the matrix corresponding to the TMG of the considered TRS and $\{\boldsymbol{x}_{-q} | q = 0, m\}$ is the set of initial vectors.

Equation (37) can be simplified using slack variables (Braker, 1993). Let an arc (i, j) with weight w_{ij} be associated with the relation

$$x_j(k) \geq x_i(k-q) + w_{ij}, \quad 1 \leq q \leq m \tag{38}$$

This arc can be partitioned into q arcs with weights

$$w_{ij}^{(q)} = \frac{w_{ij}}{q} \tag{39}$$

and eqn. (38) can be written in the form

$$\begin{cases} x_{ij}^{(r)}(k) \geq x_{ij}^{(r-1)}(k-1) + w_{ij}^{(q)}, \quad r = 1, \ldots, q \\ x_{ij}^{(0)}(k) = x_i(k) \\ x_{ij}^{(q)}(k) = x_j(k) \end{cases} \tag{40}$$

After such modifications the state equation (37) takes the form

$$\begin{cases} \boldsymbol{x}^*(k) = \left(\boldsymbol{A}_0^* \otimes \boldsymbol{x}^*(k)\right) \oplus \left(\boldsymbol{A}_1^* \otimes \boldsymbol{x}^*(k-1)\right) \\ \boldsymbol{x}^*(0) = \boldsymbol{x}_0^* \end{cases} \tag{41}$$

where $\boldsymbol{A}_0^*$, $\boldsymbol{A}_1^*$, and $\boldsymbol{x}^*(k)$ are of new sizes (owing to inclusion of the slack variables). It expands to

$$\begin{aligned} \boldsymbol{x}^*(k) &= \left(\boldsymbol{A}_0^* \otimes \boldsymbol{x}^*(k)\right) \oplus \left(\boldsymbol{A}_1^* \otimes \boldsymbol{x}^*(k-1)\right) \\ &= \left(\boldsymbol{A}_0^* \otimes \left[\left(\boldsymbol{A}_0^* \otimes \boldsymbol{x}^*(k)\right) \oplus \left(\boldsymbol{A}_1^* \otimes \boldsymbol{x}^*(k-1)\right)\right]\right) \oplus \left(\boldsymbol{A}_1^* \otimes \boldsymbol{x}^*(k-1)\right) \\ &= \left[\left(\boldsymbol{A}_0^*\right)^2 \otimes \boldsymbol{x}^*(k)\right] \oplus \left[\left(\boldsymbol{A}_0^* \oplus \boldsymbol{E}\right) \otimes \boldsymbol{A}_1^* \otimes \boldsymbol{x}^*(k-1)\right] = \cdots \\ &= \left[\left(\boldsymbol{A}_0^*\right)^p \otimes \boldsymbol{x}^*(k)\right] \\ &\quad \oplus \left[\left(\left(\boldsymbol{A}_0^*\right)^{p-1} \oplus \cdots \oplus \left(\boldsymbol{A}_0^*\right)^2 \oplus \boldsymbol{A}_0^* \oplus \boldsymbol{E}\right) \otimes \boldsymbol{A}_1^* \otimes \boldsymbol{x}^*(k-1)\right] = \cdots \\ \boldsymbol{x}^*(0) &= \boldsymbol{x}_0^* \end{aligned}$$

which then can be rewritten in the compact form

$$\begin{cases} \boldsymbol{x}^*(k) = \underline{\boldsymbol{A}} \otimes \boldsymbol{x}^*(k-1) \\ \boldsymbol{x}^*(0) = \boldsymbol{x}_0^* \end{cases} \tag{42}$$

where

$$\underline{A} = \left(\bigoplus_{p=0}^{\infty} (A_0^*)^p \right) \otimes A_1^*, \quad (A_0^*)^p = (A_0^*)^{p-1} \otimes A_0^*, \quad (A_0^*)^0 = E \tag{43}$$

E is the identity matrix, i.e. $(E)_{ii} = e$ and $(E)_{ij} = \varepsilon$ for $i \neq j$.

Lemma 4. *Consider a TRS and assume that an SS for each intersection is given. Moreover, assume that the "green lines" are selected. If the vector $x^*(k)$ in (42) is an eigenvector of $\underline{A}$ in (43), then T_G attains its minimum.*

Proof. The "green lines" and the set of SS determines explicitly the state equation (43). This equation describes relations included in the TMG of the TRS considered, i.e. the periodicity of TLPs and the quasi-rendez-vous synchronisation of TLPs associated with a given "green line". Since TLPs are cyclic, the steady state of the system is periodic (Braker, 1993; Obuchowicz and Banaszak, 1995). Hence the state vector has to possess periodical behaviour and has to satisfy the quasi-rendez-vous synchronisation. Thus

$$\underline{A} \otimes x^*(k-1) = x^*(k) = T \otimes x^*(k-1) \tag{44}$$

Accordingly, the vector $x^*(k-1)$ is an eigenvector of $\underline{A}$ and the period T is its eigenvalue. If $\underline{A}$ is irreducible, then there exists a solution to (44), the elements of the obtained eigenvector satisfy (35) and T_G is minimised. ■

By solving (44), the TRS period T and the traffic-signal timetable $x^*(k-1)$ are directly obtained. In this way, the $(\max, +)$ algebraic model turns out to be a very useful tool to allocate traffic parameters referred to in the problem formulation (Section 2).

Solving the spectral equation is not straightforward. But if the matrix is irreducible (the graph corresponding to the matrix is strongly-connected, see Remark 2), then a unique eigenvalue (Proposition 1) and the corresponding eigenvector can be obtained using a $(\max, +)$ algebraic version of the power algorithm (Braker, 1993).

5.3. Allocation of Signal Timings

The following procedure provides a signal timing plan (STP) allocation within a given configuration of the TRS in order to minimise the overall waiting time T_G of the vehicles at intersections:

Algorithm 1.

Step 1. Choose a set of SS.

Step 2. For each selected SS construct the corresponding state-order graphs and calculate the signal timings and relative phases (43).

Step 3. For each plan calculate T_G and find the best one T_G^*.

Step 4. If $T_G^* < T_G^{\text{limit}}$, where T_G^{limit} is a satisfactory limit of T_G^*, then stop, otherwise create a new set of SS and go to Step 2.

The time T_G can be calculated (Step 3) using the $(\min, +)$ algebraic model (Section 4). Let Δ be a set of "green routes" and $\alpha \in \Delta$. Let $x_{\alpha 0}^g$ and $x_{\alpha 0}^r$ be respectively the time moments at which the green and red signals for vehicles moving along the route α start at the first intersection of α. Here $x_{\alpha 0}^g$ can be treated as the starting time moment of the vehicle which moves at the beginning of some vehicles packet. Similarly, $x_{\alpha 0}^r$ can be treated as the starting time moment of the vehicle ending this vehicle packet. From Theorem 1, the travel times y_α^g and y_α^r along the route α of these two vehicles can be calculated. The minimum travel time $t_{G\alpha}$ along this route can also be calculated as

$$t_{G\alpha} = \sum_{(i,j)\in\alpha} t_{ij} \tag{45}$$

where t_{ij} is the average travel time between the neighbouring intersections i and j. Then the time T_G can be obtained in the following way:

$$T_G = \sum_{\alpha\in\Delta} \left(|y_\alpha^g - t_{G\alpha}| + |y_\alpha^r - t_{G\alpha}| \right) \tag{46}$$

where the sum is taken over all "green" routes in the TRS.

Example 6. Consider the TRS configuration shown in Fig. 1 with two types of SS (see SS-A and SS-B in Fig. 2). Routes 1, 2 and 3 are "green" and the modified the TMGs of TRSs are presented in Figs. 10 and 11 (Example 5). Assume the following data:

- the distances between the intersections (ft): $l_{I,II} = 2200$; $l_{I,III} = 1800$;
- the average speed (miles per hour): $v = v_{I,II} = v_{I,III} = 30$;
- the relevant data for Fig. 10 are (sec):

  SS-A: $g_{I,a1} = 40$, $g_{I,a2} = 30$, $g_{I,a3} = 40$, $g_{I,a4} = 20$

  $g_{II,b1} = 40$, $g_{II,b2} = 35$, $g_{II,b3} = 15$

  $g_{III,a1} = 10$, $g_{III,a2} = 40$, $g_{III,a3} = 10$, $g_{III,a4} = 10$

- the relevant data for Fig. 11 are (sec):

  SS-B: $g_{I,c1} = 40$, $g_{I,c2} = 20$, $g_{I,c3} = 30$, $g_{I,c4} = 10$

  $g_{II,d1} = 40$, $g_{II,d2} = 10$, $g_{II,d3} = 15$

  $g_{III,c1} = 10$, $g_{III,c2} = 10$, $g_{III,c3} = 40$, $g_{III,c4} = 10$

The average travel times can be calculated from (19) (sec): $t_{I,II} = 50$, $t_{I,III} = 40$.

The matrix $\boldsymbol{A}_A$ corresponding to the graph in Fig. 10 has the form (the indices of the columns: $Ia1$, $Ia2$, $Ia3$, $Ia4$, $IIb1$, $IIb2$, $IIb3$, $IIIa1$, $IIIa2$, $IIIa3$, $IIIa4$):

$$\boldsymbol{A}_A = \begin{bmatrix} \varepsilon & \varepsilon & \varepsilon & 20 & -50 & \varepsilon & \varepsilon & 40 & \varepsilon & \varepsilon & \varepsilon \\ 40 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & 30 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & 40 & \varepsilon & \varepsilon & \varepsilon & 50 & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 15 & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & 50 & -50 & \varepsilon & 40 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 35 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 10 \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 10 & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 40 & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & -40 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 10 & \varepsilon \end{bmatrix}$$

In the case of the graph shown in Fig. 11 we have (the indices of the columns: $Ic1$, $Ic2$, $Ic3$, $Ic4$, $IId1$, $IId2$, $IId3$, $IIIc1$, $IIIc2$, $IIIc3$, $IIIc4$):

$$\boldsymbol{A}_B = \begin{bmatrix} \varepsilon & \varepsilon & \varepsilon & 10 & -50 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ 40 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 50 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & 20 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & 30 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 40 \\ -50 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 15 & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & 50 & \varepsilon & \varepsilon & 40 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 10 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 10 \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 10 & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & -40 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 10 & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 40 & \varepsilon \end{bmatrix}$$

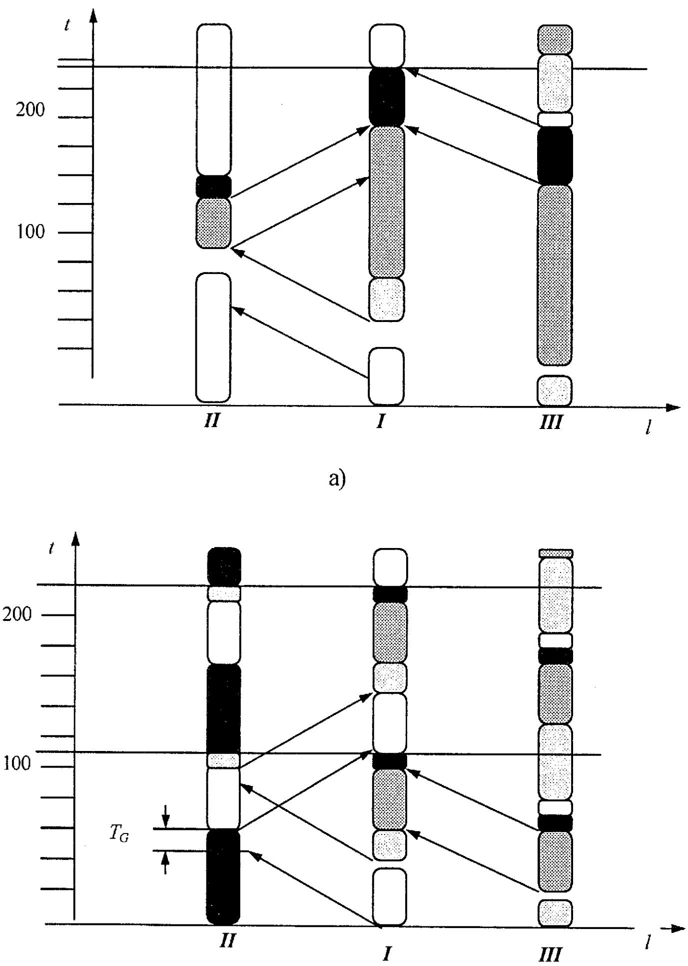
In both the cases the system equation has the form

$$\boldsymbol{x}(k) = \big(\boldsymbol{A}_0 \otimes \boldsymbol{x}(k)\big) \oplus \big(\boldsymbol{A}_1 \otimes \boldsymbol{x}(k-1)\big), \quad \boldsymbol{x}(0) = \boldsymbol{x}_0$$

without using slack variables, because the average travel times are of the same order as the minimal periods of TLPs.

First, we have to select the initial states of TLPs. This operation determines which columns of the matrix $\boldsymbol{A}_A$ ($\boldsymbol{A}_B$) will be included into $\boldsymbol{A}_0$ and $\boldsymbol{A}_1$. For SS-A the initial states are $Ia1$, $IIb2$ and $IIIa3$. For SS-B the initial states are $Ic1$, $IId3$ and $IIIc3$. The columns corresponding to the states preceding the above-mentioned states are included in $\boldsymbol{A}_1$ and the others in $\boldsymbol{A}_0$. Using (42) we can obtain $\underline{\boldsymbol{A}}$ and solving the spectral equation (43) we can calculate the timetable of TLPs and the system period.

The results are presented in Fig. 12. It can be seen that the solution to SS-A guarantees that the overall waiting time $T_G = 0$, while the system period $T_A = 215$.



b)

Notation:

- states a) *I* a1; *II* b1; *III* a1; b) *I* c1; *II* d1; *III* c1;

- states a) *I* a2; *II* b2; *III* a2; b) *I* c2; *II* d2; *III* c2;

- states a) *I* a3; *III* a3; b) *I* c3;; *III* c3;

- states a) *I* a4; *II* b3; *III* a4; b) *I* c4; *II* d3; *III* c4;

→ - represents the travel time between two neighbouring intersections.

Fig. 12. Gantt's charts of a signal timings realisation in the TRS considered: (a) SS-A, (b) SS-B.

In the case of SS-B, we have $T_B = 110$, while the overall waiting time $T_G = 10$. This is contradictory with eqn. (34), which represents a quasi-rendez-vous of TLPs and which forces T_G to be zero. It is easy to see that in Fig. 12(b) the timings $x_I^g(k)$ and $x_{II}^g(k)$ do not satisfy eqn. (34), but $x_I^g(k)$ and $x_{II}^g(k-1)$ do. Hence this is the problem whether the TMG arc corresponding to this timings should be represented in $\boldsymbol{A}_1$ or $\boldsymbol{A}_0$, c.f. (41). If this is the case in $\boldsymbol{A}_1$, then the above results are valid, otherwise the eigenvalue of (44) (the system period) is equal to infinity. The green time interval at intersection I is enclosed by the green time interval at intersection II. When a green line is designed in both directions, the green time intervals enclose each other. In general, there is no finite solution when the time distance between these two intersections is not equal to half the system period. It is easy to see that if the mean travel time between intersections I and II were slightly higher ($t_{I,II} = 55$ instead $t_{I,II} = 50$), then T_G would be equal to zero. Since $T_B < T_A$, the average vehicle flow through the arterial network in Case B is higher then in Case A. In consequence, if the speed limit between intersections I and II decreased so that the mean travel time $t_{I,II} = 55$, then the best solution for the TRS considered would be found for SS-B. ♦

6. Concluding Remarks

Some algebraic models of a collision-free traffic route system are proposed. The first model is based on the (min, +)-algebra approach. It solves the problems regarding determination of the travel time along a given route in the street network and the shortest route in the sense of the travel time among all possible routes joining two given points. This method allows us to evaluate various settings of signal timings for a specific combination of traffic lights with the same computation complexity as is required by Dijkstra's algorithm determining the shortest path in a graph.

The extension of the above results to the problem of real-time traffic-light control so as to guarantee a " smooth" flow of vehicles along presumed routes without violating timings presumed for other routes is our main concern. A method for the adjustment of the traffic signal timings so as to minimise the waiting travel time in an urban traffic system is also considered. The approach is based on the (max, +) algebra which provides a framework for building an executable performance-oriented model of an arterial network.

Further open problems regard e.g. the tasks ranging from routing school buses in the single and multi-school environment, through railway and airway routings, to special vehicles, e.g. routing and scheduling ambulances, fire guards, and convoys.

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