Appl. Math. and Comp. Sci., 1998, Vol.8, No.4, 733–754

SOURCE IDENTIFICATION IN DISTRIBUTED PARAMETER SYSTEMS

MIKHAIL SKLIAR*, W. FRED RAMIREZ**

In this paper, we solve the problem of the pointwise source identification of the convection-diffusion transport processes. This is done by converting the identification problem into an optimization problem of finding a spatial location and the capacity of a point source which results in the best match of model-predicted measurements to actual observed measurements.

1. Introduction

Identification of the source function in distributed parameter systems (DPS) is an example of the inverse problem. Though identification of DPS has been an active area of research for the last three decades (Kubrusly, 1977; Tzafestas, 1982), the effort was mostly concentrated on parameter identification. A number of proposed source function identification algorithms are not readily applicable in the case of complex spatial geometry and boundary conditions. In some cases, proposed methods are specific to the problems in one or two spatial dimensions. Other methods may work only with a certain type of measurements. For instance, it is common to limit permissible measurements to boundary observations. Finally, all available methods are usually specific to a particular type of partial differential equations.

Silva Nato and Özişik (1993; 1994) considered the identification of the spatial location and dynamics of a heat source in a one-dimensional heat transport problem and proposed a solution based on the conjugate gradient method. For a one-dimensional time-varying diffusion equation with a single unknown pointwise heat source and boundary measurements at both end points, they develop an iterative procedure for finding spatial location and capacity of the source which minimizes the difference between actual and model-predicted measurements.

Ohnaka and Uosaki (1989) proposed an identification method for multiple pointwise sources of the diffusion equation which uses an integral equation approach to minimize the sum of squares of relative errors between the model-predicted and observed boundary measurements. The number of unknown point sources is determined

* Department of Chemical and Fuels Engineering, University of Utah, Salt Lake City, Utah 84112, e-mail: mikhail.skliar@m.cc.utah.edu.

** Department of Chemical Engineering, University of Colorado, Boulder, Colorado 80309, e-mail: fred.ramirez@colorado.edu.

as a number for which the relative identification error becomes insensitive to the increase in the numbers of identified sources.

A source function identification for a one-dimensional diffusion equation on a semi-infinite line $x \geq 0$, where x is the spatial variable, was considered by Lin and Ewing (1989). They assumed that an unknown source is in the form $f(x)g(t)$, where a time-dependent capacity is known and only spatial distribution of the source action $f(x)$ has to be identified. They further assumed that the boundary value of the state variable and its first derivative are known. Under some additional conditions, they prove existence and uniqueness of the solution to this identification problem. They also proposed the numerical algorithm for determination of $f(x)$ on the closed interval $x \in [0, 1]$.

Newsam and Enting (1988), and Enting and Newsam (1990) considered the problem of estimating surface sources of trace atmospheric constituents (such as CO_2 sources) from surface concentration data. They have used a three-dimensional diffusion equation in spherical coordinates as a model of the transport process. A particular form of the model equation and chosen boundary conditions allow for the analytical solution of concentration distribution as a function of model parameters and strengths of the sources. They used the analytical solution to analyze the influence of different factors on our ability to invert measurement data to obtain source estimations. In particular, they examined the effect of measurement errors on the accuracy of source identification. They considered different types of measurements, including surface measurements, high altitude measurements and height averaged measurements. They concluded that the determination of the boundary source capacities based on the given boundary measurements is a mildly ill-posed problem, in that the measurement errors and the estimation errors are linearly correlated. They further showed that the estimation of bulk (non-boundary) sources is a more ill-posed problem, and that there is a quadratic dependence between measurement and estimation errors. Enting (1993) used a simplified semi-analytical model of atmospheric transport to obtain error estimates for sources deduced from the spatial distribution of greenhouse gases. He suggested that there is a trade off between the resolution and variance in an inversion problem.

In this paper, we propose a single pointwise source identification method for the distributed parameter systems with complex geometry and arbitrary distribution of model parameters. The solution is based on the iterative application of the least-squares source-capacity estimation and gradient search for the source location. The proposed method is applied to the identification of an unknown source in a two-dimensional convection-diffusion problem with complex velocity distribution. A number of numerical experiments are conducted to study the effect of various factors on the accuracy of the identification results.

2. Source Isolation as an Optimization Problem

In this paper, the source identification problem is approached by converting it into an optimization problem. This allows us to abstract from the mathematically ill-posed nature of the source identification problem and concentrate on the development of the practical method of finding an unknown source function which minimizes the error between measurements and model predictions.

Problem statement: Given

(a) the model of the process

$$\frac{\partial X(t,z)}{\partial t} = \mathcal{N}(X,t,z)X(t,z) + F_n(t,z) + F_u(t,z), \quad z \in \Omega \tag{1}$$

with the appropriate initial and boundary conditions, where $\mathcal{N}(\cdot)$ is the convection-diffusion operator, $z = (z_1, z_2, z_3) \in \Omega$, $X(\cdot)$ is a state function, $F_n(\cdot)$ describes known inputs (sources) to the system and $F_u(\cdot)$ is an unknown source function;

(b) the model of the measurement system which relates the state function $X(\cdot)$ to the model-predicted time sampled pointwise measurements $\boldsymbol{Z}^M(k)$;

(c) current process measurements $\boldsymbol{Z}(k)$, $k = 1, 2, \ldots$;

(d) the time of an unknown source application τ.

Find the spatial location $z^u = (z_1^u, z_2^u, z_3^u)$ and the capacity $f(t)$ of an unknown source function

$$F_u = f(t)\delta(z_1 - z_1^u)\delta(z_2 - z_2^u)\delta(z_3 - z_3^u) \tag{2}$$

such that when F_u is used in the model, it minimizes a norm of the errors between the observed and the model-predicted measurements, i.e.

$$\min_{F_u} \|\boldsymbol{Z}(k) - \boldsymbol{Z}^M(k)\| \tag{3}$$

This problem statement deserves commentary.

(1) *This is a single point source identification problem.*

In many application areas, source functions of interest can often be adequately characterized as pointwise sources. When the actual source function is spatially distributed, its identification as an "equivalent" pointwise source can provide useful information: the location of the "equivalent" point source will usually be within the region affected by an unknown distributed source; the capacity of the "equivalent" point source is often a good approximation of the overall capacity of a distributed source.

(2) *Perfect process measurements are assumed.*

Information about an unknown source is carried exclusively by process measurements. As will be seen from the numerical experiment, noisy measurements severely limit our ability for early and accurate identification of an unknown source.

(3) *Application time of an unknown source is assumed to be known.*

The time τ of an unknown source application has a strong influence on source identification. If τ is not known, then it must be estimated as a part of the source identification. The effect of misidentified τ on the source identification will be examined later in the paper.

(4) *One step identification.*

The source identification is posed as a "one-shot" problem. When new measurements $\boldsymbol{Z}(k)$ become available, the identification procedure starts anew, and the source identification is based only on the latest set of measurements. The identification results from $k-1$ step are used only as an initial identification guess. Such formulation allows us to address the problem of identifying moving source with varying capacity. This approach, however, is sensitive to measurement noise. An alternative is to identify a source such that the following objective is minimized:

$$\min_{F_u} \sum_{i=0}^{k} \left\| \boldsymbol{Z}(i) - \boldsymbol{Z}^M(i) \right\| \tag{4}$$

This is more appropriate for identification of a stationary time-invariant source based on noisy measurements.

(5) *Nonlinear mixed-integer optimization.*

After the discretization, the model can be written as

$$\boldsymbol{x}(k+1) = \boldsymbol{A}(k)\boldsymbol{x}(k) + \boldsymbol{f}_n(k) + \boldsymbol{f}_u(k) \tag{5}$$

where the state vector $\boldsymbol{x}(k)$ is an approximation of the state function $X(k\Delta t, \boldsymbol{z})$, the matrix $\boldsymbol{A}(k)$ approximates the spatial operator $\mathcal{N}(X, k\Delta t, \boldsymbol{z})$, the input vector $\boldsymbol{f}_n(k)$ is an approximation of $F_n(k\Delta t, \boldsymbol{z})^1$ and Δt is the time discretization step. The identifiable location of an unknown source function is now limited to the mesh nodes used to approximate the distributed model. Therefore $\boldsymbol{f}_u$ can be written as

$$\boldsymbol{f}_u(k) = f(k) \cdot \boldsymbol{\delta} \tag{6}$$

where $f(k)$ describes temporal dynamic of the source and $\boldsymbol{\delta} = \{\delta_i | \{0, 1\}\}$ is the vector describing spatial location of the source. For a single point source, we have $\sum_i \delta_i = 1$.

The source identification amounts to the estimation of $f(k)$ and the integer valued vector $\boldsymbol{\delta}$. Since $f(k)$ and $\boldsymbol{\delta}$ enter the problem nonlinearly, the source identification in this formulation is a nonlinear mixed-integer optimization (NLMIO) problem (Floudas, 1995).

(6) *Global versus local solution.*

In this paper, we will consider a local solution to the source identification problem.

[1] $\boldsymbol{f}_n(k)$ also includes the contribution from the boundary conditions.

3. Basic Identification Algorithm

After spatial and temporal discretization, the source identification problem is formulated as follows:

$$\min_{f(k),\boldsymbol{\delta}} \left\| \boldsymbol{z}(k+1) - \boldsymbol{z}^M(k+1) \right\| \tag{7}$$

$$\boldsymbol{z}^M(k+1) = \boldsymbol{H}(k+1)\boldsymbol{x}(k+1) \tag{8}$$

subject to

$$\boldsymbol{x}(k+1) = \boldsymbol{A}(k)\boldsymbol{x}(k) + \boldsymbol{f}_n(k) + f(k) \cdot \boldsymbol{\delta} \tag{9}$$

$$\boldsymbol{\delta} = \left\{ \delta_i \mid \delta_i = \{0,1\},\ i = \overline{1,n} \right\}, \qquad \sum_i^n \delta_i = 1 \tag{10}$$

where (9) is the discrete analog of (1), and (8) is a result of the discretization of the model of the measurement system.

The methods of the mixed-integer optimization can now be used to find $f^*(k)$ and $\boldsymbol{\delta}^*$ which minimizes (7). Instead, we develop an optimization method based on a least-squares solution for the capacity $f(k)$, coupled with a gradient search algorithm for the location $\boldsymbol{\delta}$. The basic structure of the method consists of the iterative application of the following two stages:

1. Given a value of $\boldsymbol{\delta}$, find $f(k)$ which minimizes the following objective:

$$\min_{f(k)} \left\| \boldsymbol{z}(k+1) - \boldsymbol{H}(k+1)\boldsymbol{x}(k+1) \right\|_2 \tag{11}$$

   subject to the dynamic constraints of the model equation (9).

2. Use a gradient improvement to find the next value of the source location $\boldsymbol{\delta}$.

The detailed algorithms for these two steps are considered in what follows.

4. Least-Squares Estimation of the Source Capacity

Assume that $\boldsymbol{f}_u(k) = f(k) \cdot \boldsymbol{\delta}$ is an unknown but stationary time-invariant source. Then

$$
\begin{aligned}
\boldsymbol{z}(k+1) - \boldsymbol{z}^M(k+1) &= \boldsymbol{z}(k+1) - \boldsymbol{H}(k+1)\boldsymbol{x}(k+1) \\
&= \boldsymbol{z}(k+1) - \boldsymbol{H}(k+1)\big[\boldsymbol{A}(k)\boldsymbol{x}(k) + \boldsymbol{f}_n(k) + f \cdot \delta\big] \\
&= z(k+1) - \boldsymbol{H}(k+1)\Big[\boldsymbol{A}(k)\boldsymbol{A}(k-1)\cdots\boldsymbol{A}(0)\boldsymbol{x}(0) \\
&\quad + \boldsymbol{A}(k)\boldsymbol{A}(k-1)\cdots\boldsymbol{A}(1)\boldsymbol{f}_n(0) \\
&\quad + \boldsymbol{A}(k)\boldsymbol{A}(k-1)\cdots\boldsymbol{A}(2)\boldsymbol{f}_n(1) + \cdots \\
&\quad + \boldsymbol{A}(k)\boldsymbol{f}_n(k-1) + \boldsymbol{f}_n(k) \\
&\quad + \big[\boldsymbol{A}(k)\boldsymbol{A}(k-1)\cdots\boldsymbol{A}(1) + \boldsymbol{A}(k)\boldsymbol{A}(k-1)\cdots\boldsymbol{A}(2) \\
&\quad + \cdots + \boldsymbol{A}(k) + \boldsymbol{I}\big] f \cdot \delta\Big]
\end{aligned}
$$

Let

$$\delta\boldsymbol{z}(k+1) = \boldsymbol{z}(k+1) - \boldsymbol{H}(k+1)\boldsymbol{x}_{ns}(k+1) \tag{12}$$

where

$$\boldsymbol{x}_{ns}(k+1) = \boldsymbol{A}(k)\boldsymbol{x}_{ns}(k) + \boldsymbol{f}_n(k) \tag{13}$$

with

$$\boldsymbol{x}_{ns}(0) = \boldsymbol{x}(0)$$

The system (13) is the model of the process driven only by *known* sources, and the state variable $\boldsymbol{x}_{ns}$ describes the process evolution in the absence of unknown sources. With these definitions we obtain that

$$
\begin{aligned}
\boldsymbol{z}(k+1) - \boldsymbol{z}^M(k+1) &= \delta\boldsymbol{z}(k+1) - \boldsymbol{H}(k+1)\big[\boldsymbol{A}(k)\boldsymbol{A}(k-1)\cdots\boldsymbol{A}(1) \\
&\quad + \boldsymbol{A}(k)\boldsymbol{A}(k-1)\cdots\boldsymbol{A}(2) + \cdots + \boldsymbol{A}(k) + \boldsymbol{I}\big] f \cdot \delta
\end{aligned}
$$

Suppose that an unknown source function $\boldsymbol{f}_u$ is applied at $t = \tau = m \cdot \Delta t$, i.e.

$$\boldsymbol{f}_u(k) = \begin{cases} 0 & \text{if } k < m \\ f \cdot \boldsymbol{\delta} & \text{if } k \geq m \end{cases} \tag{14}$$

Then

$$
\begin{aligned}
\boldsymbol{z}(k+1) - \boldsymbol{z}^M(k+1) &= \delta\boldsymbol{z}(k+1) - \boldsymbol{H}(k+1)\big[\boldsymbol{A}(k)\boldsymbol{A}(k-1)\cdots\boldsymbol{A}(m+1) \\
&\quad + \boldsymbol{A}(k)\boldsymbol{A}(k-1)\cdots\boldsymbol{A}(m+2) + \cdots \\
&\quad + \boldsymbol{A}(k) + \boldsymbol{I}\big] f \cdot \delta
\end{aligned} \tag{15}
$$

For a given δ, our objective is to find f such that least-squares error between the actual and the model-predicted measurements is attained, i.e.

$$\left\| z(k+1) - z^M(k+1) \right\|_2 \longrightarrow \min$$

This, in turn, amounts to a least-squares solution of the following linear equation:

$$z(k+1) - z^M(k+1) = 0$$

which, as we have seen, is equivalent to

$$fg = \delta z(k+1) \tag{16}$$

where

$$g = H(k+1)\big[A(k)A(k-1)\cdots A(m+1) + \cdots + A(k) + I\big]\delta \tag{17}$$

The least-squares solution of (16) gives

$$f = \left[\frac{\delta z^T(k+1)\delta z(k+1)}{g^T g}\right]^{1/2} \tag{18}$$

provided that $g^T g \neq 0$.

Sometimes an unknown source function affects the process through the known input transition matrix B. The model of the process in this case can be written as

$$x(k+1) = A(k)x(k) + f_n(k) + fB\delta \tag{19}$$

and the corresponding least-squares solution is equal to

$$\begin{aligned} f &= \left[\delta z^T(k+1)\delta z(k+1)\right]^{1/2} \\ &\quad \times \Big[\delta^T B^T \big(A(k)A(k-1)\cdots A(m+1) + \cdots + A(k) + I\big)^T H^T \\ &\quad \times H(k+1)\big(A(k)A(k-1)\cdots A(m+1) + \cdots + A(k) + I\big)B\delta\Big]^{-1/2} \end{aligned} \tag{20}$$

If the system matrix A is time invariant, then obtain the following simplified expression for f:

$$f = \left[\frac{\delta z^T(k+1)\delta z(k+1)}{\delta^T B^T \big(A^{(k-m)} + \cdots + A + I\big)^T H^T H \big(A^{(k-m)} + \cdots + A + I\big)B\delta}\right]^{1/2} \tag{21}$$

where we again assumed that the denominator is not equal to zero. Note that from a computational point of view, the calculation of f involves only vector-vector and vector-matrix multiplications and, therefore, requires only $O(n^2)$ flops for its implementation.

4.1. Identification of Time Varying Capacity

For a time varying capacity (and location) of an unknown source, we obtain that

$$\begin{aligned} z(k+1) - \boldsymbol{H}(k+1)\boldsymbol{x}(k+1) = \delta z(k+1) \\ - \boldsymbol{H}(k+1)\big[f(0)\boldsymbol{A}(k)\boldsymbol{A}(k-1)\cdots\boldsymbol{A}(1)\boldsymbol{B}(0)\boldsymbol{\delta}_0 \\ + f(1)\boldsymbol{A}(k)\boldsymbol{A}(k-1)\cdots\boldsymbol{A}(2)\boldsymbol{B}(1)\boldsymbol{\delta}_1 + \cdots \\ + f(k-1)\boldsymbol{A}(k)\boldsymbol{B}(k-1)\boldsymbol{\delta}_{k-1} + f(k)\boldsymbol{B}(k)\boldsymbol{\delta}_k\big] \end{aligned} \tag{22}$$

Assume that an unknown source has been applied at $t = \tau = m \cdot \Delta t$ so that

$$\boldsymbol{f}_u(k) = \begin{cases} 0 & \text{if } k < m \\ f(k) \cdot \boldsymbol{\delta}_k & \text{if } k \geq m \end{cases} \tag{23}$$

and the simplification of equation (22) yields

$$\begin{aligned} z(k+1) - \boldsymbol{H}(k+1)\boldsymbol{x}(k+1) = \delta z(k+1) \\ - \boldsymbol{H}(k+1)\big[f(m)\boldsymbol{A}(k)\boldsymbol{A}(k-1)\cdots\boldsymbol{A}(m+1)\boldsymbol{B}(m)\boldsymbol{\delta}_m \\ + f(m+1)\boldsymbol{A}(k)\boldsymbol{A}(k-1)\cdots\boldsymbol{A}(m+2)\boldsymbol{B}(m+1)\boldsymbol{\delta}_{m+1} + \cdots \\ + f(k-1)\boldsymbol{A}(k)\boldsymbol{B}(k-1)\boldsymbol{\delta}_{k-1} + f(k)\boldsymbol{B}(k)\boldsymbol{\delta}_k\big] \end{aligned} \tag{24}$$

Define the following dynamic system:

$$\begin{cases} \boldsymbol{x}_d(k+1) = \boldsymbol{A}(k)\boldsymbol{x}_d(k) + f(k)\boldsymbol{B}(k)\boldsymbol{\delta}_k, & k \geq m \\ \boldsymbol{x}_d(m) = \boldsymbol{0} \end{cases} \tag{25}$$

According to the definition, the system (25) is the model of the process with *zero* initial conditions driven only by the point source, which was *identified* by us in an attempt to match the observed symptoms, created by an unknown source.

Using the definition of $\boldsymbol{x}_d$, eqn. (24) can be written as

$$\begin{aligned} z(k+1) - \boldsymbol{H}(k+1)\boldsymbol{x}(k+1) = \delta z(k+1) \\ - \boldsymbol{H}(k+1)\big[\boldsymbol{A}(k)\boldsymbol{x}_d(k) + f(k)\boldsymbol{B}(k)\boldsymbol{\delta}_k\big] \end{aligned} \tag{26}$$

The capacity $f(k)$ can be found as a least-squares solution to the following equation:

$$\boldsymbol{H}(k+1)\big[\boldsymbol{A}(k)\boldsymbol{x}_d(k) + f(k)\boldsymbol{B}(k)\boldsymbol{\delta}_k\big] = \delta z(k+1) \tag{27}$$

It is straightforward to obtain that

$$f(k) = \left[\frac{\Delta z^T(k+1)\Delta z(k+1)}{\boldsymbol{\delta}_k^T \boldsymbol{B}(k)^T \boldsymbol{H}^T(k+1)\boldsymbol{H}(k+1)\boldsymbol{B}(k)\boldsymbol{\delta}_k}\right]^{1/2} \tag{28}$$

where

$$\Delta z(k+1) = z(k+1) - \boldsymbol{H}(k+1)\big[\boldsymbol{x}_{ns}(k+1) + \boldsymbol{A}(k)\boldsymbol{x}_d(k)\big] \tag{29}$$

and the denominator in (28) is assumed to be non-zero.

We can now formulate the algorithm for computing the least-squares estimate of the capacity $f(k)$ of an unknown source. Given the process measurements $z(k+1)$, the estimation of the source capacity on the previous time step, $f(k-1)$, and the current time step location $\boldsymbol{\delta}_k$,

1. Find $\boldsymbol{x}_d(k)$ by propagating eqn. (25) for the given values $f(k-1)$ and $\boldsymbol{\delta}_{k-1}$.
2. Calculate $\boldsymbol{x}_{ns}(k+1)$ using the model with only known sources, eqn. (13).
3. Compute $\Delta z(k+1)$ according to eqn. (29).
4. Use eqn. (28) to find the next value of the source capacity $f(k)$.

As in the case of time-invariant capacity, this is an $O(n^2)$ algorithm.

5. Search for Source Location

Identification of the spatial location of an unknown source amounts to the determination of a vector $\boldsymbol{\delta}^*$ which minimizes the difference between the observed and model-predicted measurements. There is a natural relationship between two vectors $\boldsymbol{\delta}_a$ and $\boldsymbol{\delta}_b$ based on whether they correspond to spatial neighbors or not. This ordering can be exploited to develop a simple but effective search algorithm for determining $\boldsymbol{\delta}^*$.

As an illustration, let us consider the case of a two-dimensional transport problem with regular mesh, Fig. 1. The current identified location of an unknown source is B and $\boldsymbol{\delta}_B$ is the corresponding location vector. Point B has four immediate neighbors, which are associated with vectors $\boldsymbol{\delta}_A$, $\boldsymbol{\delta}_C$, $\boldsymbol{\delta}_D$ and $\boldsymbol{\delta}_E$.

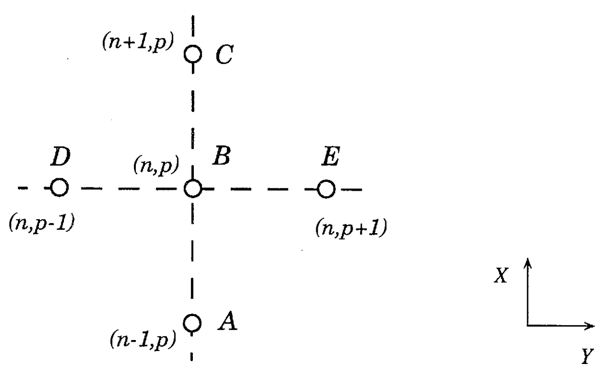

Fig. 1. Mesh point B and its neighbors.

In terms of the components of the measurement vectors, define the following objective function:

$$J = \max_i \frac{|z_i(k+1) - z_i^M(k+1)|}{z_i(k+1)} \tag{30}$$

If B is the currently identified location of an unknown source, then let J_B denote the corresponding value of the objective (30). Figure 2 depicts graphically the values of $1/J$ for all neighbors of B. Each arrow points in the direction of decreasing J. Figure 2 corresponds to the case when the identification error decreases if the location of an unknown source is moved from point B to either C or E. The maximum error decrease is observed when the sensor location is moved to point C.

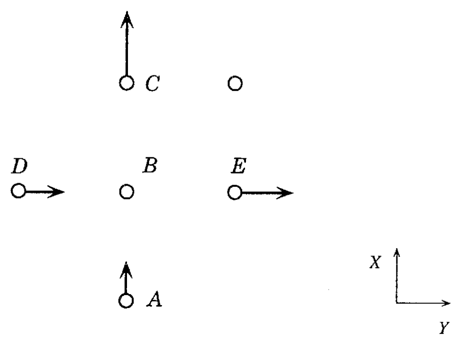

Fig. 2. A relative source identification error in the neighboring points.

The following algorithm can be used to find a new source location which reduces the identification error. Given the current source location δ_B and the associated identification error,

1. Choose one of the spatial neighbors of B as the location of an unknown source. The spatial points which lead to closed-loop cycles must be excluded from consideration.
2. For the chosen location, find the least-squares solution for source capacity $f(k)$. For the new location and capacity of the source, calculate the value of J, eqn. (30).
3. Repeat Steps 1 and 2 for all neighbors of B which do not lead to closed-loop cycles.
4. Compare the calculated values of the objective function J. As a next approximation of an unknown source location, choose a neighboring point which leads to the maximum reduction in the relative identification error.
5. Return to Step 1 if the identification error is not within allowable limits.

Step 4 in the above algorithm can be replaced by a number of alternative rules for choosing the next spatial source location. The example of Fig. 2 shows that the identification error decreases in both the $\vec{X}$ and $\vec{Y}$ directions. Therefore, the next source location can be determined by the summation of vectors associated with points C and E. Following this rule, we will choose the unmarked upper right mesh point of Fig. 2 as the next source location.

The next location can also be chosen beyond immediate neighbors of B. Let (x, y, z) be the current source location. In general, we can find the coordinates of the next location using the following gradient-like iterative scheme:

$$\begin{cases} x_{\text{next}} = x + \mathtt{Integer}\big[\epsilon_x(J_B - J_C)\big] \\ y_{\text{next}} = y + \mathtt{Integer}\big[\epsilon_y(J_B - J_E)\big] \\ z_{\text{next}} = z + \mathtt{Integer}\big[\epsilon_z(J_B - J_G)\big] \end{cases} \tag{31}$$

where $\epsilon_{x,y,z}$ controls the length of the jump to a new source location.

The relative location of sensors and a source can be taken into account when selecting the next source location. Let us illustrate one possible search strategy. Suppose that the current source location leads to the maximum relative identification error equal to $|z_\alpha(k+1) - z_\alpha^M(k+1)|/z_\alpha(k+1)$, where the location of the sensor α is depicted in Fig. 3. We may choose to search for the next source location along the line specified by the location of the sensor α and the current location of the source. We can move the source along this line until the desired reduction in the identification error is achieved. Once the desired location along the line is determined, it must be approximated by the nearest mesh point to obtain the next source location.

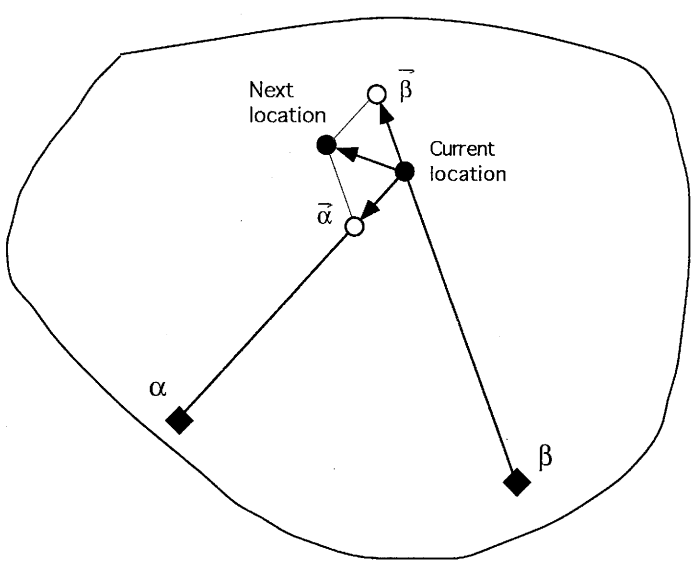

Fig. 3. A search algorithm based on relative sensors-source location.

The same basic idea can also be used with more than one sensor. Figure 3 illustrates how to find the next location using the directional search specified by two sensors (α and β) associated with the largest relative identification errors.

Note that the proposed algorithm uses a search strategy to minimize the relative ℓ_∞ norm of the identification error, while the source capacity is found by minimizing the ℓ_2 error norm. This choice was motivated by the results of numerical experiments which showed a better performance of the identification algorithm in the case of complex velocity distribution in the convection-diffusion process.

6. Examples of Source Identification

6.1. Time Invariant Source

Consider a two-dimensional convection-diffusion air-borne contaminant transport process (Skliar, 1996; Skliar and Ramirez, 1996). Figure 4 depicts the air velocity distribution within the spatial domain. Assume that the contaminant concentration is measured by twelve boundary gas sensors, the location of which is shown in Fig. 5. Suppose that the initial contaminant concentration is zero and that at time $t = 0$ a contaminant is introduced into the cabin with the inlet air stream at a known rate (Table 1).

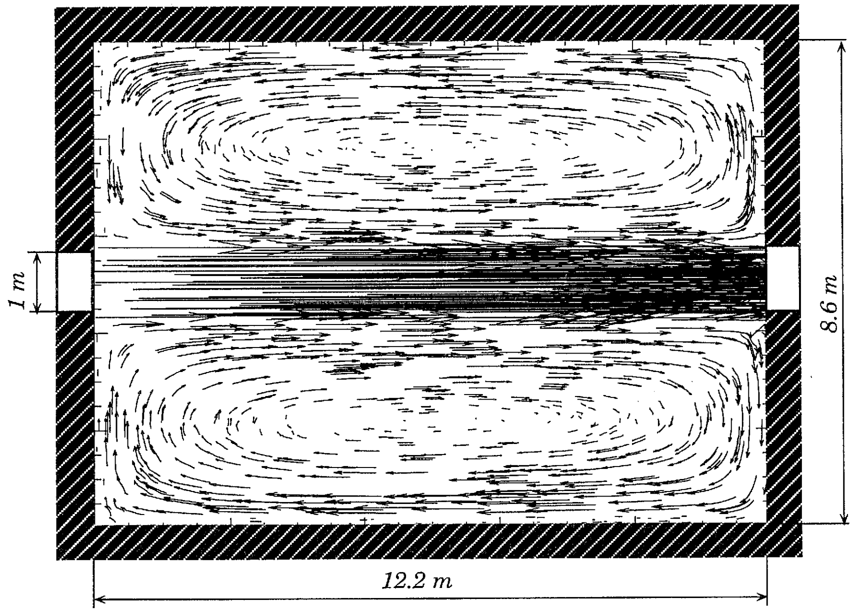

Fig. 4. Air velocity distribution.

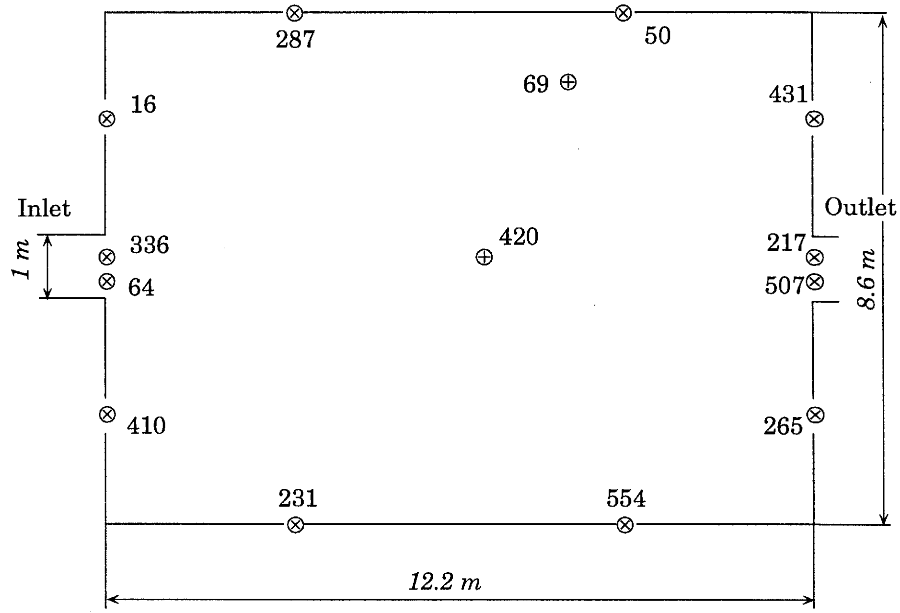

Fig. 5. Spatial location of gas sensors.

Table 1. Input data used in simulation.

Parameters:	
Diffusivity	$D_T = 23\,\mathrm{cm}^2/\mathrm{s}$
Temperature	$T = 20^{\circ}\,\mathrm{C}$
Density	$\rho = 1.200\,\mathrm{g/l}$
Viscosity	$\mu = 1.834 \cdot 10^{-5}\,\mathrm{Pa \cdot s}$
Kinematic viscosity	$\nu = 1.528 \cdot 10^{-5}\,\mathrm{m^2/s}$
Bulk velocity $\boldsymbol{U}$	Fig. 4
Spatial domain:	
Geometry	Fig. 5
Mesh	62-by-44
Discretization in x direction	$\Delta x = 0.2\,\mathrm{m}$
Discretization in y direction	$\Delta y = 0.2\,\mathrm{m}$
Time step	$\Delta t = 1\,\mathrm{s}$
Boundary conditions:	
Impermeable walls	
Known flux at inlet duct	
Permeable wall at outlet duct	

This describes a *known* contamination source. The model-predicted effect of a known source on the contaminant concentration distribution at $t = 640\,\mathrm{s}$ is depicted in Fig. 6. Assume that at $t = 640\,\mathrm{s}$ a new and *unknown* pointwise source of contamination is applied to the system. The location and capacity of this source is listed in Table 2.

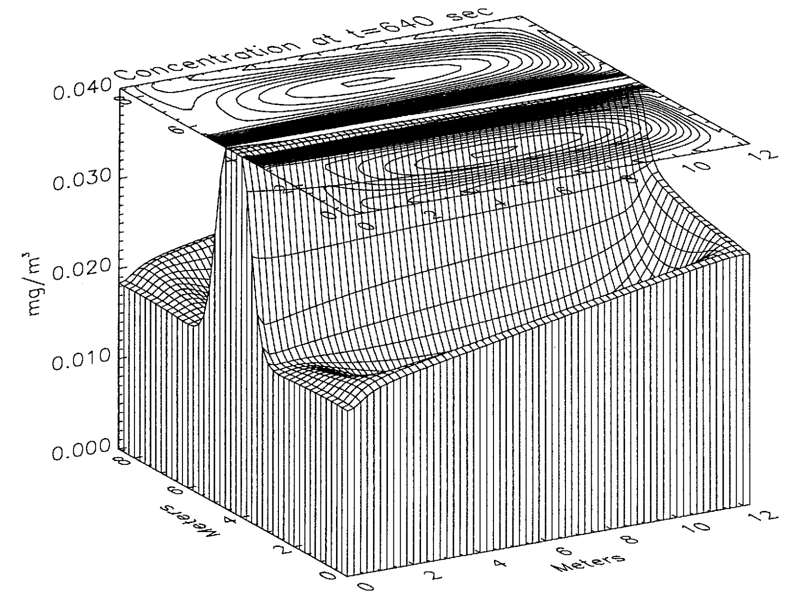

Fig. 6. Model-predicted contaminant concentration 10 min and 40 s after the beginning of the emission.

Table 2. Capacity and location of an unknown source.

Unknown Source	
Location	Spatial point 43
Capacity	$0.02\,\mathrm{mg}/(\mathrm{m}^3\cdot\mathrm{s})$

Two different process models are used in this computer experiment. The first model accounts for both "known" and "unknown" sources. It is used exclusively to generate process measurements $z(k+1)$ affected by all sources. Perfect (noise-free) measurements are assumed to be available. The second model is used as the reflection of our (incomplete) knowledge about the real process and incorporates no information about the "unknown" pointwise source. The objective is to identify the capacity and location of the unknown source based on the (second) model of the process and process measurements $z(k+1)$.

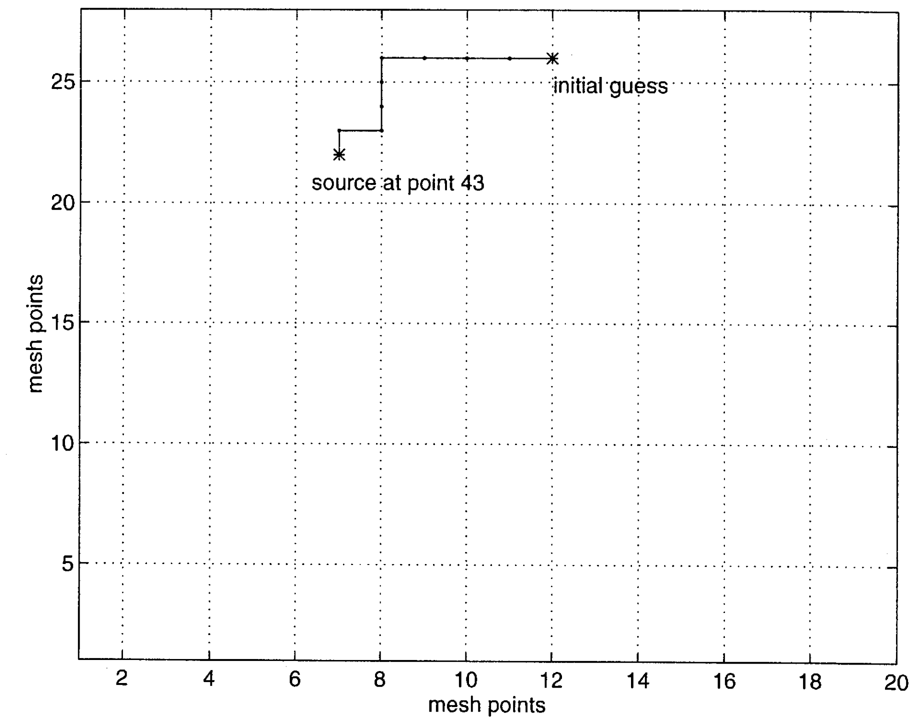

Fig. 7. The search history for an unknown source location.

It is assumed that the identification procedure is triggered after a detection system ascertained that the contamination event had occured. Assuming that the detection system correctly estimated the source application time as $\tau = 640\,\mathrm{s}$, the unknown source identification was performed based on the set of sensor measurements taken at $t = 650\,\mathrm{s}$. Figure 7 shows the search history for the location of an unknown source. The initial guess for the source location was chosen upstream from the sensor associated with the largest difference between measured and model-predicted contaminant concentration. Figure 8 depicts the corresponding source capacity estimation history as a function of the source location. The correct location and capacity of an unknown source are identified in 10 iterations.

6.2. Effect of Misidentified Source Application Time

The proposed source identification method requires the precise estimation of time τ when an unknown source is applied to the process. Let us examine the effect of misidentified τ on the identification process. To accomplish this, we adopt the same contamination scenario as in the previous section, but now we assume that it was erroneously determined that the contamination event has occurred at $\tau = 642\,\mathrm{s}$, or two seconds later than it actually happened.

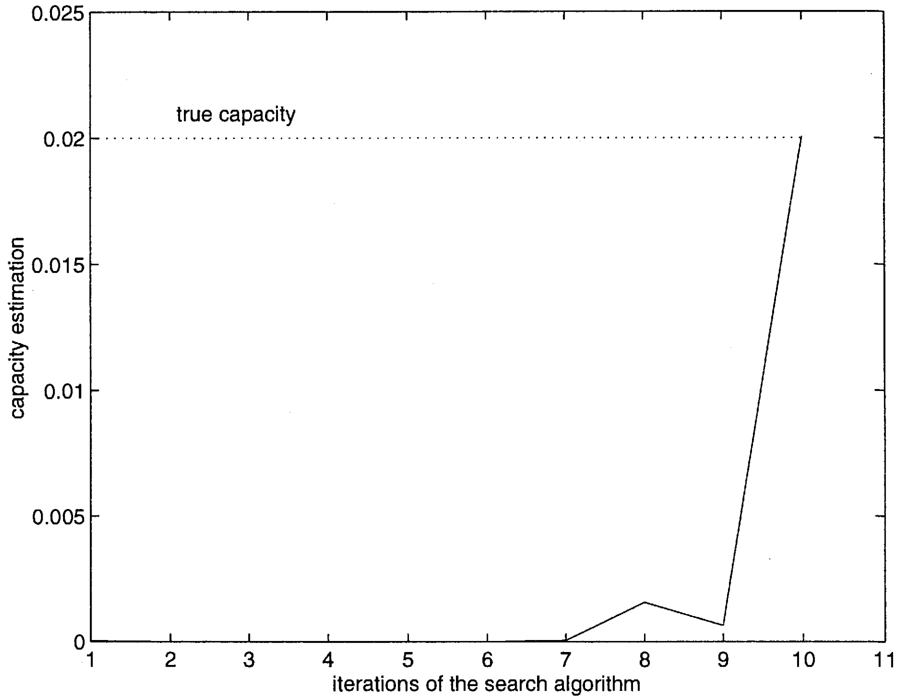

Fig. 8. The source capacity estimation history.

The search for the source location is identical to the results depicted in Fig. 7, and the correct location is identified in 10 iterations. The results, however, are very different for the capacity estimation; as we can see from Fig. 9, the estimated source capacity is over four times larger than the actual value.

The importance of the precise estimation of τ decreases as the influence of an unknown source tends to a steady state for $t \gg \tau$. However, when fast and accurate identification of an unknown source is required, an accurate estimation of τ must be obtained. The location of the source is not sensitive to errors in τ, which suggests the following modification of the proposed algorithm: for a given estimate of τ, perform the identification of source location and capacity; for the identified source location, obtain the least-squares estimates of both f and τ.

6.3. Identification Based on Noisy Measurements

We now consider the effect of measurement noise on the single point source identification problem. Figure 10 shows the set of measurements from six different sensors. These measurements (dashed line) were obtained by first removing the effect of known sources and superimposing the noise-free signal induced by an unknown source (solid line) and a zero-mean white Gaussian noise with standard deviation $\sigma = 10^{-7}$. When the resulted reading is negative, the sensor output is set to zero. Note that even 100 s after the time of an unknown source application, the signal-to-noise ratio for some sensors remains very low. The ten-second sample of the measurements from $t = 659$ to $t = 669$ s is used for source identification.

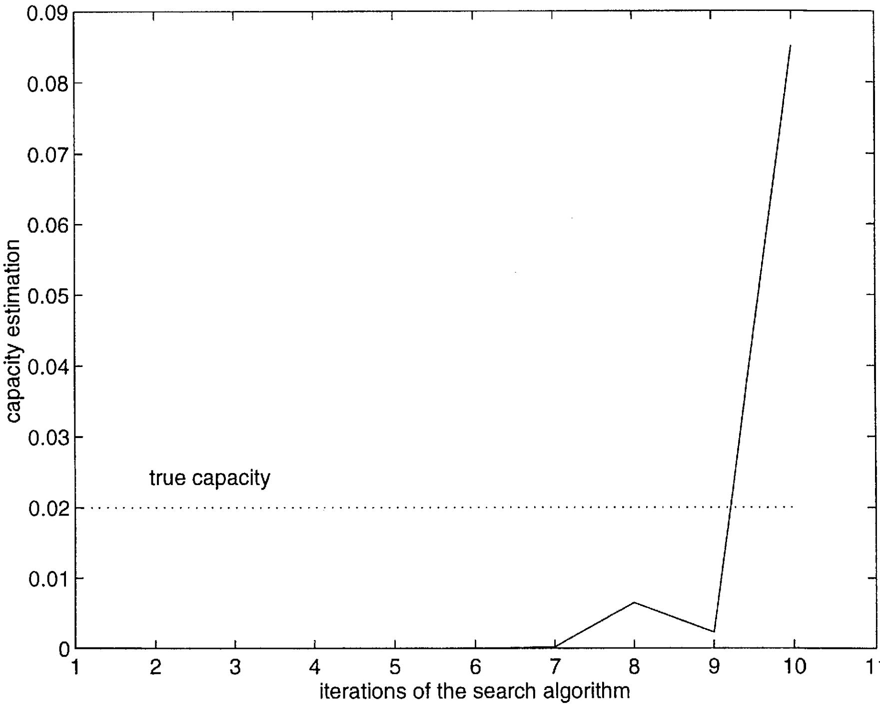

Fig. 9. The source capacity estimation history when τ is misidentified.

Figure 11 depicts the results of identification of the spatial location of an unknown source. Each location is numbered according to the measurement sample used in its identification. Figure 12 shows the estimated values of the source capacity. The maximum relative identification error, eqn. (30), is plotted in Fig. 13. For comparison, the identification error based on the perfect measurements was 3.37×10^{-3}.

As expected, the measurement noise has an adverse effect on source identification. It leads to misidentification of both the spatial location and the capacity of an unknown source. The accuracy of the source identification critically depends on the noise-to-signal ratio; an accurate source identification may be delayed until the effect of an unknown source substantially exceeds the level of measurement noise.

6.4. Distributed Source Identification

Suppose that an unknown spatially distributed source is applied to the system. Assume that after spatial discretization, this unknown distributed source can be approximated as a combination of four clustered pointwise sources listed in Table 3.

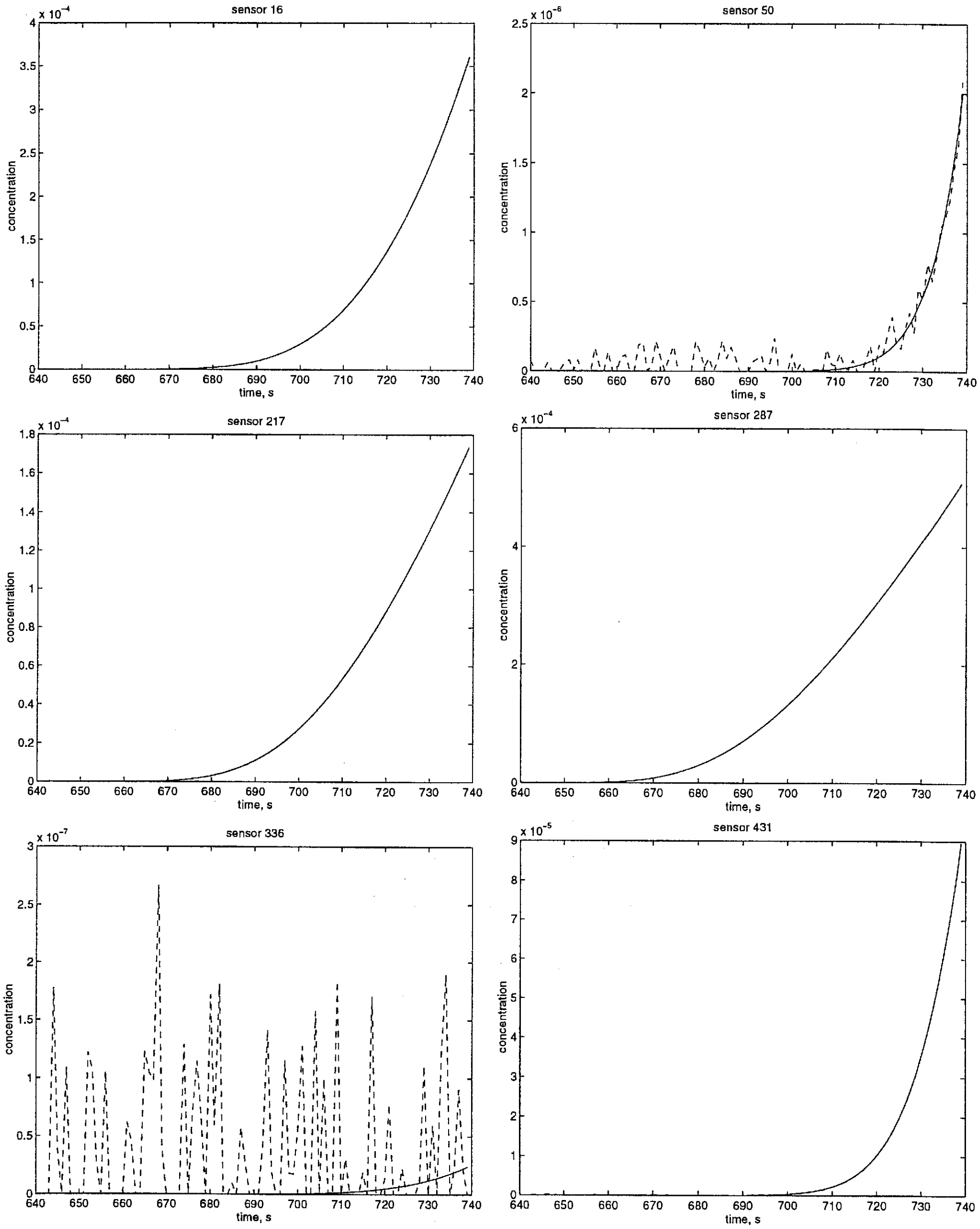

Fig. 10. The output of the sensors induced by an unknown source and corrupted by measurement noise.

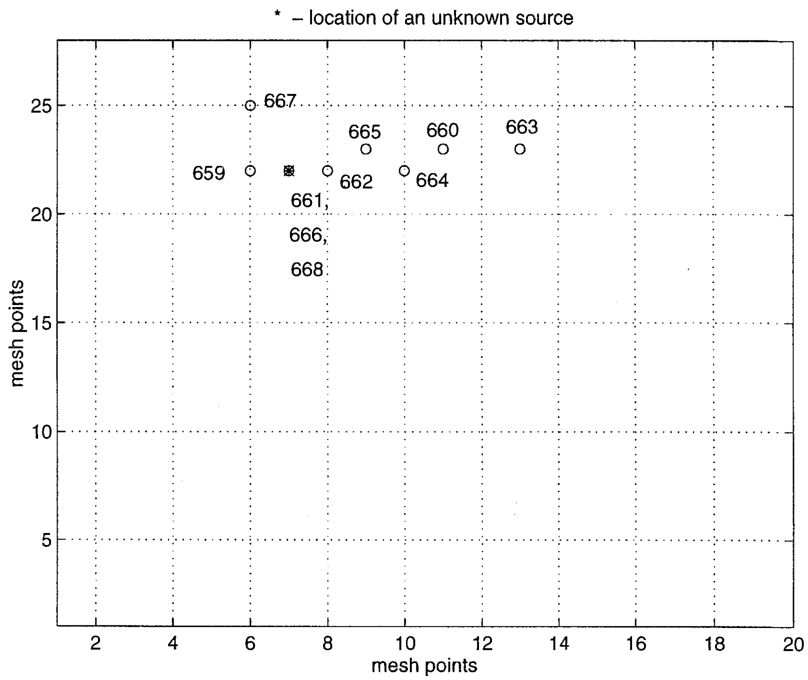

Fig. 11. Identified location of an unknown source based on noisy measurements.

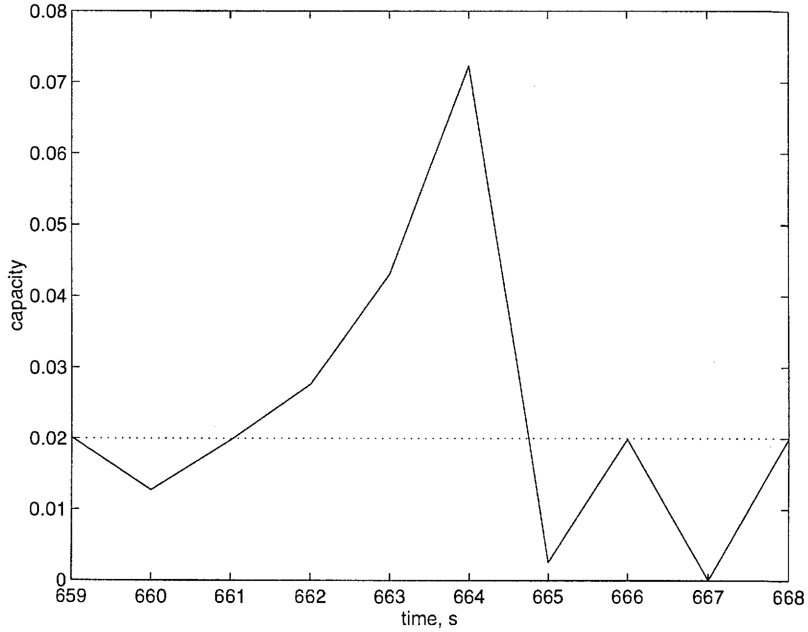

Fig. 12. Identified capacity of an unknown source based on noisy measurements.

Table 3. Approximation of the distributed source by a set of multiple pointwise sources.

	Location	**Capacity**
1	Spatial point 43	$0.02\,\mathrm{mg}/(\mathrm{m}^3\cdot \mathrm{s})$
2	Spatial point 44	$0.02\,\mathrm{mg}/(\mathrm{m}^3\cdot \mathrm{s})$
3	Spatial point 317	$0.02\,\mathrm{mg}/(\mathrm{m}^3\cdot \mathrm{s})$
4	Spatial point 330	$0.02\,\mathrm{mg}/(\mathrm{m}^3\cdot \mathrm{s})$

Our objective is to find a single point source which best approximates the symptoms induced by sources of Table 3. This problem is solved using noise-free measurements obtained 10 s after the application of four point sources. It was also assumed that the source application time τ had been correctly estimated. Figure 14 shows the search history for the location of the sought point source. The identified location of a single source lies within the spatial area of the distributed source application. The identified capacity of a single point source is $f = 0.053\,\mathrm{mg}/(\mathrm{m}^3\cdot \mathrm{s})$ which gives a reasonable idea about the total capacity of the unknown distributed source.

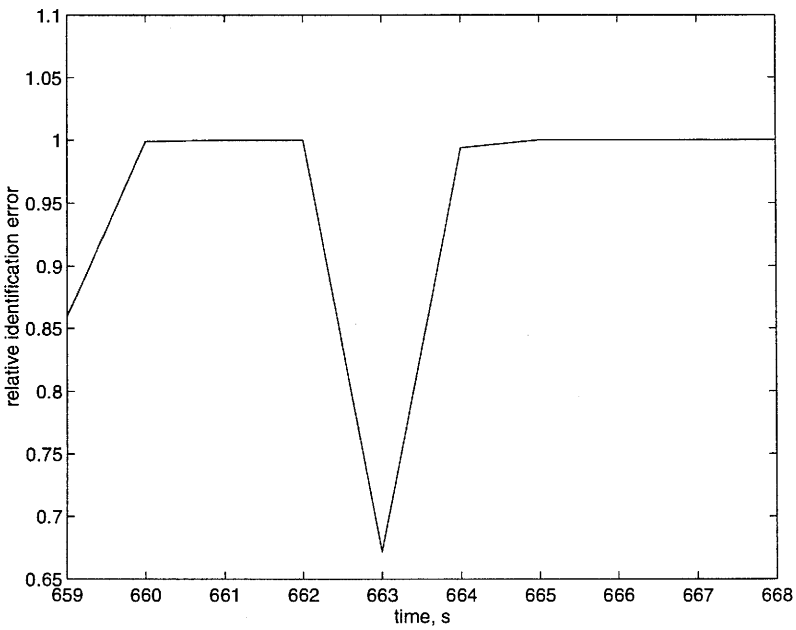

Fig. 13. Relative identification error when noisy measurements are used.

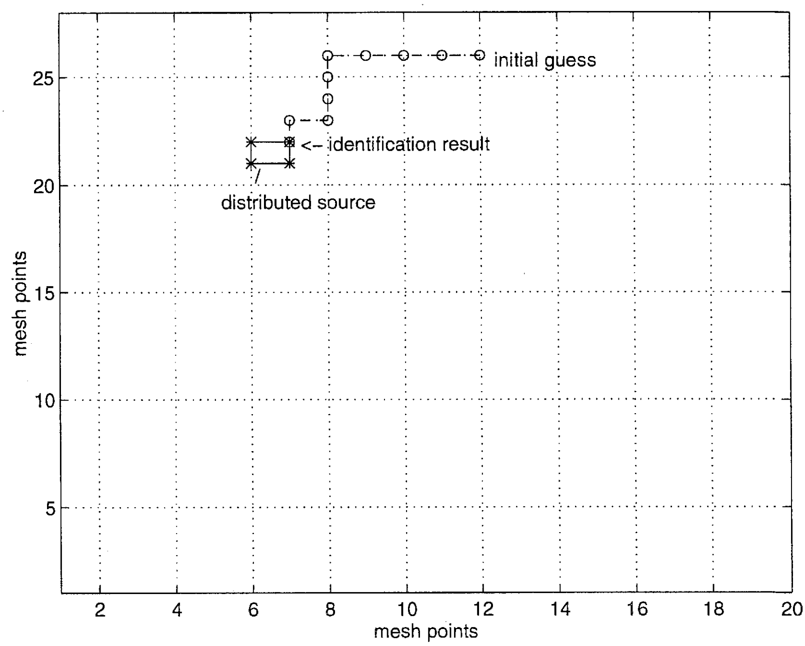

Fig. 14. Result of the identification of an unknown distributed source. Stars (∗) are used to indicate the location of four point sources which approximate the distributed source. Circles (∘) denote the search history for the spatial location of a single point source which best reproduces the symptoms induced by the unknown distributed source.

7. Conclusions

The proposed method allows for identification of an unknown pointwise source function with time varying strength and spatial location. A series of carefully designed numerical experiments demonstrated the ability of the method to precisely identify an unknown source based on perfect measurement. It was also determined that source location is relatively insensitive to the (usually unknown) source application time, suggesting the modification of the method to include simultaneous identification of source location, capacity and time of application. It was also found that the proposed method often provides us with useful information about an unknown spatially distributed source. For identification based on noisy measurements, our results suggest that there exists a trade-off between an early source identification and the fidelity of the identification results, and that further research focused on the development of the conditional source estimation methods is needed.

Acknowledgment

The authors acknowledge the support of the NASA Specialized Center for Research and Training on Environmental Health under grant No. 89988–G004, and NASA support under grant No. NRA93–OLMSA–07.

References

Enting I.G. (1993): *Inverse problems in atmospheric constituent studies: III. Estimating errors in surface sources.* — Inv. Problems, Vol.9, No.6, pp.649–665.

Enting I.G. and Newsam G.N. (1990): *Inverse problems in atmospheric constituent studies: II. Sources in the free atmosphere.* — Inv. Problems, Vol.6, No.3, pp.349–362.

Floudas C.A. (1995): *Nonlinear and Mixed-Integer Optimization.* — New York: Oxford University Press.

Kubrusly C.S. (1977): *Distributed Parameter Systems Identification—A Survey.* — Int. J. Contr., Vol.26, No.5, pp.509–535.

Lin T. and Ewing R.E. (1989): *A Note on Source Term Identification for Parabolic Equations.* — Int. Series of Numerical Mathematics, Vol.91, Basel: Birkhäuser.

Newsam G.N. and Enting I.G. (1988): *Inverse problems in atmospheric constituent studies: I. Determination of surface source under a diffusive transport approximation.* — Inv. Problems, Vol.4, No.11, pp.1037–1054.

Ohnaka K. and Uosaki K. (1989): *Integral equation approach for identification of external force of distributed parameter systems*, In: Distributed Parameter Systems: Modelling and Simulation (Futagami T., Tzafestas S.G. and Sunahara Y., Eds.). — New York: Elsevier Science Publishers.

Silva Nato A.J. and Özişik M.N. (1994): *The estimation of space and time dependent strength of a volumetric heat source in one-dimensional plate.* — Int. J. Heat Mass Transfer, Vol.37, No.6, pp.909–915.

Silva Nato A.J. and Özişik M.N. (1993): *Simultaneous estimation of location and timewise-varying strength of a plane heat source.* — Numerical Heat Transfer, Part A, Vol.24, No.4, pp.467–477.

Skliar M. (1996): *State Estimation and Fault Diagnosis for Distributed Parameter Transport Processes with Application to Air Contamination Control.* — Ph.D. Thesis, University of Colorado, Boulder.

Skliar M. and Ramirez W.F. (1996): *Air quality monitoring and detection of air contamination in an enclosed environment.* — J. Spacecraft and Rockets, Vol.34, No.4, pp.522–532.

Tzafestas S.G. (Ed.) (1982): *Distributed Parameter Control Systems.* — Oxford: Pergamon Press.