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OBSERVER DESIGN FOR DISTRIBUTED-PARAMETER DISSIPATIVE BILINEAR SYSTEMS

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This paper deals with the problem of observer synthesis for a class of infinite-dimensional dissipative bilinear systems working for a class of inputs. First, a simple observer (the estimation error converges strongly asymptotically to zero) for strongly persistent input signals is presented. Next, sufficient conditions are given which guarantee the existence of an exponential observer for skew-adjoint bilinear systems based on the time-varying differential Riccati (or Lyapunov) equations working for strongly regularly persistent inputs. These results are illustrated by means of partial-differential systems.

1. Introduction

Many authors have studied the observer synthesis problem for bilinear control systems. In the finite-dimensional case, the first results are due to Funahashi (1979) and Hara and Furuta (1976) who have studied the cases when the system has no "bad inputs" (inputs which make the system unobservable) or when the system has a bad input and this leaves the system asymptotically stable.

Over the last decade, Bornard *et al.* (1988) and Celle *et al.* (1989) have shown that an observer can be constructed for general finite-dimensional bilinear systems provided that the inputs are "persistent" or "regular." In (Celle *et al.*, 1989), it is suggested that a finite-dimensional complete nonlinear system with a finite-dimensional Lie Algebra can be embedded into an infinite-dimensional one which is bilinear and skew-adjoint. The authors also construct a rather simple infinite-dimensional observer system to estimate the states of the original nonlinear system. Furthermore, the bad inputs give rise to singularities in the observer design. In (Bornard and Gauthier, 1982), the class of bilinear control systems which are observable for any input is characterized.

For infinite-dimensional bilinear systems which are generated by a dissipative drift, an exponential Kalman-like observer which works for some bounded inputs (Bounit and Hammouri, 1996) was proposed. By extending the regularly persistent notion of inputs to the infinite-dimensional case, Gauthier *et al.* (1995) established, for general infinite-dimensional skew-adjoint bilinear systems, a simple observer such that the estimation error decays weakly to zero. The authors also showed that the finite-dimensional structure assumption is not necessary for the construction of this

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observer. In other words, the observer exhibited in Celle *et al.* (1989) still works for skew-adjoint systems whose dynamics is not necessarily related to some finite-dimensional Lie group. In this paper, we show how to construct a simple strong Luenberger-like (resp. Kalman-like) observer for dissipative (resp. skew-adjoint) bilinear systems. Nonhomogenous bilinear systems are not considered here since the extension of the given results to this class of systems is straightforward. Our paper essentially deals with strongly persistent (resp. regularly strongly persistent) inputs which are introduced here.

An outline of the paper is as follows: In Section 2, we introduce some definitions and notations. In Section 3, we give a simple observer for dissipative bilinear systems which works for any positive strongly persistent inputs. Then, we establish the strong convergence of the estimation error for any positive strongly persistent input. In Section 4, we consider regularly strongly persistent (not necessarily positive) inputs and give an exponential Kalman-like observer for skew-adjoint bilinear systems which are based on some differential Riccati equations. In Section 5, the above theoretical results are illustrated with two examples whose models are expressed by means of partial-differential equations.

2. Definitions and Notations

In this paper, we study the observation problem for bilinear systems in an infinite-dimensional state space. Let us consider the following bilinear system:

$$\begin{cases} \dot{x}(t) = Ax(t) + u(t)Bx(t) \\ y(t) = Cx(t) \end{cases} \tag{1}$$

where $t \geq 0$. The element $x_0 \in H$ is called the initial state, $x(t)$ is said to be the state at time t, H denotes a real Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and the norm $\|\cdot\|$, $u(t)$ is an $\mathbb{R}$-valued control, $y(t) \in Y$ stands for the known output function of (1) and Y is a real Hilbert space of observation. Throughout this paper, the operator A is an infinitesimal generator of a linear C_0-semigroup of contractions on H denoted by e^{tA} (in particular, $\|e^{tA}x\| \leq \|x\|, x \in D(A)$), B is a bounded linear dissipative operator from H into itself (i.e. $B \in \mathcal{L}(H,H) = \mathcal{L}(H)$) and C is a bounded linear operator from H into Y (i.e. $C \in \mathcal{L}(H,Y)$. $\Sigma^+(H) = \{T \in \mathcal{L}(H)/\ T^* = T$ and $\langle Tx, x\rangle \geq 0,\ \forall x \in H\}$ and $\Sigma^{\#}(H) = \{T \in \Sigma^+(H)/\exists\ \alpha > 0$ s.t. $\langle Tx, x\rangle \geq \alpha\|x\|^2,\ \forall x \in H\}$. If $E, F \in \Sigma^+(H)$ and $E - F \in \Sigma^+(H)$, then we shall write $E \geq F$. An admissible operator on $[0,T]$ is a function $t \longrightarrow G(t)$, $t \in [0,T]$, with values in $\mathcal{L}(H)$ and the following properties: $G(t)$ is bounded on $[0,T]$ and $t \longrightarrow G(t)x$ is strongly measurable on $[0,T]$ for all $x \in H$. We will denote by $L^\infty(\mathbb{R}^+, \mathbb{R}) = L^\infty(\mathbb{R}^+)$ the space of measurable $\mathbb{R}$-valued functions u on $\mathbb{R}^+$ such that $\|u\|_\infty = \sup_{t\in\mathbb{R}^+}|u(t)| < +\infty$ and write $L^\infty([t_0,t_1],\mathbb{R}) = L^\infty[t_0,t_1]$ for the restriction of $L^\infty(\mathbb{R}^+)$ to the interval $[t_0,t_1]$. $L^p(\mathbb{R}^+)$ $(1 \leq p < +\infty)$ signifies the set of strongly measurable p-integrable $\mathbb{R}$-valued functions on $\mathbb{R}^+$ and $L^p[t_0,t_1]$ is the restriction of $L^p(\mathbb{R}^+)$ to the interval $[t_0,t_1]$.

Now, let $u \in L^\infty(\mathbb{R}^+)$ and $t_0 \geq 0$. Then, for any $T > 0$, the unique mild solution on $[t_0, T]$ to system (1) resulting from an initial condition $x_0 \in H$ at $t = t_0$ is given by the integral equation:

$$x(t) = e^{(t-t_0)A}x_0 + \int_{t_0}^{t} e^{(t-s)A}u(s)Bx(s)\,\mathrm{d}s = \phi_u(t, t_0)x_0$$

where $\phi_u(t,s)$, $0 \leq s \leq t \leq T$ is the unique mild evolution operator associated with $A_u(t) = (A + u(t)B)$ (see e.g. Curtain and Pritchard, 1978) which satisfies the following conditions:

1. $\phi_u(t,t) = I_d$ for all $t \in [t_0, T]$,
2. $\phi_u(t,s)\phi_u(s,r) = \phi_u(t,r)$ for $T \geq t \geq s \geq r \geq 0$,
3. $\phi_u(\cdot, s)$ and $\phi_u(t, \cdot)$ are strongly continuous on $[t_0, T]$.

Remark 1. From the dissipativity of A and B, it is easy to verify that for every positive input u the associated operators $\phi_u(t,s)$ are contractive for all $t \geq s$ (i.e. $||\phi_u(t,s)x|| \leq ||x||$, $\forall x \in H$).

Now, consider the so-called observability Grammian associated with system (1):

$$W(u, t_0, t_1) = \int_{t_0}^{t_1} \phi_u^*(s, t_0)C^*C\phi_u(s, t_0)\,\mathrm{d}s$$

where $\phi_u^*(s, t_0)$ denotes the adjoint operator of the mild evolution operator $\phi_u(t, t_0)$ such that for all $x, y \in H$ we have

$$\langle W(u, t_0, t_1)x, y\rangle = \int_{t_0}^{t_1} \langle C\phi_u(s, t_0)x, C\phi_u(s, t_0)y\rangle\,\mathrm{d}s$$

Definition 1. We call $u \in L^\infty(\mathbb{R}^+)$ a *universal (U)* input on $[t_0, T]$, $T > t_0$ if and only if $\forall x \in H$; $W(u, t_0, T)x = 0$ implies that $x = 0$.

This definition means that the map from H into $L^2([t_0, T], Y)$ defined by $x \longrightarrow C\phi_u(\cdot, t_0)x$ is one to one.

Definition 2. A function $u \in L^\infty(\mathbb{R}^+)$ is a *strongly universal (SU)* input on $[t_0, T]$, $T > t_0$ for system (1) if and only if

$$\exists \eta > 0;\ \forall x \in H,\ \langle W(u, t_0, T)x, x\rangle \geq \eta ||x||^2$$

Remark 2. Clearly, Definitions 1 and 2 are equivalent for the finite-dimensional bilinear systems of the form (1). However, for the infinite-dimensional case, strong universal inputs are universal and the converse is not always true since $W(u, 0, T)$ can be a compact operator and therefore it cannot be bounded from below. In the particular case where the input u is constant on some $[0, T]$, $T > 0$ the universality (resp. strong universality) of u on $[0,T]$ is equivalent to the condition that the pair $(C, A + uB)$ is initially (resp. L^2-continuously initially) observable on $[0,T]$ (Curtain and Pritchard, 1978; Russel and Weiss, 1994). For example, when e^{tA} or C is compact (in particular, $\mathrm{rank}(C) < \infty$) (Gauthier *et al.*, 1995), any universal input cannot be a strong one. Hence a necessary (but not sufficient) condition which gives rise to the existence of a strong universal input is the noncompactness of C and e^{tA}.

On the other hand, it is well-known that a bounded set of $L^\infty[0,T]$ is precompact with respect to the weak* topology. In other words, given a bounded sequence u_n, we can extract a subsequence u_{n_k} with the property that there exists $u^* \in L^\infty[0,T]$ such that for each $f \in L^1[0,T]$

$$\int_0^T \Big(u_n(s) - u(s)\Big) f(s)\,\mathrm{d}s \xrightarrow[k\to+\infty]{} 0$$

i.e. for each $f \in L^1([0,T], H)$ (Gauthier *et al.*, 1995):

$$\left\| \int_0^T \Big(u_n(s) - u(s)\Big) f(s)\,\mathrm{d}s \right\|_H \xrightarrow[k\to+\infty]{} 0$$

In the sequel, we shall use the following definition which is the modified one from (Celle *et al.*, 1989).

Definition 3. Given an input $u \in L^\infty(\mathbb{R}^+)$, we set $u_{[\theta]}(t) = u(t+\theta)$. We say that u is a *strongly persistent (SP)* input for (1) iff there exists an increasing sequence $(\theta_n)_{n\in\mathbb{N}}$, with $\lim\limits_{n\to+\infty} \theta_n = +\infty$, and $T > 0$ such that

1. the restriction of $u_{[\theta_n]}$ to $[0,T]$ converges to u^* in the weak* topology,
2. $u^* \in L^\infty(\mathbb{R}^+)$ is SU for (1) on $[0,T]$.

If, in addition, the sequence $(\theta_n)_n$ is such that $\theta_{n+1} - \theta_n$ is bounded, u is said to be a *strongly regularly persistent (SRP)* input.

This means, as in the finite-dimensional case, that $u_{[\theta_n]}$ tends to make the system observable in the same way as u^*. Since $\theta_{n+1} - \theta_n$ is bounded, it also means that the size of the interval on which u makes the system unobservable does not increase. In particular, every T-periodic input function u which is SU is SRP.

3. An Asymptotic Observer for Dissipative Bilinear Systems

Let us consider the dynamical system

$$\begin{cases} \dot{\hat{x}}(t) = A\hat{x}(t) + u(t)B\hat{x}(t) - C^*\Big(C\hat{x}(t) - y(t)\Big) \\ y(t) = C\hat{x}(t) \end{cases} \tag{2}$$

where A and B are as in Section 2. In this section, we will show that the system (2) is an observer for (1).

The Cauchy problem associated with (2) admits a unique weak solution which is defined for all $t \geq 0$ and $u \in L^\infty(\mathbb{R}^+)$:

$$\begin{cases} \dot{\hat{x}}(t) = A\hat{x}(t) + u(t)B\hat{x}(t) - C^*C\hat{x}(t) + C^*y(t) \\ \hat{y}(t) = C\hat{x}(t) \end{cases} \tag{3}$$

As is the case above, one can show that there exists a unique mild evolution operator $\psi_u(t,s)$ associated with $A + u(t)B - C^*C$. Since $C^*Cx(\cdot) \in L^1([0,T];H)$ for every $T \geq 0$, the unique solution to (3) is well-defined for any $x_0 \in H$, $t \geq 0$ (Curtain and Pritchard, 1978).

Now, we state our main result:

Theorem 1. *For an SP positive input $u \in L^\infty(\mathbb{R}^+)$, the estimation error $\varepsilon(t) = \hat{x}(t) - x(t)$ converges strongly to zero in H.*

To prove it, we need some preliminary technical results. Namely, consider $L^\infty[0,T]$ equipped with the weak* topology. The proof of the following Lemma is standard and it can be obtained by using Gronwall's Lemma (see e.g. Celle *et al.*, 1989; Xu *et al.*, 1995):

Lemma 1. *The map from $L^\infty[0,T]$ equipped with the weak* topology to $\mathcal{L}(H)$ (resp. to $L^\infty([0,T];\mathcal{L}(H))$):*

$$u \longrightarrow W(u,0,T) \quad (\textit{resp. } u \longrightarrow \phi_u(\cdot,0))$$

is locally Lipschitz for any $T > 0$.

Remark 3. The strong universality of positive inputs is an open property for the weak* topology.

Proof of Theorem 1. Let $\varepsilon(\cdot) = \hat{x}(\cdot) - x(\cdot)$ be the estimation error. Then we have

$$\dot{\varepsilon}(t) = A\varepsilon(t) + u(t)B\varepsilon(t) - C^*C\varepsilon(t), \quad \varepsilon(0) = \varepsilon_0 \tag{4}$$

As above, the unique solution to (4) is given by the following integral equation:

$$\varepsilon(t) = e^{tA}\varepsilon_0 + \int_0^t e^{(t-s)A}\Big(u(s)B - C^*C\Big)\varepsilon(s)\,\mathrm{d}s \tag{5}$$

The basic idea of the proof is to show that there exists a ball $\mathcal{B}(0,\sigma)$ centred at the origin and of radius σ, which attracts any solution of eqn. (5) and then that $\mathcal{B}(0,\sigma)$ is reduced to the origin ($\sigma = 0$).

Let $T_0 > 0$ and define

$$f(\cdot) : [0,T_0] \longrightarrow H : t \longrightarrow \Big(u(t)B - C^*C\Big)\varepsilon(t)$$

Since $u \in L^\infty[0,T_0]$, we can pick a sequence of continuous functions $u_n \in C^1[0,T_0]$ such that

$$\|u_n - u\|_{L^1[0,T_0]} = \int_0^{T_0} \big|u_n(s) - u(s)\big|\,\mathrm{d}s \xrightarrow[n\to+\infty]{} 0$$

Now, let us take $(\varepsilon_{0n})_{n\in\mathbb{N}} \subset D(A)$ such that $\varepsilon_{0n} \xrightarrow[n\to+\infty]{} \varepsilon_0$ (since $\overline{D(A)} = H$) and consider

$$\varepsilon_n(t) = e^{tA}\varepsilon_{0n} + \int_0^t e^{(t-s)A} f_n(s)\,\mathrm{d}s \tag{6}$$

where

$$f_n(\cdot) : [0, T_0] \longrightarrow H : t \longrightarrow \Big(u_n(t)B - C^*C\Big)\varepsilon_n(t)$$

According to the regularity theorem (Pazy, 1983, Theorem (1.5), pp.187), $\varepsilon_n(t) \in D(A)$ for any $t > 0$ and $\varepsilon_n(t) \in C^1[0, T_0]$. Moreover, $\varepsilon_n(\cdot)$ satisfies the differential equation

$$\dot{\varepsilon}_n(t) = A\varepsilon_n(t) + f_n(t), \quad \varepsilon_n(0) = \varepsilon_{0n} \tag{7}$$

Differentiating the Lyapunov function $V(\varepsilon) = (1/2)\|\varepsilon\|^2$ along the trajectories of (7) gives

$$p - \dot{V}\big(\varepsilon_n(t)\big) = \big\langle A\varepsilon_n(t) + f_n(t), \varepsilon_n(t)\big\rangle$$

It follows that

$$V\big(\varepsilon_n(t)\big) - V\big(\varepsilon_n(0)\big) \le \int_0^t \big\langle A\varepsilon_n(s) + f_n(s), \varepsilon_n(s)\big\rangle\,\mathrm{d}s \tag{8}$$

Setting $e_n(\cdot) = \varepsilon_n(\cdot) - \varepsilon(\cdot)$ on $[0, T_0]$ and combining (5), (6), we obtain

$$e_n(t) = e^{tA}e_{0n} + \int_0^t e^{(t-s)A}\Big(f_n(s) - f(s)\Big)\,\mathrm{d}s$$

with $e_{0n} = \varepsilon_{0n} - \varepsilon_0$. Consequently,

$$\begin{aligned}\|e_n(t)\| &\le \|e_{0n}\| + \int_0^t \Big|\Big(u_n(s) - u(s)\Big)\Big|\|B\| + \|C^*C\|\,\big\|e_n(s)\big\|\,\mathrm{d}s\\ &\le \|e_{0n}\| + \|u_n - u\|_{L^1[0,T_0]} + \|C^*C\|\int_0^t \big\|e_n(s)\big\|\,\mathrm{d}s\end{aligned}$$

for all $t \in [0, T_0]$. Since $(\|e_{0n}\|, \|u_n - u\|_{L^1[0,T_0]}) \xrightarrow[n\to+\infty]{} (0,0)$, Gronwall's inequality gives

$$e_n(t) \xrightarrow[n\to+\infty]{} 0, \;\; \forall\, t \in [0, T_0],\; T_0 > 0 \tag{9}$$

Passing to the limit in (8) and taking into account eqn. (9) and that A is dissipative on $D(A)$, we obtain that for every $T_0 > 0$ and every $t \in [0, T_0]$

$$V\big(\varepsilon(t)\big) - V(\varepsilon_0) \le \int_0^t \big\langle f(s), \varepsilon(s)\big\rangle\,\mathrm{d}s \le -\int_0^t \big\|C\varepsilon(s)\big\|^2\,\mathrm{d}s$$

Thus

$$\int_0^{+\infty} \big\|C\varepsilon(s)\big\|^2\,\mathrm{d}s < +\infty \tag{10}$$

and $\varepsilon(t)$ is attracted by the ball $\mathcal{B}(0,\sigma)$ with $\sigma^2 = 2\Big(V(\varepsilon_0) - \int_0^{+\infty} \|C\varepsilon(s)\|^2\,\mathrm{d}s\Big)$. It remains only to show that $\sigma = 0$. To do this, it suffices to show that $\varepsilon(t_n) \xrightarrow[n\to\infty]{} 0$ for some sequence $(t_n)_{(n\in\mathbb{N})}$ with $t_n \xrightarrow[n\to\infty]{} 0$.

From the definition of the SP input of u, there exists a sequence $(\theta_n)_{n\in\mathbb{N}}$ $(\theta_n \xrightarrow[n\to+\infty]{} +\infty)$ such that

1. $u_{[\theta_n]}$ converges to u^* in the weak* topology on $[0,T]$,
2. $u^* \in L^\infty(\mathbb{R}^+)$ is SU for (1), i.e.

$$\exists\,\eta > 0;\ \forall x \in H;\ \langle W(u^*,0,T)x, x\rangle \geq \eta\|x\|^2$$

To end the proof, it suffices to show that $W(u^*,0,T)\varepsilon(\theta_n) \xrightarrow[n\to\infty]{} 0$. From (10), we deduce

$$\int_{\theta_n}^{\theta_n+T} \|C\varepsilon(t)\|^2\,\mathrm{d}t \xrightarrow[n\to\infty]{} 0 \tag{11}$$

Using a counterpart of (5), we obtain

$$\varepsilon(t) = \phi_u(t,\theta_n)\varepsilon(\theta_n) - \int_{\theta_n}^{t} \phi_u(s,t)C^*C\varepsilon(s)\,\mathrm{d}s$$

Thus

$$\begin{aligned}\|C\varepsilon(t)\|^2 &= \|C\phi_u(t,\theta_n)\varepsilon(\theta_n)\|^2 + \Big\|\int_{\theta_n}^{t} C\phi_u(\tau,\theta_n)C^*C\varepsilon(\tau)\,\mathrm{d}\tau\Big\|^2 \\ &\quad - 2\Big\langle C\phi_u(t,\theta_n)\varepsilon(\theta_n), \int_{\theta_n}^{t} C\phi_u(\tau,\theta_n)C^*C\varepsilon(\tau)\,\mathrm{d}\tau\Big\rangle\end{aligned}$$

and

$$\begin{aligned}\int_{\theta_n}^{\theta_n+T} \|C\varepsilon(t)\|^2\,\mathrm{d}t &= \langle W(u_{[\theta_n]},0,T)\varepsilon(\theta_n), \varepsilon(\theta_n)\rangle \\ &\quad + \int_{\theta_n}^{\theta_n+T} \Big\|\int_{\theta_n}^{t} C\phi_u(\tau,\theta_n)C^*C\varepsilon(\tau)\,\mathrm{d}\tau\Big\|^2\,\mathrm{d}t \\ &\quad - 2\int_{\theta_n}^{\theta_n+T} \Big\langle C\phi_u(t,\theta_n)\varepsilon(\theta_n), \int_{\theta_n}^{t} C\phi_u(\tau,\theta_n)C^*C\varepsilon(\tau)\,\mathrm{d}\tau\Big\rangle\,\mathrm{d}t \\ &\geq \langle W(u_{[\theta_n]},0,T)\varepsilon(\theta_n), \varepsilon(\theta_n)\rangle \\ &\quad - 2\int_{0}^{T} \Big\langle C\phi_u(t,\theta_n)\varepsilon(\theta_n), \int_{\theta_n}^{s} C\phi_u(\tau,\theta_n)C^*C\varepsilon(\tau)\,\mathrm{d}\tau\Big\rangle\mathrm{d}t\end{aligned} \tag{12}$$

Using the Cauchy-Holder inequality yields

$$\int_0^T \Big\langle C\phi_u(s,\theta_n)\varepsilon(\theta_n), \int_{\theta_n}^s C\phi_u(\tau,\theta_n)C^*C\varepsilon(\tau)\,\mathrm{d}\tau\Big\rangle\,\mathrm{d}s$$

$$\le \int_0^T \|C\phi_u(s,\theta_n)\varepsilon(\theta_n)\|\,\Big\|\int_{\theta_n}^s C\phi_u(\tau,\theta_n)C^*C\varepsilon(\tau)\,\mathrm{d}\tau\Big\|\mathrm{d}s$$

$$\le \int_0^T \|C\phi_u(s,\theta_n)\varepsilon(\theta_n)\|\Big(\int_{\theta_n}^s \|C\phi_u(\tau,\theta_n)C^*\|^2\,\mathrm{d}\tau\Big)^{1/2}\Big(\int_{\theta_n}^s \|C\varepsilon(\tau)\|^2\,\mathrm{d}\tau\Big)^{1/2}\mathrm{d}s$$

$$\le \int_0^T \|C\phi_u(s,\theta_n)\varepsilon(\theta_n)\|\,\mathrm{d}s\Big(\int_{\theta_n}^{T+\theta_n} \|C\phi_u(\tau,\theta_n)C^*\|^2\,\mathrm{d}\tau\Big)^{1/2}$$

$$\times\Big(\int_{\theta_n}^{T+\theta_n}\|C\varepsilon(\tau)\|^2\mathrm{d}\tau\Big)^{1/2} \le \alpha(T)\Big(\int_{\theta_n}^{\theta_n+T}\|C\varepsilon(t)\|^2\,\mathrm{d}t\Big)^{1/2} \tag{13}$$

with $\alpha(T) = T^{3/2}\|C\|^2$. Combining (12) and (13), we obtain

$$\int_{\theta_n}^{\theta_n+T}\|C\varepsilon(t)\|^2\,\mathrm{d}t + 2\alpha(T)\left(\int_{\theta_n}^{\theta_n+T}\|C\varepsilon(t)\|^2\,\mathrm{d}t\right)^{1/2} \ge \langle W(u_{[\theta_n]},0,T)\varepsilon(\theta_n),\varepsilon(\theta_n)\rangle$$

Finally, using (12), we deduce that

$$\langle W(u_{[\theta_n]},0,T)\varepsilon(\theta_n),\varepsilon(\theta_n)\rangle \xrightarrow[n\to\infty]{} 0 \tag{14}$$

In addition, we have

$$\begin{aligned}\langle W(u^*,0,T)\varepsilon(\theta_n),\varepsilon(\theta_n)\rangle &= \langle W(u_{[\theta_n]},0,T)\varepsilon(\theta_n),\varepsilon(\theta_n)\rangle \\ &\quad + \Big\langle\Big[W(u^*,0,T) - W(u_{[\theta_n]},0,T)\Big]\varepsilon(\theta_n),\varepsilon(\theta_n)\Big\rangle\end{aligned} \tag{15}$$

Using Lemma 1 with (14), (15) and the fact that $(\varepsilon(\theta_n))_{n\in\mathbb{N}}$ is bounded, we conclude

$$\langle W(u^*,0,T)\varepsilon(\theta_n),\varepsilon(\theta_n)\rangle \xrightarrow[n\to+\infty]{} 0 \tag{16}$$

From (16), the fact that $(1/2)\|\varepsilon(t)\|^2$ converges and u^* is SU for (1), it follows that

$$\eta\|\varepsilon(\theta_n)\|^2 \le \langle W(u^*,0,T)\varepsilon(\theta_n),\varepsilon(\theta_n)\rangle$$

which yields

$$\varepsilon(\theta_n) \xrightarrow[n\to+\infty]{} 0$$

■

Remark 4. Recall that in the finite-dimensional case the study of universal inputs was initiated by Sontag (1979) for the discrete case. Moreover, for nonlinear analytic systems, it is shown in (Sussmann, 1979) that C^ω-universal inputs do exist and are generic in C^ω equipped with C^ω-topology on $[0,T]$, $T > 0$. But for infinite-dimensional dissipative bilinear control systems where the pair (C, A) is L^2-exactly observable on some $[0,T]$ (Curtain and Pritchard, 1978) L^∞-strongly universal inputs do exist. In fact,

$$\exists \eta > 0;\ \forall\, x \in H,\ \int_0^T \|Ce^{sA}x\|^2\, \mathrm{d}s \geq \eta \|x\|^2$$

which exactly implies that $u = 0$ is an SU input for (1) on $[0,T]$. Furthermore, one can see that the set of L^∞-SU inputs equipped with the weak * topology on $[0,T]$, $T > 0$ is open (see Remark 3).

4. An Exponential Observer for Skew-Adjoint Bilinear Systems

In Section 3, we have synthesized a simple asymptotic observer with constant gain for dissipative bilinear systems with positive inputs. However, in many engineering problems, the use of exponential observers is preferred to that of asymptotic ones. In this section, we are mainly concerned with this problem. In other words, we shall omit the positivity condition on the inputs and wish to design an exponential observer for a skew-adjoint bilinear system.

In what follows, we assume that

(A_1) Both A and B are skew-adjoint on operators H.

Our aim here is to prove that the system

$$\dot{\hat{x}}(t) = A\hat{x}(t) + u(t)B\hat{x}(t) - R_u(t)C^*\Big(C\hat{x}(t) - y(t)\Big) \tag{17}$$

$$\begin{aligned}\dot{R}(t) &= \theta R_u(t) + \Big(A + u(t)B\Big)R_u(t) \\ &\quad + R_u(t)\Big(A + u(t)B\Big)^* - R_u(t)C^*CR_u(t)\end{aligned} \tag{18}$$

where $\theta > 0$ is a parameter and $R(0)$ is a coercive self-adjoint operator on $H (R(0) \in \Sigma^{\#}(H))$, is an exponential observer for (1) which converges strongly as soon as u is a strongly regularly persistent input.

Before proceeding with the proof of this last fact, we need some preliminary results. In particular, we will show the existence and uniqueness of the solution of the Riccati equation (18) and that $R_u(t) = R^*(t) \leq \alpha I$ for all $t \geq 0$, for some constant $\alpha > 0$, whenever $R(0) \in \Sigma^{\#}(H)$.

For that purpose, consider the following time-varying Riccati operator equation:

$$\begin{cases} \dot{R}(t) = \theta R_u(t) + A_u(t)R_u(t) + R_u(t)A_u^*(t) - R_u(t)C^*CR_u(t) \\ R(0) = R_0 \end{cases} \tag{19}$$

A solution to (19), if it exists, is taken in the following inner-product Riccati equation:

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\langle R_u(t)x, y\rangle = \theta\langle R_u(t)x, y\rangle + \langle R_u(t)x, A_u^*(t)y\rangle \\ \qquad\qquad + \langle A_u^*(t)x, R_u(t)y\rangle - \langle R^*(t)C^*CR_u(t)x, y\rangle \\ R(0) = R_0 \end{cases} \tag{20}$$

for all $x, y \in D(A^*)$ and for almost every $t \geq 0$, where $\theta > 0$ is a parameter.

Under some hypotheses (see Lemma 2 below), we shall prove that any solution to (20) starting at $t = 0$ from $\Sigma^\#(H)$ is invertible and its inverse satisfies the following Lyapunov operator equation:

$$\dot{S}_u(t) = -\theta S_u(t) - A_u^*(t)S_u(t) - S_u(t)A_u(t) + C^*C, \quad S(0) = S_0 \tag{21}$$

As for (19), solutions to (21) are taken in the following weak sense:

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\langle S_u(t)x, y\rangle = -\theta\langle S_u(t)x, y\rangle - \langle S_u(t)x, A_u(t)y\rangle \\ \qquad\qquad - \langle A_u(t)x, S_u(t)y\rangle + \langle C^*Cx, y\rangle \\ S(0) = S_0 \end{cases} \tag{22}$$

for all $x, y \in D(A)$ and for almost every $t \geq 0$.

Remark 5. From Hypothesis (A_1), it is easy to check that the operator $\phi_u(t,s)$ is isometric for any $s, t \in [0, T]$ and $\phi_u^{-1}(t,s) = \phi_u^*(t,s) = \phi_u(s,t)$. In particular, we have

$$\phi_{u_{[t]}}(s, 0) = \phi_u(t+s, 0)\phi_u^{-1}(t, 0) = \phi_u(t+s, t) \quad \text{for every } t, s$$

where the input u is not necessarily positive.

Lemma 2. *Under Hypothesis (A_1), for any $u \in L^\infty(\mathbb{R}^+)$, $S_0 \in \Sigma^\#(H)$, the Lyapunov equation (22) admits a unique solution $S_u(t)$ in $\Sigma^\#(H)$ with $S(0) = S_0$. Moreover, its inverse $R_u(t)$ is the unique solution to (19) resulting from $R(0) = S_0^{-1}$.*

Proof. Existence. Let $S_0 \in \Sigma^\#(H)$ and

$$S_u(t) = e^{-\theta t}\left\{\phi_u^*(0,t)S_0\phi_u(0,t) + \int_0^t e^{\theta s}\phi_u^*(s,t)C^*C\phi_u(s,t)\,\mathrm{d}s\right\}$$

Since $\phi_u(s,t)$ is invertible and the integral term is a bounded self-adjoint positive operator, it follows that $S_u(t) \in \Sigma^\#(H)$ for every $t \geq 0$.

Now, let us show that S(t) satisfies (22). For $x, y \in D(A)$, we have

$$\langle S_u(t)x, y\rangle = e^{-\theta t}\langle S_0\phi_u(0,t)x, \phi_u(0,t)y\rangle + \int_0^t e^{\theta s}\langle C\phi_u(s,t)x, C\phi_u(s,t)y\rangle\,\mathrm{d}s \tag{23}$$

The assumption $u \in L^\infty(\mathbb{R}^+)$ yields that $G(\cdot) = u(\cdot)B$ is an admissible operator on $[0, T]$. From Theorem 2.34 of (Curtain and Pritchard, 1978) this means that the mild evolution operator $\phi_u(s,t)$ is differentiable with respect to t and we have

$$\forall x \in D(A);\ \frac{\partial\phi_u(s,t)}{\partial t}x = -\phi_u(s,t)A_u(t)x \quad \text{a.e.} \tag{24}$$

Differentiating (23), term by term, and using (24), we obtain (22).

Uniqueness. Write $A_u^\theta(t) = A_u(t) + \frac{1}{2}\theta I_d$. Let $S_1(t)$ and $S_2(t)$ be two solutions to (22) and $\tilde{S}_u(t) = S_1(t) - S_2(t)$. It is easy to see that $\tilde{S}_u(t)$ satisfies the equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle \tilde{S}_u(t)x, y\rangle = -\langle \tilde{S}_u(t)x, A_u^\theta(t)y\rangle - \langle A_u^\theta(t)x, \tilde{S}_u(t)y\rangle, \quad \tilde{S}(0) = 0$$

a.e. in t and for every $x, y \in D(A)$.

Now, consider $K(t,s) = \Pi_\theta(t-s)\tilde{S}_u(t)\Pi_\theta^*(t-s)$, where $\Pi_\theta(t) = e^{t(A+\frac{1}{2}\theta)}$ is the C_0-group generated by $A_\theta = A + \frac{1}{2}\theta I_d$. Then, for every $x, y \in D(A)$, $\langle K(t,s)x, y\rangle$ is a.e. differentiable w.r.t. t and satisfies the following differential equation:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\langle K(t,s)x, y\rangle &= \langle \tilde{S}_u(t)\Pi_\theta(t-s)A_\theta x, \Pi_\theta(t-s)y\rangle \\ &\quad + \langle \tilde{S}_u(t)\Pi_\theta(t-s)x, \Pi_\theta(t-s)A_\theta y\rangle \\ &\quad - \langle \tilde{S}_u(t)\Pi_\theta(t-s)x, A_u^\theta(t)\Pi_\theta(t-s)y\rangle \\ &\quad - \langle \tilde{S}_u(t)\Pi_\theta(t-s)y, A_u^\theta(t)\Pi_\theta(t-s)x\rangle\end{aligned}$$

In particular, for $x = y$, it is easy to see that

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle K(t,s)x, x\rangle = -2\,\langle u(t)\tilde{S}_u(t)\Pi_\theta(t-s)x, B\Pi_\theta(t-s)x\rangle$$

Since $K(0,s) = 0, s \geq 0$ and $\overline{D(A)} = H$, it follows that

$$\langle K(t,s)x, x\rangle = -2\int_0^t \langle u(\tau)\tilde{S}_u(\tau)\Pi_\theta(\tau-s)x, B\Pi_\theta(\tau-s)x\rangle\,\mathrm{d}\tau \tag{25}$$

for all $x \in H$. Passing to the limit $(s \longrightarrow t)$ in (25), we obtain

$$\langle K(t,t)x, x\rangle = \langle \tilde{S}_u(t)x, x\rangle = -2\int_0^t \langle u(\tau)\tilde{S}_u(\tau)\Pi_\theta(\tau-t)x, B\Pi_\theta(\tau-t)x\rangle\,\mathrm{d}\tau$$

Since $\tilde{S}_u(t) \in \Sigma^+(H)$, we have

$$\begin{aligned}\|\tilde{S}_u(t)\| &= \sup_{\|x\|=1}\left|\langle \tilde{S}_u(t)x, x\rangle\right| \\ &\leq \|B\|\,\|u\|_\infty \sup_{\|x\|=1}\int_0^t \|\tilde{S}_u(\tau)\|\,\|\Pi_\theta(\tau-s)\|\,\|x\|^2\,\mathrm{d}\tau \\ &\leq \|B\|\,\|u\|_\infty \int_0^t e^{\frac{1}{2}\theta\tau}\|\tilde{S}_u(\tau)\|\,\mathrm{d}\tau \ \ (A^* = -A)\end{aligned}$$

Hence, by Gronwall's inequality, it follows that $\|\tilde{S}_u(t)\| = 0$ or $S_1(t) = S_2(t)$. Consequently, (22) has a unique solution. Using the above expression for $S_u(t)$

and the fact that $\phi_u(s,t)$ is invertible yield that $S_u(t)$ is also invertible and $R_u(t) = S^{-1}(t) \in \Sigma^{\#}(H)$ is the unique solution to (20) with $R(0) = S^{-1}(0)$. ■

Lemma 3. *Given an SRP input u for (1), there exist times t_u, $\tilde{T} > 0$ and a constant $\eta > 0$ such that*

$$\forall t \geq t_u;\ \forall x \in H,\ \ \langle W(u_{[t]}, 0, \tilde{T})x, x \rangle \geq \eta \|x\|^2$$

Proof. From the definition of the SRP input u, there exists an increasing sequence $(\theta_n)_{n\in\mathbb{N}}$ with $\theta_n \xrightarrow[n\to\infty]{} \infty$ and $\sup_{n\in\mathbb{N}}(\theta_{n+1} - \theta_n) < \infty$. Further, there exists $T > 0$ such that the restriction $u_{[\theta_n]}{}_{/[0,T]} = u_{[\theta_n]}$ converges weakly* to $u^* \in L^\infty[0,T]$ which is SU on $[0,T]$.

Now, set $\Delta = \sup_{n\in\mathbb{N}}(\theta_{n+1} - \theta_n) < \infty$, $\tilde{T} = 2\max(\Delta, T)$ and $\theta_{n(t)} = \inf\{\theta_n; \theta_n \geq t\}$. Then a straightforward calculation shows that

$$\forall t \geq 0;\ [\theta_{n(t)}, \theta_{n(t)} + T] \subset [t, t + \tilde{T}]$$

Using the positivity of the observability Grammian, we get

$$W(u_{[t]}, 0, \tilde{T}) \geq W(u_{[\theta_{n(t)}]}, 0, T) \tag{26}$$

Since $u_{[\theta_n]} \xrightarrow[n\to\infty]{} u^*$ weakly* and u^* is SU on $[0,T]$, it follows that

$$u_{[\theta_n(t)]} \xrightarrow[n\to\infty]{} u^* \ \text{weakly*}\ \text{and}\ \exists \eta_0 > 0,\ W(u^*, 0, T) \geq \eta_0 I \tag{27}$$

Combining Lemma 1 and formulae (26), (27), we get

$$\exists \eta > 0;\ \exists t_u \geq 0;\ \forall x \in H;\ \forall t \geq t_u;\ \langle W(u_{[t]}, 0, \tilde{T})x, x \rangle \geq \eta \|x\|^2$$

This completes the proof. ■

Remark 6. As in the finite-dimensional case, it is easy to verify that the converse of Lemma 3 also remains true.

Lemma 4. *Under Hypothesis (A_1) for an SRP input $u \in L^\infty(\mathbb{R}^+)$, $S_0 \in \Sigma^{\#}(H)$ and $\theta > 0$, there exists $\lambda_\theta > 0$ such that*

$$S_u(t) \geq \lambda_\theta I_d \quad \text{for all} \quad t \geq 0$$

where $S_u(t)$ is the unique solution to (22) resulting from S_0 at $t = 0$.

Proof. From Lemma 2, we know that for any $S_0 \in \Sigma^{\#}(H)$, the unique solution to (22) resulting from S_0 at $t = 0$ is given by the expression

$$S_u(t) = e^{-\theta t}\phi_u^*(0,t) S_0 \phi_u(0,t) + \int_0^t e^{\theta(s-t)} \phi_u^*(s,t) C^* C \phi_u(s,t)\, \mathrm{d}s$$

Since u is SRP, there exist $\tilde{T} > 0$, $t_u > 0$ and $\eta > 0$ (see Lemma 3) such that

$$\forall t \geq t_u;\ W(u_{[0,T]}, 0, \tilde{T}) \geq \eta I_d \tag{28}$$

As the operator $\int_{t_0}^{t} e^{\theta(s-t)}\phi_u^*(s,t)C^*C\phi_u(s,t)\,\mathrm{d}s$ is positive, for all $t \in [0, t_u + \tilde{T}]$ we have

$$S_u(t) \geq e^{-\theta t}\phi_u^*(0,t)S_0\phi_u(0,t) \geq e^{-\theta(t_u+\tilde{T})}\phi_u^*(0,t)S_0\phi_u(0,t)$$

Now, using the fact that $\phi_u^*(s,t)\phi_u(s,t) = I_d$ and $S_0 \geq \eta_0 I_d$ for some $\eta_0 > 0$, we obtain

$$S_u(t) \geq \eta_0 e^{-\theta(t_u+\tilde{T})} I_d \;\; \forall t \in [0, t_u + \tilde{T}]$$

For $t > t_u + \tilde{T}$, we have

$$\begin{aligned} S_u(t) &\geq \int_0^t e^{\theta(s-t)}\phi_u^*(s,t)C^*C\phi_u(s,t)\,\mathrm{d}s \\ &= \int_0^{t-\tilde{T}} e^{\theta(s-t)}\phi_u^*(s,t)C^*C\phi_u(s,t)\,\mathrm{d}s + \int_{t-\tilde{T}}^{t} e^{\theta(s-t)}\phi_u^*(s,t)C^*C\phi_u(s,t)\,\mathrm{d}s \\ &\geq e^{-\theta\tilde{T}}\phi_u^*(t-\tilde{T},t)\left\{\int_{t-\tilde{T}}^{t}\phi_u^*(s,t-\tilde{T})C^*C\phi_u(s,t-\tilde{T})\,\mathrm{d}s\right\}\phi_u(t-\tilde{T},t) \end{aligned}$$

Hence

$$S_u(t) \geq e^{-\theta\tilde{T}}\phi_u^*(t-\tilde{T},t)\left\{\int_0^{\tilde{T}}\phi_{u_{[t-\tilde{T}]}}^*(s,0)C^*C\phi_{u_{[t-\tilde{T}]}}(s,0)\,\mathrm{d}s\right\}\phi_u(t-\tilde{T},t) \quad (29)$$

as $\phi_u(s,t) = \phi_u(s,t-\tilde{T})\phi_u(t-\tilde{T},t)$.

Combining (28) and (29), we get

$$S_u(t) \geq \eta e^{-\theta\tilde{T}} I_d, \quad \forall t \geq t_u + \tilde{T}$$

Finally, setting $\lambda_\theta = e^{-\theta T}\min(\eta_0 e^{-\theta t_u}, \eta)$, we obtain

$$S_u(t) \geq \lambda_\theta I_d, \quad \forall t \geq 0$$

■

Claim 1. *Let $u_n \in L^\infty[0,T]$ such that u_n converges weakly* to u. Then we have*

$$\sup_{s\in[0,T]} \left\| S_{u_n}^{-1}(s) - S_u^{-1}(s) \right\|_{\mathcal{L}(H)} \xrightarrow[n\to+\infty]{} 0$$

Proof. This claim can be obtained by a direct application of Lemma 1. In fact, from Lemma 2 we know that the unique solution $S_u(t)$ to the Lyapunov equation (22) is of the form

$$S_u(t) = e^{-\theta t}\left\{\phi_u^*(0,t)S_0\phi_u(0,t) + \int_0^t e^{\theta s}\phi_u^*(s,t)C^*C\phi_u(s,t)\,\mathrm{d}s\right\}$$

It follows that

$$S_{u_n}(t) - S_u(t) = e^{-\theta t}\Big\{\phi^*_{u_n}(0,t)S_0\phi_{u_n}(0,t) - \phi^*_u(0,t)S_0\phi_u(0,t)\Big\}$$
$$+ \int_0^t e^{\theta(s-t)}\Big\{\phi^*_{u_n}(s,t)C^*C\phi_{u_n}(s,t) - \phi^*_u(s,t)C^*C\phi_u(s,t)\Big\}\mathrm{d}s$$

Then a straightforward computation gives

$$\begin{aligned}\big\|S_{u_n}(t) - S_u(t)\big\|_{\mathcal{L}(H)} &= \sup_{\|x\|=1} \big|\langle S_{u_n}(t)x - S_u(t)x, x\rangle\big| \\ &\le 2\|S_0\|\,\big\|\phi_{u_n}(0,t) - \phi_u(0,t)\big\| \\ &\quad + \big\|W(u_n,0,t) - W(u,0,t)\big\|_{\mathcal{L}(H)}\end{aligned}$$

Notice that we have used above the fact that $\phi_u(s,t)$ is unitary. This implies that the map from $L^\infty[0,T]$ equipped with the weak* topology to $\Sigma^\#(H)$ equipped with the usual topology of the operator $u \longrightarrow S_u$ is continuous. Now, using the fact that the map from $\Sigma^\#(H)$ into itself $F \longrightarrow F^{-1}$ is continuous yields the desired result. ■

Before stating our main theorem, let us show that for every $\hat{x}_0 \in H$ the system (17) admits a unique solution. Indeed, from Lemmas 2 and 4, we know that for each $R_0 \in \Sigma^\#(H)$ the system (18) admits a unique solution $R_u(t) \in \Sigma^\#(H)$ which is uniformly bounded from above. By definition, for all $u \in L^\infty(\mathbb{R}^+)$ the map associated with an element $x_0 \in H$ defined from $\mathbb{R}^+$ into H by $t \longrightarrow R_u(t)x_0$ is continuous and therefore it becomes strongly measurable. Define $G(t) = u(t)B - R_u(t)C^*C$ as a perturbed operator. It is obvious that the operator G is admissible on $[0,T]$ for all $T > 0$. Moreover, we have for each $x_0 \in H$, $R_u(\cdot)C^*Cx(\cdot) \in L^1([0,T];H)$ for all $T > 0$, where $x(t)$ is a unique weak solution to (1). It follows from Sections 2.4 and 2.5 of (Curtain and Pritchard, 1978) that, for all $\hat{x}_0 \in H$, the first equation of (3.28) of (Curtain and Pritchard, 1978) has a unique weak solution $\hat{x}(t)$ resulting from $\hat{x}_0$, which is given by the following integral equation:

$$\hat{x}(t) = \psi(t,0)\hat{x}(0) + \int_0^t \psi(t,s)R_u(s)C^*y(s)\,\mathrm{d}s$$

where $\psi_u(t,s)$ is the mild evolution operator associated with $A + G(t)$.

Now we state our main theorem:

Theorem 2. *Under Hypothesis (A_1), for an SRP input $u \in L^\infty(\mathbb{R}^+)$, the system (17), (18) is an exponential observer for (1). More precisely,*

$$\exists\mu(\theta) > 0 \;\; s.t. \;\; \forall\hat{x}(0), x(0) \in H;\; \forall R(0) \in \Sigma^\#(H):$$

$$\big\|\hat{x}(t) - x(t)\big\|^2 \le \mu(\theta)\big\langle R^{-1}(0)\hat{x}(0) - x(0), \hat{x}(0) - x(0)\big\rangle e^{-\theta t}$$

Proof. In fact, by setting $S_u(t) = R^{-1}(t)$, one can show that the observer (17), (18) can be rewritten as

$$\begin{cases} \dot{\hat{x}}(t) = A\hat{x}(t) + u(t)B\hat{x}(t) - S^{-1}(t)C^*\Big(C\hat{x}(t) - y(t)\Big) \\ \dot{S}_u(t) = -\theta S_u(t) - A_u^*(t)S_u(t) - S_u(t)A_u(t) + C^*C \\ \hat{x}(0) \in H, S(0) = R^{-1}(0) \in \Sigma^{\#}(H) \end{cases} \tag{30}$$

Indeed, we shall show that the system (30) is an exponential observer for (1).

Setting $\varepsilon(t) = \hat{x}(t) - x(t)$ as the error estimation, we get

$$\dot{\varepsilon}(t) = A\varepsilon(t) + u(t)B\varepsilon(t) - S^{-1}(t)C^*C\varepsilon(t), \quad \varepsilon(0) = \varepsilon_0 \in H \tag{31}$$

In much the same way as above, for all $\varepsilon_0 \in H$ the system (31) has a unique weak solution which is given by

$$\varepsilon(t) = e^{tA}x_0 + \int_0^t e^{(t-s)A}G(s)\varepsilon(s)\,\mathrm{d}s$$

Now assume that $\varepsilon(0) \in D(A)$ and $u \in C^1[0, T_0]$ for all $T_0 > 0$. Again, according to the regularity theorem (Pazy, 1983, Theorem 1.5, pp.187), $\varepsilon_n(t) \in D(A)$ for any $t > 0$ and $\varepsilon_n(\cdot) \in C^1[0, T_0]$.

Now, consider the Lyapunov function $V(t, \varepsilon) = \langle S_u(t)\varepsilon, \varepsilon\rangle$. Differentiating $V(\cdot, \varepsilon(\cdot))$ along the trajectories to (31) gives

$$\begin{aligned} \dot{V}(t) &= 2\big\langle S_u(t)\varepsilon(t), \dot{\varepsilon}(t)\big\rangle + \big\langle \dot{S}_u(t)\varepsilon(t), \varepsilon(t)\big\rangle \\ &= 2\big\langle S_u(t)\varepsilon(t), (A{+}G(t)\varepsilon(t)\big\rangle + \big\langle C^*C\varepsilon(t), \varepsilon(t)\big\rangle + \big\langle A_u(t)\varepsilon(t), S_u(t)\varepsilon(t)\big\rangle \\ &\quad - \theta\big\langle S_u(t)\varepsilon(t), \varepsilon(t)\big\rangle - \big\langle S_u(t)\varepsilon(t), A_u(t)\varepsilon(t)\big\rangle \end{aligned}$$

so

$$\dot{V}(t, \varepsilon(t)) \le -\theta\big\langle S_u(t)\varepsilon(t), \varepsilon(t)\big\rangle - \big\langle C^*C\varepsilon(t), \varepsilon(t)\big\rangle \tag{32}$$

Note that $\langle C^*C\varepsilon(t), \varepsilon(t)\rangle > 0$ which implies

$$\dot{V}\big(t, \varepsilon(t)\big) \le -\theta V\big(t, \varepsilon(t)\big) \quad \text{a.e. for} \quad t \ge 0$$

It follows that

$$V\big(t, \varepsilon(t)\big) \le V\big(0, \varepsilon(0)\big)e^{-\theta t} \quad \text{for all} \quad t \ge 0$$

As u is SRP, there exists $\lambda_\theta > 0$ (see Lemma 4) such that

$$S_u(t) \ge \lambda_\theta I_d \quad \text{for all} \quad t \ge 0$$

Hence

$$\|\varepsilon(t)\|^2 \le \frac{1}{\lambda_\theta}\langle S_u(t)\varepsilon, \varepsilon\rangle \le \frac{1}{\lambda_\theta}\langle S(0)\varepsilon(0), \varepsilon(0)\rangle e^{-\theta t}$$

Now, let $\varepsilon_0 \in H, u \in L^\infty(\mathbb{R}^+)$ and $(\varepsilon(t), S_u(t))$ be a solution to (31) associated with (ε_0, u). As the reader may remark, using exactly the same denseness argument as in the proof of Theorem 1 gives the required result. Indeed,

1. $S_u(t) \geq \lambda_\theta I, \quad \forall t \geq 0,$

2. $S_{u_n}(t) \xrightarrow[n \to \infty]{} S_u(t)$ (by Claim 1),

3. $\sup_{n\in\mathbb{N}}\Big\{\sup_{t\in\mathbb{R}^+} \big\|S_{u_n}^{-1}(t)\big\|\Big\} < +\infty,$

4. $\varepsilon_n(t) \xrightarrow[n \to +\infty]{} \varepsilon(t),$

5. $$\begin{aligned} \big\langle S_u(t)\varepsilon(t), \varepsilon(t)\big\rangle &= \lim_{n\to+\infty} \big\langle S_{u_n}(t)\varepsilon_n(t), \varepsilon_n(t)\big\rangle \\ &\leq \lim_{n\to+\infty} \big\langle S(0)\varepsilon_{0n}, \varepsilon_{0n}\big\rangle e^{-\theta t} \\ &= \big\langle S(0)\varepsilon_0, \varepsilon_0\big\rangle e^{-\theta t} \end{aligned}$$

This is the desired conclusion. ■

5. Examples

Example 1. The example given here is similar to the one considered in (Taylor *et al.*, 1984; Slemrod, 1989). The authors have studied the stabilizing control problem for a vibrating beam. We complete these studies with the observation problem. Consider a dynamic boundary control system of a vibrating beam (M) which is described by

$$\begin{cases} \dfrac{\partial^2\varphi(x,t)}{\partial t^2} = -\dfrac{\partial^4\varphi(x,t)}{\partial x^4}, & x \in]0, L[, \quad t \geq 0 \\ \varphi(0,t) = \dfrac{\partial\varphi(0,t)}{\partial t} = 0, & t \geq 0 \\ \dfrac{\partial^2\varphi(L,t)}{\partial t^2} + \dfrac{\partial^3\varphi(L,t)}{\partial t^2 \partial x} = u(t)\dfrac{\partial\varphi(L,t)}{\partial t}, & t \geq 0 \\ \dfrac{\partial^3\varphi(L,t)}{\partial x^3} = \dfrac{\partial^2\varphi(L,t)}{\partial t^2} & \\ y(t) = \dfrac{\partial\varphi(L,t)}{\partial t} & \end{cases} \tag{33}$$

Here $\varphi(x,t)$ denotes the displacement of the beam (M) and $u(t)$ is an applied scalar control which acts on the free boundary of (M). The term $y(t)$ is the output function (the velocity of the beam (M) on the free boundary).

The system (33) has the following first-order differential version in H (Slemrod, 1989):

$$\begin{cases} \dot{Z}(t) = AZ(t) + u(t)BZ(t) \\ y(t) = CZ(t) = \langle c, Z(t)\rangle_H \end{cases} \tag{34}$$

where H is the Hilbert space given by

$$H = \Big\{(z_1, z_2, z_3, z_4) \in H^2(0,L) \times L^2(0,L) \times \mathbb{R} \times \mathbb{R};\ z_1 = \mathrm{d}z_1/\mathrm{d}x = 0 \text{ at } x = 0\Big\}$$

with the inner product

$$\Big\langle (z_1, z_2, z_3, z_4), (\bar{z}_1, \bar{z}_2, \bar{z}_3, \bar{z}_4)\Big\rangle = \int_0^L \frac{\mathrm{d}^2 z_1(x)}{\mathrm{d}x^2}\frac{\mathrm{d}^2 \bar{z}_1(x)}{\mathrm{d}x^2} + z_2(x)\bar{z}_2(x)\,\mathrm{d}x + z_3\bar{z}_3 + z_4\bar{z}_4$$

and

$$Z = \begin{pmatrix} \varphi \\ \dfrac{\partial\varphi(x,t)}{\partial t} \\ \dfrac{\partial\varphi(L,t)}{\partial t} \\ \dfrac{\partial^2\varphi(L,t)}{\partial t\partial x} \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad c = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

A being an unbounded linear operator in H such that

$$A\begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix} = \begin{pmatrix} z_2 \\ -\dfrac{\mathrm{d}^4 z_1}{\mathrm{d}x^4} \\ \dfrac{\mathrm{d}^3 z_1(L)}{\mathrm{d}x^3} \\ \dfrac{\mathrm{d}^2 z_1(L)}{\mathrm{d}x^2} \end{pmatrix}$$

and

$$D(A) = \Big\{(z_1, z_2, z_3, z_4) \in H^4(0,L) \times H^2(0,L) \times \mathbb{R} \times \mathbb{R};$$

$$z_1 = \frac{\mathrm{d}z_1}{\mathrm{d}x} = 0,\ z_2 = \frac{\mathrm{d}z_2}{\mathrm{d}x} = 0 \text{ at } x = 0,\ z_2 = z_3,\ \frac{\mathrm{d}z_2}{\mathrm{d}x} = z_4 \text{ at } x = L\Big\} \tag{35}$$

It can be shown (Slemrod, 1989) that A is a skew-adjoint, infinitesimal generator of a linear C_0-group of contractions e^{tA} with a compact resolvent.

Claim 2. *There exist $T > 0$, $\lambda > 0$ such that for all negative inputs u with $\|u\|_\infty \le \lambda$, $t \ge 0$, u is SU for the system (34) on $[t, t+T]$.*

Proof. It suffices to show that $u = 0$ is strongly universal, i.e. (C, A) is L^2-exactly observable on some $[0, T]$, $T > 0$ (Curtain and Pritchard, 1978) and this is equivalent to showing that the linear control system

$$\dot{W}(t) = AW(t) + C^* v(t) \tag{36}$$

is L^2-exactly controllable on $[0, T]$ in H.

To prove this result, we use the Hilbert Uniqueness Method (HUM) (Lions, 1986). Let us consider the systems

$$\dot{\phi}(t) = A\phi(t), \quad 0 < t < T, \quad \phi(T) = \phi_0$$

and

$$\dot{\psi}(t) = A\psi(t) + c\langle c, \phi(t)\rangle, \quad 0 < t < T, \quad \psi(0) = 0$$

Consider the operator Λ defined by $\Lambda\phi_0 = \psi(T)$. Then we have

$$\langle \Lambda\phi_0, \phi_0 \rangle = \int_0^T \langle c, e^{(s-T)A}\phi_0 \rangle^2 \, \mathrm{d}s \tag{37}$$

Now, we recall the following result (Bensoussan, 1989):

Theorem 3. *If the application defined by*

$$\phi_0 \longrightarrow \langle \Lambda\phi_0, \phi_0 \rangle$$

is a norm on H, then the system (35) is L^2-exactly controllable on $[0, T]$.

Now, let us verify that (36) defines a norm on H. It is well-known (Slemrod, 1989) that the complete orthonormal system of eigenfunctions of A, i.e. ϕ_n, $n = 1, 2,$ associated with the eigenvalues λ_n such that $A\phi_n = \lambda_n \phi_n$ (the multiplicity of λ_n is 1) are given by

$$\phi_n = \begin{pmatrix} w_n \\ i\mu_n^2 w_n \\ i\mu_n^2 w_n(L) \\ i\mu_n^2 \dfrac{\mathrm{d}w_n(x)}{\mathrm{d}x} \end{pmatrix}, \quad \text{associated with} \quad \lambda_n = \mu_n^2, \ n = 1, 2, \ldots$$

and

$$\phi_n = \begin{pmatrix} w_n \\ -i\mu_n^2 w_n \\ -i\mu_n^2 w_n(L) \\ -i\mu_n^2 \dfrac{\mathrm{d}w_n(x)}{\mathrm{d}x} \end{pmatrix}, \quad \text{associated with} \quad \lambda_{-n} = -\mu_n^2, \ n = 1, 2, \ldots$$

$$0 < \mu_1 \le \mu_2 \le \cdots \le \mu_n \longrightarrow \infty \quad \text{for} \quad n = 1, 2, \ldots$$

where

$$w_n(x) = \alpha_n\Big(\sin(\mu_n x) - sh(\mu_n x)\Big) + \beta_n\Big(\cos(\mu_n x) - \cos(\mu_n x)\Big), \quad \forall x \in [0, L] \tag{38}$$

Furthermore, these functions verify

$$\begin{cases} \dfrac{d^4 w_n(x)}{dx^4} + \lambda_n^2 w_n(x) = 0 & \text{if } 0 < x < L \\ w_n(x) = \dfrac{dw_n(x)}{dx} = 0 & \text{at } x = 0 \\ -\dfrac{d^3 w_n(x)}{dx^4} + \lambda_n^2 w_n(x) = 0 & \text{at } x = L \\ \dfrac{d^2 w_n(x)}{dx^2} + \lambda_n^2 \dfrac{dw_n(x)}{dx} = 0 & \text{at } x = L \end{cases} \tag{39}$$

Relation (36) defines a norm on H iff (Curtain and Pritchard, 1978)

$$r_n = \langle c, \phi_n \rangle \neq 0, \quad \forall n = 1, 2, \ldots$$

In fact, suppose that there exists an $n_0 \geq 1$ such that $r_{n_0} = 0$. This implies that $w_{n_0}(L) = 0$. Now, using the boundary equations (38) yields

$$\begin{cases} -\beta_{n_0} \sin(\mu_{n_0}) - \mathrm{sh}(\mu_{n_0}) + \alpha_{n_0}\Big(\cos(\mu_{n_0}) + \cos(\mu_{n_0})\Big) = 0 \\ (\alpha_{n_0} - \beta_{n_0}^3) \sin(\mu_{n_0}) + \mathrm{sh}(\mu_{n_0}) + (\beta_{n_0} + \alpha_{n_0}\mu_{n_0}^3) \cos(\mu_{n_0}) \\ \qquad + (\beta_{n_0} - \alpha_{n_0}\mu_{n_0}^3)\,\mathrm{ch}(\mu_{n_0}) = 0 \end{cases} \tag{40}$$

A direct computation shows that the system (39) has a unique solution $\alpha_{n_0} = \beta_{n_0} = 0$.

Using (37), we deduce that $w_{n_0}(x)$ vanishes on $[0, L]$ which is in contradiction to the fact that ϕ_{n_0} is an eigenfunction of A. This proves the L^2-exact controllability of (34) on $[0, T]$.

Using Remark 3 and the fact that e^{tA} is a group of isometrics yield

$$\exists r > 0;\ \forall t \geq 0;\ \forall u \leq 0,\ \|u\|_\infty \leq r,\ u \text{ is SU on } [t, t+T] \text{ for (34)}$$

From Remark 6 and Theorem (1), the Luenberger-like observer of (33) for these inputs is of the form

$$\begin{cases} \dfrac{\partial \hat{\varphi}_1(x,t)}{\partial t} = \hat{\varphi}_2(x,t), & x \in]0, L[, \quad t \geq 0 \\ \dfrac{\partial \hat{\varphi}_2(x,t)}{\partial t} = -\dfrac{\partial^4 \hat{\varphi}_1(x,t)}{\partial x^4}, & t \geq 0 \\ \dfrac{d\hat{\varphi}_3(t)}{dt} + \dfrac{\partial^3 \hat{\varphi}_1(L,t)}{\partial x^3} = u(t)\hat{\varphi}_3(t) - \left(\hat{\varphi}_3(t) - \dfrac{\partial \varphi(L,t)}{\partial t}\right), & t \geq 0 \\ \dfrac{d\hat{\varphi}_4(t)}{dt} = \dfrac{\partial^2 \hat{\varphi}_1(L,t)}{\partial x^2} & \\ \hat{\varphi}_1(0,t) = \dfrac{\partial \hat{\varphi}_1(0,t)}{\partial t} = 0, & t \geq 0 \\ \hat{y}(t) = \hat{\varphi}_3(t) & \end{cases}$$

♦

Example 2. Consider the following bilinear system in $H = L^2(0,L) \times L^2(0,L)$:

$$\begin{cases} \dfrac{\partial^2 \varphi(x,t)}{\partial t^2} = \dfrac{\partial^2 \varphi(x,t)}{\partial x^2} + iu(t)\dfrac{\partial \varphi(x,t)}{\partial t}, & x \in]0,L[, \quad t \geq 0 \\ \varphi(0,t) = \varphi(L,t) = 0, & t \geq 0 \\ y(x,t) = \dfrac{\partial \varphi(x,t)}{\partial t}, \quad x \in]0,L[, & t \geq 0 \end{cases} \tag{41}$$

The system (40) can be represented as a first-order differential equation in $H = H_0^1(0,L) \times L^2(0,L)$:

$$\begin{cases} \dot{Z}(t) = AZ(t) + u(t)BZ(t) \\ y(t) = CZ(t) \end{cases} \tag{42}$$

where

$$Z = \begin{pmatrix} \varphi(x,\cdot) \\ \dfrac{\partial \varphi(x,\cdot)}{\partial t} \end{pmatrix}, \quad A = \begin{pmatrix} 0 & I \\ \bar{A} & 0 \end{pmatrix}, \quad \bar{A} = \frac{\partial^2}{\partial x^2}, \quad B = i\begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}, \quad C = \begin{pmatrix} 0, & I \end{pmatrix}$$

Here $D(\bar{A}) = H_0^1(0,L) \cap H^2(0,L)$.

The operator A is skew-adjoint ($A^* = -A$) with $D(A) = D(\bar{A}) \times D(\bar{A}^{1/2})$ and it generates a C_0-group $(e^{tA})_{t \geq 0}$ of isometrics on H with the following inner product:

$$\langle Z, \bar{Z} \rangle_H = \int_0^L \frac{\partial Z^1(x)}{\partial x}\frac{\partial \bar{Z}^1(x)}{\partial x}\,\mathrm{d}x + \int_0^L Z^2(x)\bar{Z}^2(x)\,\mathrm{d}x \quad \forall Z, \bar{Z} \in H$$

From (Curtain and Pritchard, 1978), the pair (C, A) is L^2-exactly observable on $[0,T]$ for all $T > 0$. So, as above, it is easy to see that

$$\exists r > 0;\ \forall t \geq 0;\ \forall u;\ \|u\|_\infty \leq r,\ u \text{ is SU on } [t, t+T] \text{ for (41)}$$

Again, from Remark 6 and Theorem 2, the Kalman-like observer of (40) for these inputs is as follows:

$$\begin{cases} \dfrac{\partial \hat{\varphi}_1(x,t)}{\partial t} = \hat{\varphi}_2(x,t) - R_u^2(t)\Big(\hat{\varphi}_2(x,t) - \varphi_2(x,t)\Big), \quad x \in]0,L[, \ t \geq 0 \\ \dfrac{\partial \hat{\varphi}_2(x,t)}{\partial t} = \dfrac{\partial^2 \hat{\varphi}_1(x,t)}{\partial x^2} + u(t)\hat{\varphi}_2(x,t) - R_u^4(t)\Big(\hat{\varphi}_2(x,t) - \varphi_2(x,t)\Big) \\ \hat{\varphi}_1(0,t) = \hat{\varphi}_1(L,t) = 0, \qquad t \geq 0 \\ \dot{S}_u(t) = -\theta S_u(t) - A_u^*(t)S_u(t) - S_u(t)A_u(t) + C^*C \end{cases} \tag{43}$$

where

$$S_u^{-1}(t) = \begin{pmatrix} R_u^1(t) & R_u^2(t) \\ R_u^3(t) & R_u^4(t) \end{pmatrix}$$

♦

6. Conclusion

In this paper, the observer synthesis of bilinear systems has been discussed:

- The first observer is simple and its gain is constant. This observer is simply a Luenberger-like one.
- The gain of the second observer depends on the systems inputs. This observer converges for a class of strongly regularly persistent inputs. Note that if the input is not a regularly persistent one, the convergence of the observer is not guaranteed, even in a finite-dimensional case. One of the fundamental problems consists in the following question:

> Given an observable dissipative bilinear system (1) (this means that for every initial state $x \neq \bar{x}$ there exists an input $u \in L^{\infty}([0,T],\mathbb{R})$ such that $y(x,u,t)$ is not identically equal to $y(\bar{x},u,t)$ on $[0,T]$, where $y(x,u,t)$ is the output associated with the input u and the initial state x), does the system (1) admit a persistent (resp. regularly persistent) input?

To answer this question, we must show the existence of a universal (strongly universal) input. Using the stratification of subanalytic sets, the author showed in (Sussmann, 1979) that every observable analytic system, on a finite-dimensional manifold, admits a universal input and that the set of analytic and universal inputs is a countable intersection of an open dense subset of $C^{\omega}([0,T])$ equipped with the Whitney topology. The idea for the construction of universal inputs is based on the fact that the stratification process is finite. In the infinite-dimensional case, this is not the case. Hence the method used in (Sussmann, 1979) cannot be applied for (1).

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