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TIME-OPTIMAL CONTROL OF A PARABOLIC SYSTEM WITH MULTIPLE TIME-VARYING LAGS

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In this paper, the time-optimal control problem for a distributed parabolic system in which different multiple time-varying lags appear both in the state equation and in the boundary condition is presented. Some particular properties of the optimal control are discussed.

1. Introduction

Various optimization problems associated with the optimal control of distributed-parameter systems with time lags appearing in the boundary conditions have been studied recently in (Kowalewski, 1987a; 1987b; 1988a; 1988b; 1988c; 1990a; 1990b; 1990c; 1990d; 1991; 1993a; 1993b; 1993c; 1993d; 1995; 1998; Kowalewski and Duda, 1992; Kowalewski and Krakowiak, 1994; Knowles, 1978; Wang, 1975; Wong, 1987).

In this paper, we consider the time-optimal control problem for a linear parabolic system in which different multiple time-varying lags appear both in the state equation and in the Neumann boundary condition. This equation constitutes in a linear approximation a universal mathematical model for many diffusion processes in which time-delayed feedback signals are introduced on the boundary of a system's spatial domain. For example, in the area of plasma control (Kowalewski and Duda, 1992), it is of interest to confine a plasma in a given bounded spatial domain Ω by introducing a finite electric potential barrier or a "magnetic mirror" surrounding Ω. For a collision-dominated plasma, its particle density can be described by a parabolic equation. Due to particle inertia and finiteness of electric potential barrier or the magnetic-mirror field strength, the particle reflection on the domain boundary is not instantaneous. Consequently, the particle flux on the boundary of Ω at any time depends on the flux of particles which escaped earlier with various velocities and reflected back into Ω at a later time. This leads to boundary conditions involving time-varying lags.

The existence and uniqueness of solutions of such parabolic equations are proved. The optimal control is characterized by the adjoint equation. By using this characterization, particular properties of the optimal control are proved.

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2. Existence and Uniqueness of Solutions

Consider now the distributed-parameter system described by the following parabolic equation:

$$\frac{\partial y}{\partial t} + A(t)y + \sum_{i=1}^{m} b_i(x,t)y\big(x,t-h_i(t)\big) = u, \quad x \in \Omega,\ t \in (0,T) \tag{1}$$

$$y(x,t') = \Phi_0(x,t'), \qquad x \in \Omega,\ t' \in \big[-\Delta(0),0\big) \tag{2}$$

$$y(x,0) = y_0(x), \qquad x \in \Omega \tag{3}$$

$$\frac{\partial y}{\partial \eta_A} = \sum_{s=1}^{l} c_s(x,t)y\big(x,t-k_s(t)\big) + v, \qquad x \in \Gamma,\ t \in (0,T) \tag{4}$$

$$y(x,t') = \Psi_0(x,t'), \qquad x \in \Gamma,\ t' \in \big[-\Delta(0),0\big) \tag{5}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded, open set with boundary Γ, which is a C^∞-manifold of dimension $(n-1)$. Locally, Ω is totally on one side of Γ. Furthermore,

$$y \equiv y(x,t;u), \quad u \equiv u(x,t), \quad v \equiv v(x,t), \quad Q \equiv \Omega \times (0,T)$$

$$\bar{Q} = \bar{\Omega} \times [0,T], \quad Q_0 = \Omega \times \big[-\Delta(0),0\big), \quad \Sigma = \Gamma \times (0,T)$$

$$\Sigma_0 = \Gamma \times \big[-\Delta(0),0\big)$$

T is a specified positive number representing a time horizon, b_i are given real C^∞ functions defined on $\bar{Q}$, c_i are given real C^∞ functions defined on $\sum$, $h_i(t)$ and $k_s(t)$ are functions representing time-varying lags, Φ_0 and Ψ_0 are initial functions defined on Q_0 and Σ_0, respectively. Moreover,

$$\Delta(0) = \max\big\{h_1(0), h_2(0), \ldots, h_m(0), k_1(0), k_2(0), \ldots, k_l(0)\big\}$$

The parabolic operator $(\partial/\partial t) + A(t)$ in the state equation (1) satisfies the hypothesis of Section 1, Chapter 4 of Lions and Magenes (1972, Vol.2, p.2) and $A(t)$ is given by

$$A(t)y = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x,t)\frac{\partial y(x,t)}{\partial x_j}\right) \tag{6}$$

where $a_{ij}(x,t)$ are real C^∞ functions defined on $\bar{Q}$ (the closure of Q) and satisfy the ellipticity condition

$$\sum_{i,j=1}^{n} a_{ij}(x,t)\varphi_i\varphi_j \geq \alpha\sum_{i=1}^{n}\varphi_i^2, \quad \alpha > 0, \quad \forall (x,t) \in \bar{Q}, \quad \forall \varphi_i \in \mathbb{R} \tag{7}$$

Equations (1)–(5) constitute a Neumann problem. The left-hand side of (4) is written in the following form:

$$\frac{\partial y}{\partial \eta_A} = \sum_{i,j=1}^{n} a_{ij}(x,t)\cos(n,x_i)\frac{\partial y(x,t)}{\partial x_j} = q(x,t), \quad x \in \Gamma, \ t \in (0,T) \tag{8}$$

where $\partial y/\partial \eta_A$ is directed towards the exterior for a normal derivative at Γ, $\cos(n, x_i)$ being the i-th direction cosine of n, and

$$q(x,t) = \sum_{s=1}^{l} c_s(x,t)y\big(x, t-k_s(t)\big) + v(x,t), \quad x \in \Gamma, \ t \in (0,T) \tag{9}$$

Let $t - h_i(t)$ for $i = 1, \ldots, m$ and $t - k_s(t)$ for $s = 1, \ldots, l$ be strictly increasing functions, $h_i(t)$ and $k_s(t)$ being nonnegative C^1 functions. Then there exist inverse functions of $t - h_i(t)$ and $t - k_s(t)$.

Let us write $r_i(t) \triangleq t - h_i(t)$ for $i = 1, \ldots, m$ and $\lambda_s(t) \triangleq t - k_s(t)$ for $s = 1, \ldots, l$. Then the inverse functions of $r_i(t)$ and $\lambda_s(t)$ have the form $t = f_i(r_i) = r_i + s_i(r_i)$ and $t = \varepsilon_s(r_s) = r_s + q_s(r_s)$, where $s_i(r_i)$ and $q_s(r_s)$ are time-varying predictions. Let $f_i(t)$ and $\varepsilon_s(t)$ be the inverse functions of $t - h_i(t)$ and $t - k_s(t)$, respectively.

We define the following iteration:

$$\begin{aligned}
\hat{t}_0 &= 0 \\
\hat{t}_1 &= \min\big\{f_1(0), f_2(0), \ldots, f_m(0), \varepsilon_1(0), \varepsilon_2(0), \ldots, \varepsilon_l(0)\big\} \\
\hat{t}_2 &= \min\big\{f_1(\hat{t}_1), f_2(\hat{t}_1), \ldots, f_m(\hat{t}_1), \varepsilon_1(\hat{t}_1), \varepsilon_2(\hat{t}_1), \ldots, \varepsilon_l(\hat{t}_1)\big\} \\
&\ \vdots \\
\hat{t}_j &= \min\big\{f_1(\hat{t}_{j-1}), f_2(\hat{t}_{j-1}), \ldots, f_m(\hat{t}_{j-1}), \varepsilon_1(\hat{t}_{j-1}), \varepsilon_2(\hat{t}_{j-1}), \ldots, \varepsilon_l(\hat{t}_{j-1})\big\}
\end{aligned}$$

First we shall prove sufficient conditions for the existence of a unique solution to the mixed initial-boundary value problem (1)–(5) for the case $u \in L^2(Q)$. For this purpose, for any pair of real numbers $r, s \geq 0$, we introduce the Sobolev space $H^{r,s}(Q)$ (Lions and Magenes 1972, Vol.2, p.6) defined by

$$H^{r,s}(Q) = H^0\big(0,T; H^r(\Omega)\big) \cap H^s\big(0,T; H^0(\Omega)\big) \tag{10}$$

which is a Hilbert space normed by

$$\left(\int_0^T \|y(t)\|^2_{H^r(\Omega)} dt + \|y\|^2_{H^s(0,T;H^0(\Omega))}\right)^{1/2} \tag{11}$$

where the spaces $H^r(\Omega)$ and $H^s(0,T;H^0(\Omega))$ are defined in Chapter 1 of (Lions and Magenes 1972, Vol.1).

The existence of a unique solution to the mixed initial-boundary value problem (1)–(5) on the cylinder Q can be proved using a constructive method, i.e., first, solving (1)–(5) on the subcylinder Q_1 and in turn on Q_2, etc. until the procedure covers the whole cylinder Q. In this way, the solution in the previous step determines the next one.

For simplicity, we introduce the following notation:

$$E_j \triangleq (\hat{t}_{j-1}, \hat{t}_j), \quad Q_j = \Omega \times E_j, \quad Q_0 = \Omega \times \big[-\Delta(0), 0\big)$$

$$\Sigma_j = \Gamma \times E_j, \qquad \Sigma_0 = \Gamma \times \big[-\Delta(0), 0\big) \quad \text{for } j = 1, \ldots$$

Using Theorem 6.1 of (Lions and Magenes, 1972, Vol.2, p.33), we can prove the following lemma.

Lemma 1. *Let*

$$u \in L^2(Q) \tag{12}$$

and

$$f_j \in L^2(Q_j) \tag{13}$$

where

$$f_j(x,t) = u(x,t) - \sum_{i=1}^{m} b_i(x,t) y_{j-1}\big(x, t - h_i(t)\big)$$

$$y_{j-1}(\cdot, \hat{t}_{j-1}) \in H^1(\Omega) \tag{14}$$

$$q_j \in H^{1/2,1/4}(\Sigma_j) \tag{15}$$

and

$$q_j(x,t) = \sum_{s=1}^{l} c_s(x,t) y_{j-1}\big(x, t - k_s(t)\big) + v(x,t)$$

Then there exists a unique solution $y_j \in H^{2,1}(Q_j)$ for the mixed initial-boundary value problem (1), (4), (14).

Proof. We observe that, for $j = 1$, $\sum_{i=1}^{m} y_{j-1}\big|_{Q_0}(x, t - h_i(t)) = \sum_{i=1}^{m} \Phi_0(x, t - h_i(t))$ and $\sum_{s=1}^{l} y_{j-1}\big|_{\Sigma_0}(x, t - k_s(t)) = \sum_{s=1}^{l} \Psi_0(x, t - k_s(t))$. Then the assumptions (13)–(15) are fulfilled if we assume that $\Phi_0 \in H^{2,1}(Q_0)$, $y_0 \in H^1(\Omega)$, $v \in H^{1/2,1/4}(\Sigma)$ and $\Psi_0 \in H^{1/2,1/4}(\Sigma_0)$. These assumptions are sufficient to ensure the existence of a unique solution $y_1 \in H^{2,1}(Q_1)$. In order to extend the result to Q_2, we have to prove that $y_1(\cdot, \hat{t}_1) \in H^1(\Omega)$, $q_2 \in H^{1/2,1/4}(\Sigma_2)$ and $f_2 \in L^2(Q_2)$. In fact, from Theorem 3.1 of (Lions and Magenes, 1972, Vol.1, p.19) $y_1 \in H^{2,1}(Q_1)$ implies that the mapping $t \to y_1(\cdot, t)$ is continuous from $[0, \hat{t}_1]$ into $H^1(\Omega)$. Thus $y_1(\cdot, \hat{t}_1) \in H^1(\Omega)$. Then using the Trace Theorem (Lions and Magenes, 1972, Vol.2, p.9), we can verify that $y_1 \in H^{2,1}(Q_1)$ implies that $y_1 \to y_1|_{\Sigma_1}$ is a linear, continuous mapping from $H^{2,1}(Q_1)$ into $H^{1/2,1/4}(\Sigma_1)$. By assuming that c_s are C^∞-functions and $v \in H^{1/2,1/4}(\Sigma)$, the condition $q_2 \in H^{1/2,1/4}(\Sigma_2)$ is fulfilled. Also, it is easy to notice that the assumption (13) follows from the fact that $y_1 \in H^{2,1}(Q_1)$ and $u \in L^2(Q)$. Then there exists a unique solution $y_2 \in H^{2,1}(Q_2)$. The foregoing result is now summarized for $j = 3, \ldots$. ■

Theorem 1. *Let y_0, Φ_0, $\Psi_0 v$, u be given with $y_0 \in H^1(\Omega)$, $\Phi_0 \in H^{2,1}(Q_0)$, $\Psi_0 \in H^{1/2,1/4}(\Sigma_0)$, $v \in H^{1/2,1/4}(\Sigma)$ and $u \in L^2(Q)$. Then there exists a unique solution $y \in H^{2,1}(Q)$ for the mixed initial-boundary value problem (1)–(5). Moreover, $y(\cdot, \hat{t}_j) \in H^1(\Omega)$ for $j = 1, \ldots$.*

3. Problem Formulation and Optimization Theorems

Now, we shall formulate the minimum-time problem for (1)–(5) in the context of Theorem 1, i.e.

$$u \in U = \{u \in L^2(Q) : \ |u(x,t)| \le 1\} \tag{16}$$

We shall define the reachable set Y such that

$$Y = \{y \in L^2(Q) : \ \|y - z_d\|_{L^2(\Omega)} \le \epsilon\} \tag{17}$$

where z_d and ϵ are given with $z_d \in L^2(\Omega)$ and $\epsilon > 0$.

The solving of the stated minimum-time problem is equivalent to hitting the target set Y in minimum time, i.e. minimizing the time t, for which $y(t;u) \in Y$ and $u \in U$.

Moreover, we assume that

$$\text{there exists } T > 0 \text{ and } u \in U \text{ with } y(T;u) \in Y \tag{18}$$

Theorem 2. *If the assumption (18) holds, then the set Y is reached in minimum time t^* by an admissible control $u^* \in U$. Moreover,*

$$\int_\Omega (z_d - y(t^*;u^*)) (y(t^*;u) - y(t^*;u^*)) \, dx \le 0 \quad \forall u \in U \tag{19}$$

Outline of the Proof. Let us define

$$t^* \stackrel{\text{df}}{=} \inf\{t : \ y(t;u) \in Y \ \text{ for some } u \in U\} \tag{20}$$

The minimum is well-defined, as (18) quarantees that this set is non-empty. By definition, we can choose $t_n \downarrow t^*$ and admissible controls $\{u_n\}$ such that

$$y(t_n; u_n) \in Y, \quad n = 1, 2, 3, \ldots \tag{21}$$

Each u_n is defined on $\Omega \times (0, t_n) \supset \Omega \times (0, t^*)$ and, to simplify the notation, we will write the restriction of u_n to $\Omega \times (0, t^*)$ again as u_n. The set of admissible controls then forms a weakly compact, convex set in $L^2(\Omega \times (0, t^*)$ and so we can extract a weakly convergent subset $\{u_m\}$ which converges weakly to some admissible control u^*.

Consequently, Theorem 1 implies that $y(t; u) \in L^2(\Omega)$ for each $u \in L^2(\Omega)$ and $t > 0$. Then, by using Theorem 1.2 of (Lions 1971, p.102) and Theorem 1, it is easy to verify that the mapping $u \to y(t^*; u)$ from $L^2(\Omega \times (0, t^*))$ into $L^2(\Omega)$ is continuous. Since any continuous linear mapping between Banach spaces is also weakly continuous (Dunford and Schwartz, 1958, Theorem V.3.15), the affine mapping $u \to y(t^*; u)$ must also be weakly continuous. Hence,

$$y(t^*; u_m) \to y(t^*; u^*) \ \text{ weakly in } \ L^2(\Omega) \tag{22}$$

Moreover, $\mathrm{d}y(u)/\mathrm{d}t \in L^2([0, t^*], H^0(\Omega))$, for each $u \in U$, by definition of $H^{2,1}(\Omega \times (0, t^*))$, and

$$\begin{aligned} \left\| y(t_m; u_m) \ - \ y(t^*; u_m) \right\|_{L^2(\Omega)} &= \left\| \int_{t^*}^{t_m} \dot{y}(\sigma; u_m) \, \mathrm{d}\sigma \right\|_{L^2(\Omega)} \\ &\le \sqrt{t_m - t^*} \left(\int_{t^*}^{t_m} \left\| \dot{y}(\sigma; u_m) \right\|^2_{L^2(\Omega)} \mathrm{d}\sigma \right)^{1/2} \end{aligned} \tag{23}$$

By applying Theorem 1.2 of (Lions, 1971) and Theorem 1 again, the set $\{\dot{y}(u_m)\}$ must be bounded in $L^2(0, t^*; H^0(\Omega))$, and so

$$\left\| y(t_m; u_m) - y(t^*; u_m) \right\|_{L^2(\Omega)} \le M\sqrt{t_m - t^*} \tag{24}$$

Combining (22) and (24) shows that

$$y(t_m; u_m) - y(t^*; u^*) = \big(y(t_m; u_m) - y(t^*; u_m)\big) + \big(y(t^*; u_m) - y(t^*; u^*)\big) \tag{25}$$

converges weakly to zero in $L^2(\Omega)$, and so $y(t^*; u^*) \in Y$ as Y is closed and convex, and hence weakly closed. This shows that Y is reached in time t^* by an admissible control. Accordingly, t^* must be the minimum time and u^* an optimal control.

We shall now prove the second part of our theorem. From Theorem 3.1 of (Lions and Magenes, 1972, Vol.1, p.19), $y(u) \in H^{2,1}(Q)$ implies that the mapping $t \to y(t; u)$ from $[0, T]$ into $L^2(\Omega)$ is continuous for each fixed u, and so $y(t^*; u) \notin \operatorname{int} Y$, for any $u \in U$, by the minimality of t^*.

From our earlier remarks, the set

$$\mathcal{A}(t^*) = \{y(t^*; u_x) \colon u_x \in U\} \tag{26}$$

is the continuous affine image of a weakly compact and convex set in $L^2(\Omega)$. Applying Theorem 21.11 of (Choquet, 1969) to the sets $\mathcal{A}(t^*)$ and Y shows that there exists a non-trivial hyperplane $z' \in L^2(\Omega)$ separating these sets, i.e.

$$\int_\Omega z' y(t^*; u) \, \mathrm{d}x \le \int_\Omega z' y(t^*; u^*) \, \mathrm{d}x \le \int_\Omega z' y \, \mathrm{d}x \tag{27}$$

for all $u \in U$ and $y \in L^2(\Omega)$ with

$$\|y - z_d\|_{L^2(\Omega)} \le \varepsilon \tag{28}$$

From the second inequality in (27), z' must support the set Y at $y(t^*; u^*)$. Since $L^2(\Omega)$ is a Hilbert space, z' must be of the form

$$z' = \lambda\big(z_d - y(t^*; u^*)\big) \quad \text{for some } \lambda > 0 \tag{29}$$

Therefore dividing (27) by λ gives the desired result (19). ■

We shall apply Theorem 2 to the control problem (1)–(5). To simplify (19), we introduce the adjoint equation and, for every $u \in U$, we define the adjoint variable $p = p(u) = p(x, t; u)$ as the solution to the equation

$$-\frac{\partial p(u)}{\partial t} + A^*(t)p(u) + \sum_{i=1}^{m} b_i\big(x, t + s_i(t)\big)\, p\big(x, t + s_i(t); u\big)\big(1 + s_i'(t)\big) = 0, \qquad x \in \Omega, \quad t \in \big(0, t^* - \Delta(t^*)\big) \tag{30}$$

$$-\frac{\partial p(u)}{\partial t} + A^*(t)p(u) = 0, \qquad x \in \Omega, \quad t \in \big(t^* - \Delta(t^*), t^*\big) \tag{31}$$

$$p(x, t^*; u) = z_d(x) - y(x, t^*; u), \qquad x \in \Omega \tag{32}$$

$$\frac{\partial p}{\partial \eta_{A^*}}(x, t) = \sum_{s=1}^{l} c_s\big(x, t + q_s(t)\big)p\big(x, t + q_s(t); u\big)\big(1 + q_s'(t)\big), \qquad x \in \Gamma, \quad t \in \big(0, t^* - \Delta(t^*)\big) \tag{33}$$

$$\frac{\partial p(u)}{\partial \eta_{A^*}}(x, t) = 0, \qquad x \in \Gamma, \quad t \in \big(t^* - \Delta(t^*), t^*\big) \tag{34}$$

where

$$\begin{cases} \Delta(t^*) = \max\big\{h_1(t^*), h_2(t^*), \dots, h_m(t^*), k_1(t^*), k_2(t^*), \dots, k_l(t^*)\big\} \\ \dfrac{\partial p(u)}{\partial \eta_{A^*}}(x, t) = \displaystyle\sum_{i,j=1}^{n} a_{ji}(x, t) \cos(n, x_i) \frac{\partial p(u)}{\partial x_j}(x, t) \\ A^*(t)p = -\displaystyle\sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}\left(a_{ij}(x, t)\frac{\partial p}{\partial x_i}\right) \end{cases} \tag{35}$$

The existence of a unique solution to the problem (30)–(34) on the cylinder $\Omega \times (0, t^*)$ can be proved using a constructive method. It is easy to notice that for given z_d and u, problem (30)–(34) can be solved backwards in time starting from

$t = t^*$, i.e. solving first (30)–(34) on the subcylinder Q_K and in turn on Q_{K-1}, etc. until the procedure covers the whole cylinder $\Omega \times (0, t^*)$. For this purpose, we may apply Theorem 1 (with an obvious change of variables).

Hence, by using Theorem 1, the following result can be proved:

Theorem 3. *Let the hypothesis of Theorem 1 be satisfied. Then, for a given $z_d \in L^2(\Omega)$ and any $u \in L^2(Q)$, there exists a unique solution $p(u) \in H^{2,1}(\Omega \times (0, t^*))$ for the adjoint problem (30)–(34).*

We simplify (19) using the adjoint equation (30)–(34). For this purpose, setting $u = u^*$ in (30)–(34), multiplying both sides of (30) and (31) by $y(u) - y(u^*)$, then integrating over $\Omega \times (0, t^* - \Delta(t^*))$ and $\Omega \times (t^* - \Delta(t^*), t^*)$, respectively, and then adding both sides of (30) and (31), we get

$$\begin{aligned}
&\int_0^{t^*}\int_\Omega \Big(- \frac{\partial p(u^*)}{\partial t} + A^*(t)p(u^*)\Big)\big(y(u) - y(u^*)\big)\,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \sum_{i=1}^m \int_0^{t^*-\Delta(t^*)} \int_\Omega b_i\big(x, t + s_i(t)\big)p\big(x, t + s_i(t); u^*\big)\big(1 + s_i'(t)\big) \\
&\quad \times \big(y(x,t;u) - y(x,t;u^*)\big)\,\mathrm{d}x\,\mathrm{d}t \\
&= - \int_\Omega p(x, t^*; u^*)(y(x, t^*; u) - y(x, t^*; u^*))\,\mathrm{d}x \\
&\quad + \int_0^{t^*}\int_\Omega p(u^*)\frac{\partial}{\partial t}\big(y(u) - y(u^*)\big)\,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \int_0^{t^*}\int_\Omega A^*(t)p(u^*)\big(y(u) - y(u^*)\big)\,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \sum_{i=1}^m \int_0^{t^*-\Delta(t^*)} \int_\Omega b_i\big(x, t + s_i(t)\big)p\big(x, t + s_i(t); u^*\big) \\
&\quad \times \big(1 + s_i'(t)\big)\big(y(x,t;u) - y(x,t;u^*)\big)\,\mathrm{d}x\,\mathrm{d}t = 0
\end{aligned} \tag{36}$$

Then, by applying (32), the formula (36) can be expressed as

$$\begin{aligned}
&\int_\Omega \big(z_d - y(t^*; u^*)\big)\big(y(t^*; u) - y(t^*; u^*)\big)\,\mathrm{d}x \\
&= \int_0^{t^*}\int_\Omega p(u^*)\frac{\partial}{\partial t}\big(y(u) - y(u^*)\big)\,\mathrm{d}x\,\mathrm{d}t + \int_0^{t^*}\int_\Omega A^*(t)p(u^*)\big(y(u) - y(u^*)\big)\mathrm{d}x\,\mathrm{d}t \\
&\quad + \sum_{i=1}^m \int_0^{t^*-\Delta(t^*)} \int_\Omega b\big(x, t + s_i(t)\big)p\big(x, t + s_i(t); u^*\big)\big(1 + s_i'(t)\big) \\
&\quad \times \big(y(x,t;u) - y(x,t;u^*)\big)\,\mathrm{d}x\,\mathrm{d}t
\end{aligned} \tag{37}$$

Based on (1), the first integral on the right-hand side of (37) can be rewritten as

$$\int_0^{t^*}\int_\Omega p(u^*)\frac{\partial}{\partial t}(y(u)-y(u^*))\,\mathrm{d}x\,\mathrm{d}t$$

$$= \int_0^{t^*}\int_\Omega p(u^*)(u-u^*)\,\mathrm{d}x\,\mathrm{d}t - \int_0^{t^*}\int_\Omega p(u^*)A(t)(y(u)-y(u^*))\,\mathrm{d}x\,\mathrm{d}t$$

$$-\sum_{i=1}^{m}\int_0^{t^*}\int_\Omega p(x,t;u^*)b_i(x,t)\Big(y\big(x;t-h_i(t);u\big)-y\big(x,t-h_i(t);u^*\big)\Big)\mathrm{d}x\,\mathrm{d}t$$

$$= \int_0^{t^*}\int_\Omega p(u^*)(u-u^*)\,\mathrm{d}x\,\mathrm{d}t - \int_0^{t^*}\int_\Omega p(u^*)A(t)(y(u)-y(u^*))\,\mathrm{d}x\,\mathrm{d}t$$

$$-\sum_{i=1}^{m}\int_{-h_i(0)}^{t^*-h_i(t^*)}\int_\Omega p\big(x,t_i+s_i(t_i);u^*\big)b\big(x,t_i+s_i(t_i)\big)\big(1+s_i'(t_i)\big)$$

$$\times\big(y(x,t_i;u)-y(x,t_i;u^*)\big)\,\mathrm{d}x\,\mathrm{d}t_i \tag{38}$$

where $t_i = t - h_i(t)$, $t = t_i + s_i(t_i)$ and $\mathrm{d}t = [1+s_i'(t_i)]\,\mathrm{d}t_i$.

The second integral on the right-hand side of (37), in view of Green's formula, can be expressed as

$$\int_0^{t^*}\int_\Omega A^*(t)p(u^*)\big(y(u)-y(u^*)\big)\,\mathrm{d}x\,\mathrm{d}t$$

$$= \int_0^{t^*}\int_\Omega p(u^*)A(t)\big(y(u)-y(u^*)\big)\,\mathrm{d}x\,\mathrm{d}t + \int_0^{t^*}\int_\Gamma p(u^*)\Big(\frac{\partial y(u)}{\partial\eta_A}-\frac{\partial y(u^*)}{\partial\eta_A}\Big)\,\mathrm{d}\Gamma\,\mathrm{d}t$$

$$-\int_0^{t^*}\int_\Gamma\frac{\partial p(u^*)}{\partial\eta_{A^*}}\big(y(u)-y(u^*)\big)\,\mathrm{d}\Gamma\,\mathrm{d}t \tag{39}$$

By applying the boundary condition (4), the second integral on the right-hand side of (39) can be expressed as

$$\int_0^{t^*}\int_\Gamma p(u^*)\left(\frac{\partial y(u)}{\partial\eta_A}-\frac{\partial y(u^*)}{\partial\eta_A}\right)\mathrm{d}\Gamma\,\mathrm{d}t$$

$$=\sum_{s=1}^{l}\int_0^{t^*}\int_\Gamma p(x,t;u^*)c_s(x,t)\Big(y\big(x,t-k_s(t);u\big)-y\big(x,t-k_s(t);u^*\big)\Big)\,\mathrm{d}\Gamma\,\mathrm{d}t$$

$$=\sum_{s=1}^{l}\int_{-k_s(0)}^{t^*-k_s(t^*)}\int_\Gamma p\big(x,t_s+q_s(t_s);u^*\big)c_s\big(x,t_s+q_s(t_s)\big)$$

$$\times\big(1+q_s'(t_s)\big)\big(y(x,t_s;u)-y(x,t_s;u^*)\big)\,\mathrm{d}\Gamma\,\mathrm{d}t_s \tag{40}$$

where $t_s = t-k_s(t), t = t_s+q_s(t_s)$ and $\mathrm{d}t = [1+q_s'(t_s)]\,\mathrm{d}t_s$.

The last component in (39) can be rewritten as

$$\int_0^{t^*}\int_\Gamma \frac{\partial p(u^*)}{\partial \eta_{A^*}}\big(y(u)-y(u^*)\big)\,\mathrm{d}\Gamma\,\mathrm{d}t = \int_0^{t^*-\Delta(t^*)}\int_\Gamma \frac{\partial p(u^*)}{\partial \eta_{A^*}}\big(y(u)-y(u^*)\big)\,\mathrm{d}\Gamma\,\mathrm{d}t + \int_{t^*-\Delta(t^*)}^{t^*}\int_\Gamma \frac{\partial p(u^*)}{\partial \eta_{A^*}}\big(y(u)-y(u^*)\big)\,\mathrm{d}\Gamma\,\mathrm{d}t \quad (41)$$

Substituting (40), (41) into (39) and then (38), (39) into (37), we obtain

$$\begin{aligned}
&\int_\Omega \big(z_d - y(t^*;u^*)\big)\big(y(t^*;u)-y(t^*;u^*)\big)\,\mathrm{d}x \\
&= \int_0^{t^*}\int_\Omega p(u^*)(u-u^*)\,\mathrm{d}x\,\mathrm{d}t - \int_0^{t^*}\int_\Omega p(u^*)A(t)(y(u)-y(u^*))\,\mathrm{d}x\,\mathrm{d}t \\
&- \sum_{i=1}^m \int_{-h_i(0)}^0\int_\Omega b_i\big(x,t+s_i(t)\big)p\big(x,t+s_i(t);u^*\big) \\
&\times \big(1+s_i'(t)\big)\big(y(x,t;u)-y(x,t;u^*)\big)\,\mathrm{d}x\,\mathrm{d}t \\
&- \sum_{i=1}^m \int_0^{t^*-h_i(t^*)}\int_\Omega b_i\big(x,t+s_i(t)\big)p\big(x,t+s_i(t);u^*\big) \\
&\times \big(1+s_i'(t)\big)\big(y(x,t;u)-y(x,t;u^*)\big)\,\mathrm{d}x\,\mathrm{d}t \\
&+ \int_0^{t^*}\int_\Omega p(u^*)A(t)(y(u)-y(u^*))\,\mathrm{d}x\,\mathrm{d}t \\
&+ \sum_{s=1}^l \int_{-k_s(0)}^0\int_\Gamma p\big(x,t+q_s(t);u^*\big)c_s\big(x,t+q_s(t)\big) \\
&\times \big(1+q_s'(t)\big)\big(y(x,t;u)-y(x,t;u^*)\big)\,\mathrm{d}\Gamma\,\mathrm{d}t \\
&+ \sum_{s=1}^l \int_0^{t^*-k_s(t^*)}\int_\Gamma p\big(x,t+q_s(t);u^*\big)c_s\big(x,t+q_s(t)\big) \\
&\times \big(1+q_s'(t)\big)big(y(x,t;u)-y(x,t;u^*)\big)\,\mathrm{d}\Gamma\,\mathrm{d}t \\
&- \int_0^{t^*-\Delta(t^*)}\int_\Gamma \frac{\partial p(u^*)}{\partial \eta_{A^*}}\big(y(u)-y(u^*)\big)\,\mathrm{d}\Gamma\,\mathrm{d}t \\
&- \int_{t^*-\Delta(t^*)}^{t^*}\int_\Gamma \frac{\partial p(u^*)}{\partial \eta_{A^*}}\big(y(u)-y(u^*)\big)\,\mathrm{d}\Gamma\,\mathrm{d}t
\end{aligned}$$

$$+\sum_{i=1}^{m}\int_{0}^{t^*-\Delta(t^*)}\int_{\Omega} b_i\big(x,t+s_i(t)\big)p\big(x,t+s_i(t);u^*\big) \times \big(1+s_i'(t)\big)\big(y(x,t;u)-y(x,t;u^*)\big)\,\mathrm{d}x\,\mathrm{d}t \tag{42}$$

From the Appendix, it is easy to verify that

$$-\sum_{i=1}^{m}\int_{t^*-\Delta(t^*)}^{t^*-h_i(t^*)}\int_{\Omega} b_i\big(x,t+s_i(t)\big)p\big(x,t+s_i(t);u^*\big) \times \big(1+s_i'(t)\big)\big(y(x,t;u)-y(x,t;u^*)\big)\,\mathrm{d}x\,\mathrm{d}t=0 \tag{43}$$

and

$$\sum_{s=1}^{l}\int_{t^*-\Delta(t^*)}^{t^*-k_s(t^*)}\int_{\Gamma} c_s\big(x,t+q_s(t)\big)p\big(x,t+q_s(t);u^*\big) \times \big(1+q_s'(t)\big)\big(y(x,t;u)-y(x,t;u^*)\big)\,\mathrm{d}\Gamma\,\mathrm{d}t=0 \tag{44}$$

Using the fact that $y(x,t;u)=y(x,t;u^*)=\Phi_0(x,t)$ for $x\in\Omega$ and $t\in(-h_i,0)$ $\forall i$ and $y(x,t;u)=y(x,t;u^*)=\Psi_0(x,t)$ for $x\in\Gamma$ and $t\in(-k_s(0),0)$ $\forall s$, and then substituting (42), (43) into (41) gives

$$\int_{\Omega}\big(z_d-y(t^*;u^*)\big)\big(y(t^*;u)-y(t^*;u^*)\big)\,\mathrm{d}x\,\mathrm{d}t=\int_0^{t^*}\int_{\Omega}p(u^*)(u-u^*)\,\mathrm{d}x\,\mathrm{d}t \tag{45}$$

Substituting (45) into (19) gives

$$\int_0^{t^*}\int_{\Omega}p(u^*)(u-u^*)\,\mathrm{d}x\,\mathrm{d}t\le 0\quad \forall u\in U \tag{46}$$

The foregoing result is now summarized.

Theorem 4. *The optimal control u^* is characterized by the condition (46). Moreover, we have*

$$u^*(x,t)=\operatorname{sign}\big(p(x,t;u^*)\big),\quad x\in\Omega,\quad t\in(0,t^*) \tag{47}$$

whenever $p(u^*)$ is non-zero.

This property leads to the following result:

Theorem 5. *If the coefficients of the operator $A(t)$ and the functions $b_i(x,t)$ for $i=1,\dots,m$ and $c_s(x,t)$ for $s=1,\dots,l$ are analytic in $\bar{\Omega}\times[0,T]$, and Ω has analytic boundary Γ, then there exists a unique optimal control for the mixed initial-boundary value problem (1)–(5). Moreover, the optimal control is bang-bang, i.e. $|u^*(x,t)|\equiv 1$ almost everywhere and the unique solution to (1)–(5), (30)–(34), (47).*

Outline of the proof. We have to verify that $p(x,t) \neq 0$ for almost all $(x,t) \in \Omega \times (0,t^*)$. We shall show this fact by contradiction. Therefore we suppose that

$$p(x,t) = 0 \text{ for } (x,t) \in K \subset \Omega \times (0,t^*) \tag{48}$$

where $K \neq \emptyset$.

Let us denote by j a non-negative integer such that

$$t^* - \hat{t}_j > 0 \text{ for } j = 1,2,\ldots$$

Moreover, we suppose that $K \cap \Omega \times \left(t^* - \hat{t}_1, t^*\right) \neq \emptyset$.

Then $p(u^*)$ satisfies the following adjoint equation in the cylinder $\Omega \times (t^* - \hat{t}_1, t^*)$:

$$-\frac{\partial p(u^*)}{\partial t} + A^*(t)p(u^*) = 0, \quad x \in \Omega, \quad t \in (t^* - \hat{t}_1, t^*) \tag{49}$$

$$\frac{\partial p(u^*)}{\partial \eta_{A^*}} = 0, \quad x \in \Gamma, \quad t \in (t^* - \hat{t}_1, t^*) \tag{50}$$

It is easy to verify (Tanabe, 1965) that $p(u^*)$ must be analytic in the cylinder $\Omega \times (t^* - \hat{t}_1, t^*)$. Then from (45) it follows that

$$p(x,t) \equiv 0 \text{ for } (x,t) \in \bar{\Omega} \times (t^* - \hat{t}_1, t^*) \tag{51}$$

Using Theorem 3.1 of (Lions and Magenes, 1972, Vol.1, p.19), we can verify that $p(u^*) \in H^{2,1}(Q)$ implies that $t \to p(t;u^*)$ is a continuous mapping from $[0,T]$ into $H^1(\Omega) \subset L^2(\Omega)$. Thus $p(t;u^*) \in L^2(\Omega)$ and so

$$p(t^*;u^*) = 0 = y(t^*;u^*) - z_d \tag{52}$$

i.e. a contradiction.

Now, we shall extend our result to any cylinder $\bar{\Omega} \times (t^* - \hat{t}_j, t^* - \hat{t}_{j-1})$, $j = 2,3,\ldots$.

It is easy to notice that $p(u^*)$ satisfies the adjoint equation

$$-\frac{\partial p(u^*)}{\partial t} + A^*(t)p(u^*) + \sum_{i=1}^{m} b_i(x,t+s_i(t))p(x,t+s_i(t))(1+s_i'(t)) = 0,$$
$$x \in \Omega, \quad t \in (t^*, t^* - \hat{t}_1) \tag{53}$$

$$\frac{\partial p(u^*)}{\partial \eta_{A^*}} = \sum_{s=1}^{l} c_s\big(x,t+q_s(t)\big)p\big(x,t+q_s(t)\big)\left(1+q_s'(t)\right),$$
$$x \in \Gamma, \quad t \in (t^* - \hat{t}_2, t^* - \hat{t}_1) \tag{54}$$

in the cylinder $\bar{\Omega} \times (t^* - \hat{t}_2, t^* - \hat{t}_1)$.

Then $\sum_{i=1}^{m} p\big|_{\Omega}(x,t+s_i'(t);u^*)$ and $\sum_{s=1}^{l} p\big|_{\Gamma}(x,t+q_s(t);u^*)$ are analytic for $x \in \bar{\Omega}$, $t \in (t^* - \hat{t}_2, t^* - \hat{t}_1)$ and $x \in \Gamma$, $t \in (t^* - \hat{t}_2, t^* - \hat{t}_1)$, respectively, and consequently $p(u^*)$ must be analytic in $\bar{\Omega} \times (t^* - \hat{t}_2, t^* - \hat{t}_1)$, since (53), (54) have analytic coefficients (Tanabe, 1965). Thus $p(u^*)$ must be analytic in any cylinder $\bar{\Omega} \times (t^* - \hat{t}_j, t^* - \hat{t}_{j-1})$ and $\bar{\Omega} \times (0, t^* - \hat{t}_j)$ for $j = 2, 3, \ldots$.

Now we suppose that

$$p(u^*) = 0 \quad \text{for} \quad (x,t) \in K \cap \Omega \times (t^* - \hat{t}_j, t^* - \hat{t}_{j-1}) \tag{55}$$

for some $j = 2, 3, \ldots$.

Then, by analyticity and continuity, from (55) it follows that

$$p(u^*) \equiv 0 \quad \text{for} \quad (x,t) \in \bar{\Omega} \times (t^* - \hat{t}_j, t^* - \hat{t}_{j-1}) \tag{56}$$

Substituting (56) into (33) gives

$$\frac{\partial p}{\partial \eta_{A^*}}(x,t) = 0 \quad \text{for} \quad (x,t) \in \Gamma \times (t^* - \hat{t}_{j-1}, t^* - \hat{t}_{j-2}) \tag{57}$$

We can observe that $p(u^*)$ satisfies

$$-\frac{\partial p(u^*)}{\partial t} + A^*(t)p(u^*) = 0, \quad x \in \Omega, \quad t \in (t^* - \hat{t}_{j-1}, t^* - \hat{t}_{j-2}) \tag{58}$$

$$\frac{\partial p(u^*)}{\partial \eta_{A^*}}(x,t) = 0, \quad x \in \Gamma, \quad t \in (t^* - \hat{t}_{j-1}, t^* - \hat{t}_{j-2}) \tag{59}$$

$$p(\cdot, t^* - \hat{t}_{j-1}; u^*) = 0 \tag{60}$$

in the cylinder $\bar{\Omega} \times (t^* - \hat{t}_{j-1}, t^* - \hat{t}_{j-2})$.

Then, using the property of backward uniqueness, we have

$$p(u^*) \equiv 0 \quad \text{in} \quad \bar{\Omega} \times (t^* - \hat{t}_{j-1}, t^* - \hat{t}_{j-2}) \tag{61}$$

Again we repeat the procedure until $p(t^*; u^*) = 0$, which contradicts the fact that $z_d \neq y(t^*; u^*)$. ∎

4. Conclusions and Some Generalizations

The results presented in the paper can be treated as a generalization of the results obtained by Kowalewski and Krakowiak (1994) onto the case of different multiple time-varying lags appearing both in the state equations and in the boundary conditions. Moreover, the time-optimal control problems presented here can be extended to certain cases of non-linear control without convexity and to certain fixed-time problems.

For this purpose, suppose that a Lebesgue measurable set-valued mapping $F : \Omega \times [0,T] \to CR$ (compact subsets of $\mathbb{R}$) is given, for which

$$\max \left| F(x,t) \right| \le g(x,t), \quad (x,t) \in \Omega \times [0,T] \tag{62}$$

and $g \in L^2(Q)$.

Let the set of admissible controls be

$$\mathcal{L}_F = \{u\colon\ u \text{ is Lebesgue measurable, } u(x,t) \in F(x,t),\ (x,t) \in Q\} \quad (63)$$

The assumption (62) implies that $\mathcal{L}_F \subset L^2(Q)$ and, by Theorem 1, $y(u) \in H^{2,1}(Q)$, for each $u \in \mathcal{L}_F$. We shall also use the fact that the set-valued mapping

$$\mathrm{co}F:\ (x,t) \to \mathrm{co}(F(x,t)) \quad (64)$$

(where co denotes the convex hull) is also Lebesgue measurable, with compact and convex values. In (Casting, 1967) the following result was proved:

Lemma 2. *$\mathcal{L}_{\mathrm{co}F}$ is a weakly compact, convex subset of $L^2(Q)$.*

Instead of (18) we will now assume the following:

$$\text{there exists a } T > 0 \text{ and } u \in \mathcal{L}_{\mathrm{co}F} \text{ with } \quad y(T;u) \in Y \quad (65)$$

Then we have the following theorem:

Theorem 6. *If (65) holds, the coefficients of the operator $A(t)$ and the functions $b_i(x,t)$ for $i = 1,\dots,m$ and $c_s(x,t)$ for $s = 1,\dots,l$ are analytic and Ω has an analytic boundary, then Y is hit in minimum time t^* by a unique admissible control $u^* \in \mathcal{L}_F$ and*

$$\int_0^{t^*}\int_\Omega p(x,t;u^*)\big(u(x,t) - u^*(x,t)\,\mathrm{d}x\,\mathrm{d}t \le 0 \quad (66)$$

for all $u \in \mathcal{L}_{\mathrm{co}F}$. Furthermore, u^* is bang-bang in the sense that $u^*(x,t) \in$ ex$(F,(x,t))$ for almost all (x,t), where ex *denotes extreme points.*

Outline of the Proof. Consider first the control problem of steering (1)–(5) to the set Y in minimum time, with controls $u \in \mathcal{L}_{\mathrm{co}F}$. For this problem, we can argue exactly as before, using the weak compactness and convexity of $\mathcal{L}_{\mathrm{co}F}$ and the regularity of $y(u) \in H^{2,1}(Q)$ to verify the existence of an optimal control $u^* \in \mathcal{L}_{\mathrm{co}F}$ hitting Y in minimum time t^*, for which (66) holds. If we can verify that $u^* \in \mathcal{L}_F$, then it must be optimal for the original problem. We shall do this by showing that u^* is bang-bang.

For that purpose, suppose that $u^* \notin \mathrm{ex}(\mathcal{L}_{\mathrm{co}F})$. Then there exists a non-zero function $f \in L^2(\Omega \times (0,t^*))$ for which $U \pm f \in \mathcal{L}_{\mathrm{co}F}$. Let $f(x,t) \neq 0$ for $(x,t) \in E \subset \Omega \times (0,t^*), E \neq \emptyset$. For any measurable set $H \subset E$, we have

$$u^* \pm f\chi_{\mathrm{H}} \in \mathcal{L}_{\mathrm{co}F}$$

and so, by (66),

$$\int_0^{t^*}\int_\Omega p(u^*)(u^* - (u^* \pm f\cdot\chi_{\mathrm{H}}))\,\mathrm{d}x\,\mathrm{d}t \le 0 \quad (67)$$

i.e.

$$\int_H p(u^*)f\ \mathrm{d}x\,\mathrm{d}t = 0 \ \text{ for any measurable set } H \subset E \quad (68)$$

The only way this can happen is for $p(u^*) = 0$ on E, but this contradicts Theorem 5. Accordingly, $u^* \in \mathrm{ex}(\mathcal{L}_{\mathrm{co}F})$ and by Theorem 7.1 of (Olech, 1966)

$$u^*(x,t) \in \mathrm{ex}(\mathrm{co}F(x,t)) \ \text{ for almost all } (x,t) \tag{69}$$

However (Dunford and Schwartz 1958),

$$\mathrm{ex}(\mathrm{co}F(x,t)) \subset \mathrm{ex}F(x,t) \tag{70}$$

and so

$$u^*(x,t) \in \mathrm{ex}F(x,t) \quad \text{a.e.} \tag{71}$$

The uniqueness of u^* now follows in the usual way. ■

Example 1. Consider now the following problem of non-linear control, which is of interest in certain induction-heating applications. Suppose that (1) is replaced by

$$\frac{\partial y}{\partial t} + A(t)y + \sum_{i=1}^{m} b_i(x,t)y\big(x,t-h_i(t)\big) = \Phi\big(x,t;u(x,t)\big) \tag{72}$$

where Φ is continuous in the third variable, measurable in the first two variables, and the controls u are measurable and take their values in a fixed compact set $U \subset \mathbb{R}$. Also suppose that

$$|\Phi(x,t,f_1)| \le g(x,t) \ \text{ for all } (x,t) \in Q, \ \text{ all } f_1 \in U \tag{73}$$

for some $g \in L^2(Q)$. Then the set-valued function $F : (x,t) \to \Phi(x,t,U)$ is measurable and has compact values. Moreover (Casting, 1967),

$$\mathcal{L}_F = \{\Phi(x,t,u(x,t)) \colon \ u \text{ is measurable and } \ u(x,t) \in U \ \text{ a.e.}\} \tag{74}$$

Consequently, if the analyticity assumptions of Theorem 6 are satisfied, the time-optimal control problem (72), (2)–(5) has a bang-bang solution.

Subsequently, we can consider the fixed-time problem, i.e.

$$\min \int_{\Omega} |y(x,T;u) - z_d(x)|^2 \,\mathrm{d}x, \quad T \text{ fixed} \tag{75}$$

subject to the mixed initial boundary value problem (1)–(5) (except for the trivial case where $z_d = y(x,T;u)$ for some admissible control u). This problem can be proven in an analogous procedure, as the necessary and sufficient conditions of optimality for this problem coincide with the adjoint problem (30)–(34) (Kowalewski, 1998). ♦

Finally, one may consider the following example.

Example 2. Using the condition (18), we will verify that the parabolic system (1)–(5) is approximately controllable in $L^2(\Omega)$ in any finite time $T > 0$, i.e. $\{y(T;u) : u \in L^2(Q)\}$ is dense in $L^2(\Omega)$. By the Hahn-Banach theorem, this will be the case if

$$\int_{\Omega} z_1(x)y(x,T;u)\,\mathrm{d}x = 0, \quad z_1 \in L^2(\Omega) \tag{76}$$

for all $u \in L^2(\Omega)$, implies that $z_1 = 0$. Let $p \in H^{2,1}(Q)$ be the unique solution to (30)–(34) with

$$p(x,T) = z_1(x), \quad x \in \Omega \tag{77}$$

In the proof of Theorem 2 we have shown that

$$\int_\Omega z_1(y(u) - y(u_1))\,\mathrm{d}x = \int_0^T \int_\Omega p(u - u_1)\,\mathrm{d}x\,\mathrm{d}t \tag{78}$$

and so, if (76) holds for all $u \in L^2(Q)$, then

$$\int_0^T \int_\Omega pu\,\mathrm{d}x\,\mathrm{d}t = 0 \tag{79}$$

$u \in L^2(Q)$, and $p = 0$ in Q. By continuity,

$$p(x,T) = z_1(x) = 0 \tag{80}$$

for almost all $x \in \Omega$. ♦

Appendix

First, we shall verify (43). By using (31), it is easy to note that

$$-\frac{\partial p(u^*)}{\partial t} A^*(t)p(u^*) = -\sum_{i=1}^m b_i\big(x, t + s_i(t)\big)p\big(x, t + s_i(t); u^*\big)\big(1 + s_i'(t)\big) = 0$$

is satisfied in the interval $(t^* - \Delta(t^*), t^*)$ and consequently in the subinterval $(t^* - \Delta(t^*), t^* - h_i(t^*))$, where

$$t^* - \Delta(t^*) \le t^* - h_i(t^*) \le t^* \quad \forall i \text{ and } x \in \Omega$$

Then

$$\begin{aligned}
-\sum_{i=1}^m \int_0^{t^*-h_i(t^*)} & \int_\Omega b_i\big(x, t + s_i(t)\big)p\big(x, t + s_i(t); u^*\big) \\
& \times\big(1 + s_i'(t)\big)\big(y(x,t;v) - y(x,t;u^*)\big)\,\mathrm{d}x\,\mathrm{d}t \\
& + \sum_{i=1}^m \int_0^{t^*-\Delta(t^*)} \int_\Omega b_i\big(x, t + s_i(t)\big)p\big(x, t + s_i(t); u^*\big) \\
& \times\big(1 + s_i'(t)\big)\big(y(x,t;v) - y(x,t;u^*)\big)\,\mathrm{d}x\,\mathrm{d}t \\
= -\sum_{i=1}^m & \int_{t^*-\Delta(t^*)}^{t^*-h_i(t^*)} \int_\Omega b_i\big(x, t + s_i(t)\big)p\big(x, t + s_i(t); u^*\big) \\
& \times\big(1 + s_i'(t)\big)\big(y(x,t;v) - y(x,t;u^*)\big)\,\mathrm{d}x\,\mathrm{d}t = 0
\end{aligned}$$

Thus, we have verified (43). Now, we shall verify (44).

Using (34), we can observe that the boundary condition

$$\frac{\partial p(u^*)}{\partial \eta_{A^*}}(x,t) = \sum_{s=1}^{l} c_s\big(x, t+q_s(t)\big)p\big(x, t+q_s(t); u^*\big)\big(1+q_s'(t)\big) = 0$$

is satisfied in the interval $(t^* - \Delta(t^*), t^*)$ and consequently in the subinterval $(t^* - \Delta(t^*), t^* - k_s(t^*))$, where

$$t^* - \Delta(t^*) \le t^* - k_s(t^*) \le t^* \quad \forall s \text{ and } x \in \Gamma$$

Then

$$\begin{aligned}
&\sum_{s=1}^{l} \int_0^{t^*-k_s(t^*)} \int_\Gamma c_s\big(x, t+q_s(t)\big)p\big(x, t+q_s(t); u^*\big) \\
&\qquad \times \big(1+q_s'(t)\big)\big(y(x,t;v) - y(x,t;u^*)\big)\,\mathrm{d}\Gamma\,\mathrm{d}t \\
&\qquad - \sum_{s=1}^{l} \int_0^{t^*-\Delta(t^*)} \int_\Gamma c_s\big(x, t+q_s(t)\big)p\big(x, t+q_s(t); u^*\big) \\
&\qquad \times \big(1+q_s'(t)\big)\big(y(x,t;v) - y(x,t;u^*)\big)\,\mathrm{d}\Gamma\,\mathrm{d}t \\
&= \sum_{s=1}^{l} \int_{t^*-\Delta(t^*)}^{t^*-k_s(t^*)} \int_\Gamma c_s\big(x, t+q_s(t)\big)p\big(x, t+q_s(t); u^*\big) \\
&\qquad \times \big(1+q_s'(t)\big)\big(y(x,t;v) - y(x,t;u^*)\big)\,\mathrm{d}\Gamma\,\mathrm{d}t = 0
\end{aligned}$$

Thus we have derived the formula (44).

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