Int. J. Appl. Math. and Comp. Sci., 1999, Vol.9, No.1, 101–128

ROBUST PREDICTIVE CONTROL USING A TIME-VARYING YOULA PARAMETER

TON J.J. VAN DEN BOOM*, ROB A.J. DE VRIES*

In this paper, a standard predictive control problem (SPCP) is formulated, which consists of one extended process description with a feedback uncertainty block. The most important finite horizon predictive control problems can be seen as special realizations of this SPCP. The SPCP and its solution are given in a state-space form. The objective of the controller is a nominal performance subject to signal constraints and robust stability with respect to a 1-norm bounded model uncertainty. The optimal controller consists of a feedforward part for nominal signal tracking and a feedback part for disturbance rejection and model error compensation. The feedforward part is realized by the predictive controller for the nominal disturbance-free case. The feedback part of the controller is realized by using the Youla parametrization. The Youla parameter is optimized at every sample time in a receding horizon setting to cope with signal constraints and (robust stability) constraints on the operator itself.

Keywords: predictive control, Youla parametrization, stability, robustness.

1. Introduction

Since the introduction of predictive control in the late 1970's, it has shown to be very successful both in theory and in practice. This last property is not surprising because predictive control is an advanced model-based control method that, contrary to many other methods, originated from industry (Cutler and Ramaker, 1979; Richalet, 1993). The main benefits of MBPC are the easy constraint handling and the fact that complex processes (e.g. multivariable or nonlinear) can be controlled without much special precautions or theoretical background.

Every predictive control problem consists of the specification of a process model (which is used to predict the future behaviour), a criterion function (which specifies the desired future behaviour) and, in all practical situations, signal constraints. Because of these many degrees of freedom in the problem specification, there exist many different predictive controllers which are all based on the same concept, but solve different predictive control problems. Soeterboek (1992) unified most important predictive control methods by using a unified process model and a unified criterion function, which could realise most of these control problems. In the present paper,

* Department of Electrical Engineering, Delft University of Technology, P.O. Box 5031, 2600 GA Delft, the Netherlands, e-mail: t.j.j.vdBoom@et.tudelft.nl.

this "unification process" is carried out one step further by formulating a standard predictive control problem (SPCP) which consists of one extended process description. The most important predictive control problems for multivariable linear processes, using quadratic criterion functions and linear constraints can be seen as special cases of the SPCP. The SPCP and its solution are given in a state-space form which is, especially in the multivariable case, easier to use, more transparent and numerically more reliable then a description in transfer function form. The solution can be seen as an extension of the work of Kinnaert (1989) for the GPC case.

A controller is called robustly stable if a small change in the system does not destabilize the system. It is said to give a robust performance if the performance does not deteriorate significantly for small changes in the system. The design and analysis of linear time-invariant (LTI) robust controllers for linear systems has been studied extensively in the last decade. In practice, a predictive controller usually can be tuned quite easily to give a stable closed loop and to be robust with respect to a model mismatch. However, in order to be able to *guarantee* stability, feasibility and/or robustness and to obtain better and easier tuning rules by an increased insight and better algorithms, the development of a general stability and robustness theory for predictive control has become an important research topic.

For the nominal case, a quite general stability (and feasibility) theory has become available over the past seven years. Two basic ways of using an infinite prediction horizon (Rawlings and Muske, 1993) or end-point constraints (Clarke and Scattolini, 1991; Kouvaritakis *et al.*, 1992; Mosca and Zhang, 1992) to guarantee nominal stability, have recently been unified in more general approaches (De Vries and van den Boom, 1996; Rossiter *et al.*, 1998). One way to obtain robustness in predictive control is by carefull tuning (Clarke and Mothadi, 1989; Lee and Yu, 1994; Soeterboek, 1992). This method gives quite satisfactory results in the unconstrained case. In the constrained case robustness analysis is much more difficult, resulting in more complex and/or conservative tuning rules (Gencelli and Nikolaou, 1993; Zafiriou, 1990). One approach to guarantee robust stability is to use an explicit contraction constraint (De Vries and van den Boom, 1997; Zheng, 1995). Two main disadvantages of this approach are the resulting non-linear optimization problem and a strong but unclear influence of the choice of the contraction constraint on the controlled system. An approach to guarantee a robust performance is to guarantee that the criterion function is a contraction by optimizing the maximum of the criterion function over all possible models (Zheng and Morari, 1993; 1994). The main disadvantages of this method are the need to use polytopic model uncertainty descriptions, the use of less general criterion functions and, especially, a difficult minimax optimization. Recently Kothare *et al.* (1996) derived an LMI-based method that circumvented all of these disadvantages. However, it may become quite conservative. A third solution is using a Youla parametrization in combination with the small-gain theorem (Vidyasagar, 1993). By optimizing the time-varying Youla parameter instead of the ouput of the controller, stability constraints on the Youla parameter, as well as signal constraints can be satisfied (De Vries and van den Boom, 1994; Kouvaritakis *et al.*, 1992).

In this paper, the nominal stability results and the tool of a time-varying Youla parametrization will be used to solve the constrained predictive control problem while remaining robustly stable. First, a basic nominally stable constrained predictive controller is tuned for optimal signal tracking in the unperturbed case. The feedforward part of the true controller is realized by this basic controller using the nominal model and assuming that there are no disturbances. Well-known stability and feasibility results can be used for this part. The feedback part of the controller is realized by the time-varying Youla parameter which is optimized to reject state-disturbances and the effects due to model uncertainty. Bounds on this Youla-parameter based on the small-gain theorem give robust stability against a 1-norm bounded model error.

One major contribution of the presented results is its generality. The only assumptions made are that the process and signal constraints are linear, that a quadratic criterion function is used and that an upper bound is known on the l_1-norm of the model uncertainty (which can be structured or unstructured, additive, output reverse, etc.). By using the small-gain theorem robust stability can be guaranteed without the difficult and limiting choice of one specific contraction constraint. The second contribution is the highly structured design procedure. In relatively simple steps, one is taken from the nominal unconstrained disturbance-free case to the constraint perturbed case. Two other advantages of the approach are that it offers a clear link with the well-known results and insights obtained in the design of LTI robust controllers and that the resulting optimization problem is convex and thereby easy to solve.

This paper is structured as follows. We start with the formulation of the constrained predictive control problem in a standard setting, followed by the related prediction model. In Section 4 the structure of the controller is given and it is shown how stability can be guaranteed. In Sections 5, 6 and 7 a solution to the SPCP problem is given, first for the unperturbed case (no noise, no model error), then for the nominal case (no model error) and finally for the robust case. The tuning of the controller is discussed in Section 8. The results are llustrated by a simulation example in Section 9, which is followed by the conclusions.

Notation: The difference operator $\Delta(q)$ is defined as $\Delta(q) = (1 - q^{-1})\, I$, where q^{-1} denotes the backward shift operator. S^n denotes the linear vector space of all sequences $\{x(k)\}$, $k \in \mathbb{Z}$, $x(k) \in \mathbb{R}^n$. For $x \in S^n$ the ∞-norm is defined as $\|x\|_\infty = \max_{k,i} |x_i(k)|$, where $x_i(k)$ denotes the i-th element of the vector $x(k) \in \mathbb{R}^n$. For $x \in S^n$ the rms-seminorm is defined as

$$\|x\|_{\rm rms} = \lim_{T\to\infty} \left(\frac{1}{2T} \sum_{k=-T}^{T} x^T(k)x(k) \right)^{1/2}$$

Let $H(q)$ be the transfer matrix of a system with Markov-parameters $h(k)$: $y(k) = H(q)\, u(k) = \sum_{\tau=0}^{\infty} h(\tau)\, u(k-\tau)$. The 1-norm of h is equal to the induced ∞-norm, so

$$\|h\|_1 = \max_i \sum_{j=1}^{n} \sum_{k=0}^{\infty} |h_{ij}(k)|$$

where $h_{ij}(k)$ is the (i,j)-th element of $h(k)$. The associated 1-norm of H is defined as

$$||H||_{(1)} = ||h||_1$$

The normed space l_∞^n of bounded sequences is defined as $l_\infty^n = \{x \in S^n : ||x||_\infty < \infty\}$. The space l_{rms}^n of power-bounded sequences is defined as $l_{\text{rms}}^n = \{x \in S^n : ||x||_{\text{rms}} < \infty\}$.

Finally, a mapping $G : S^m \to S^n$ is called BIBO or l_∞-stable if it has a bounded induced ∞-norm: $u \in l_\infty^m$ implies that $Gu \in l_\infty^n$ and there exists a finite constant ϵ_G such that $||Gu||_\infty \leq \epsilon_G \, ||u||_\infty \; \forall \, u \in l_\infty^m$. In this paper, a system is called stable if it is BIBO stable.

2. Standard Predictive Control Problem

Definition 1. (Standard Predictive Control Problem, or SPCP) Consider a system $\{\delta_2, z, \phi_E, \phi_I, y\} = \Sigma\{\delta_1, w, e, \Delta u\}$ given by the state-space realization

$$\begin{aligned}
x(k+1) &= Ax(k) + B_1\delta_1(k) + B_2w(k) + B_3e(k) + B_4\delta u(k) \\
\delta_2(k) &= C_1x(k) + D_{11}\delta_1(k) + D_{12}w(k) + D_{13}e(k) + D_{14}\delta u(k) \\
z(k) &= C_2x(k) + D_{21}\delta_1(k) + D_{22}w(k) + D_{23}e(k) + D_{24}\delta u(k) \\
\phi_E(k) &= C_3x(k) + D_{31}\delta_1(k) + D_{32}w(k) + D_{33}e(k) + D_{34}\delta u(k) \\
\phi_I(k) &= C_4x(k) + D_{41}\delta_1(k) + D_{42}w(k) + D_{43}e(k) + D_{44}\delta u(k) \\
y(k) &= C_5x(k) + D_{51}\delta_1(k) + D_{52}w(k) + D_{53}e(k)
\end{aligned} \tag{1}$$

where $w \in S^{n_2}$ is a vector containing all known signals such as the reference signal and the known disturbance signals, $e \in S^{n_3}$ is a zero-mean white noise signal, $\Delta u \in S^{n_4}$ is the increment of the control signal (so $u(k) = u(k-1) + \Delta u(k)$) and $y \in S^{m_5}$ is the process ouput measurement. Further, $\delta_1 \in S^{n_1}$ and $\delta_2 \in S^{m_1}$ are related by the mapping $\delta_1(k) = \Omega(q)\,\delta_2(k)$, where the model error $\Omega(q)$ is a stricly proper linear time-invariant transfer function in the backward shift operator q^{-1}. Finally, $z(k)$ denotes the performance signal, $\phi_E(k)$ denotes the equality constraints and ϕ_I denotes the inequality constraints. We then make the following assumptions:

1. (A, B_4) is stabilizable.
2. D_{53} is square and invertible.
3. $A - B_3D_{53}^{-1}C_5$ has all eigenvalues inside the unit circle (This implies that the pair (C_5, A) is detectable).

Further, we define

$$\widetilde{z}(k) = M_z \left[\begin{array}{cccc} \hat{z}^T(k|k) & \hat{z}^T(k+1|k) & \dots & \hat{z}^T(k+N|k) \end{array}\right]^T \in \mathbb{R}^{(N+1)m_2}$$

$$\widetilde{\phi}_E(k) = M_E \left[\begin{array}{cccc} \hat{\phi}_E^T(k|k) & \hat{\phi}_E^T(k+1|k) & \dots & \hat{\phi}_E^T(k+N|k) \end{array}\right]^T \in \mathbb{R}^{(N+1)m_3}$$

$$\widetilde{\phi}_I(k) = M_I \left[\begin{array}{cccc} \hat{\phi}_E^T(k|k) & \hat{\phi}_I^T(k+1|k) & \dots & \hat{\phi}_I^T(k+N|k) \end{array}\right]^T \in \mathbb{R}^{(N+1)m_4}$$

where $\hat{z}(k+j|k)$, $\hat{\phi}_E(k+j|k)$ and $\hat{\phi}_I(k+j|k)$ are the predictions of $z(k+j) \in \mathbb{R}^{m_2}$, $\phi_E(k+j) \in \mathbb{R}^{m_3}$ and $\phi_I(k+j) \in \mathbb{R}^{m_4}$ at time k, respectively, and M_z, M_E and M_I are constant weighting matrices with proper dimensions. Finally, we define

$$\widetilde{w}(k) = \left[\begin{array}{cccc} w^T(k) & w^T(k+1) & \dots & w^T(k+N) \end{array}\right]^T \in \mathbb{R}^{(N+1)\,n_2}$$

which is a vector containing *a-priori* known signals, and the sets $S_w^{n_2}$ and $S_e^{n_3}$ which specify the domain of w and e using *a-priori* information: $w \in S_w^{n_2} \subseteq S^{n_2}$ and $e \in S_e^{n_3} \subseteq S^{n_3}$.

The *Standard Predictive Control Problem* (SPCP) is now given as follows:

Find a stabilizing controller $\Delta u(k) = K(\widetilde{w}, y, k)$ *such that the criterion function*

$$J(k) = \widetilde{z}^T(k)\widetilde{z}(k)$$

is minimized subject to the constraints

$$\widetilde{\phi}_E(k) = 0 \quad \textit{and} \quad \widetilde{\phi}_I(k) \leq 1 \tag{2}$$

for all model errors which satisfy

$$\|\Omega\|_1 \leq \epsilon_\Omega$$

The optimal controller is denoted by

$$K = \mathcal{K}\big(\Sigma, M_z, M_E, M_I, \epsilon_\Omega, S_w^{n_2}, S_e^{n_3}, k\big)$$

where $\widetilde{\phi}_I(k) \leq 1$ means that each component of $\widetilde{\phi}_I$ is less than or equal to 1. The standard predictive control problem is visualized by the configuration of Fig. 1.

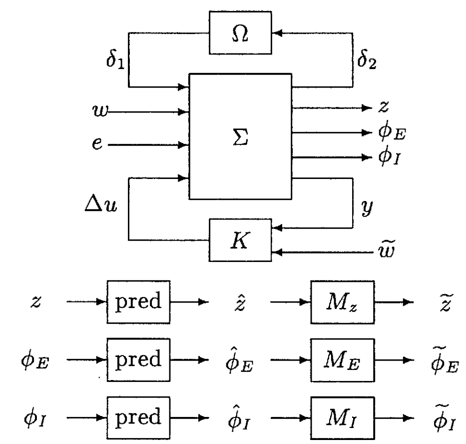

Fig. 1. Standard predictive control problem.

Remark 1. Typically, the variable $z(k)$ is a vector with a performance measure and a measure of control action. For example, GPC (Clarke and Mohtadi, 1989; Kinnaert, 1989) uses

$$z(k) = \begin{bmatrix} w(k) - \psi(k) \\ \lambda \Delta u(k) \end{bmatrix}$$

where $\psi(k) = P(q)y(k)$ and $w(k)$ is the reference signal. For MPC (Garcia *et al.*, 1989; Lee *et al.*, 1994) one chooses

$$z(k) = \begin{bmatrix} Q^{1/2}x(k) \\ R^{1/2}u(k) \end{bmatrix}$$

Definition 2. (Well-defined SPCP) A standard predictive control problem

$$K = \mathcal{K}\big(\Sigma, M_z, M_E, M_I, \epsilon_\Omega, S_w^{n_2}, S_e^{n_3}, k\big)$$

as defined in Definition 1 is called *well-defined* if the equality constrained nominal predictive control problem

$$K = \mathcal{K}\big(\Sigma, M_z, M_E, 0, 0, S^{n_2}, S^{n_3}, k\big)$$

results in a feasible stabilizing controller.

Definition 3. (Nominal feasible SPCP) A well-defined standard predictive control problem

$$K = \mathcal{K}\big(\Sigma, M_z, M_E, M_I, \epsilon_\Omega, S_w^{n_2}, S_e^{n_3}, k\big)$$

as defined in Definitions 1 and 2 is called *nominal feasible* if the constrained nominal predictive control problem

$$K = \mathcal{K}\big(\Sigma, M_z, M_E, M_I, 0, S_w^{n_2}, S_e^{n_3}, k\big)$$

results in a feasible stabilizing control law.

Remark 2. In this paper, we will assume the SPCP to be well-defined and nominal feasible. The SPCP can always be made well-defined by increasing the prediction horizon N (Rawlings and Muske, 1993), by including a terminal-state or end-point constraint (Clarke and Scattolini, 1991; Mosca and Zhang, 1992), or by using the compromise between these two approaches presented in (De Vries and van den Boom, 1996). The nominal feasibility of the SPCP depends on the equality and inequality constraints, and/or on the signal spaces $S_w^{n_2}$ and $S_e^{n_3}$ (De Vries and van den Boom, 1996; Rossiter *et al.*, 1998).

3. Prediction

In this section, we will give the prediction model for the vectors $\tilde{z}(k)$, $\tilde{\phi}_E(k)$ and $\tilde{\phi}_I(k)$ in the SPCP. As is usual in predictive control, we will use the minimum-variance prediction which is obtained by taking the conditional expectation given information up to time k. Further, we will use the certainty-equivalence principle or, in other words, the predictions are based on the nominal model which implies that nominal performance will be optimized.

Using the certainty-equivalence principle implies the assumption that $\Omega(q) = 0$ and thus that $\delta_1(k) = 0$ in the system (1). Hence it implies the assumption that the system is given by the following nominal system:

$$\begin{aligned}
\hat{x}\big(k+1\big) &= A\hat{x}(k) + B_2 w(k) + B_3\hat{e}(k) + B_4\Delta u(k) \\
\hat{z}(k) &= C_2\hat{x}(k) + D_{22}w(k) + D_{23}\hat{e}(k) + D_{24}\Delta u(k) \\
\hat{\Phi}_E(k) &= C_3\hat{x}(k) + D_{32}w(k) + D_{33}\hat{e}(k) + D_{34}\Delta u(k) \\
\hat{\Phi}_I(k) &= C_4\hat{x}(k) + D_{42}w(k) + D_{43}\hat{e}(k) + D_{44}\Delta u(k) \\
y(k) &= C_5\hat{x}(k) + D_{52}w(k) + D_{53}\hat{e}(k)
\end{aligned} \tag{3}$$

Moreover, it implies the assumption that the signal $\hat{e}(k)$ given by

$$\hat{e}(k) = D_{53}^{-1}\Big(y(k) - C_5\hat{x}(k) - D_{52}w(k)\Big) \tag{4}$$

is zero-mean white noise. This, in turn, implies that the minimum-variance prediction $\hat{e}(k+j|k) = 0 \;\; \forall\, j \geq 1$.

Remark 3. Note that the nominal system (3) in which $\hat{e}(k)$ is given by (4) is stable by Assumption 3 in the SPCP given in Definition 1, and that D_{53}^{-1} in (4) exists by Assumption 2.

Proposition 1. *Consider the standard predictive control problem given in Definition 1. Define the vector*

$$\tilde{u}(k) = \begin{bmatrix} \Delta u^T(k) & \Delta u^T(k+1) & \cdots & \Delta u^T(k+N) \end{bmatrix}^T \in \ell_\infty^{(N+1)n_4}$$

and let $E_u = [I \ 0 \ \ldots \ 0]$ *and* $E_w = [I \ 0 \ \ldots \ 0]$ *be selection matrices such that* $\Delta u(k) = E_u \tilde{u}(k)$ *and* $w(k) = E_w \tilde{w}(k)$. *Introduce the basic block matrices*

$$\mathcal{M}_u(A,B,C,D) = \begin{bmatrix} D & & & 0 \\ CB & D & & \\ CAB & CB & D & \\ \vdots & & \ddots & \\ CA^{N-2}B & & \cdots & D \end{bmatrix}$$

$$\mathcal{M}_x(A,C) = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{N-1} \end{bmatrix}, \qquad \mathcal{M}_e(A,B,C,D) = \begin{bmatrix} D \\ CB \\ CAB \\ \vdots \\ CA^{N-2}B \end{bmatrix}$$

Using these basic block matrices, define the matrices:

$$\begin{aligned}
\bar{B}_2 &= B_2 E_w, & \bar{B}_4 &= B_4 E_u \\
\bar{C}_2 &= M_z \mathcal{M}_x(A, C_2), & \bar{D}_{22} &= M_z \mathcal{M}_u(A, B_2, C_2, D_{22}) \\
\bar{D}_{23} &= M_z \mathcal{M}_e(A, B_3, C_2, D_{23}), & \bar{D}_{24} &= M_z \mathcal{M}_u(A, B_4, C_2, D_{24}) \\
\bar{C}_3 &= M_E \mathcal{M}_x(A, C_3), & \bar{D}_{32} &= M_E \mathcal{M}_u(A, B_2, C_3, D_{32}) \\
\bar{D}_{33} &= M_E \mathcal{M}_e(A, B_3, C_3, D_{33}), & \bar{D}_{34} &= M_E \mathcal{M}_u(A, B_4, C_3, D_{34}) \\
\bar{C}_4 &= M_I \mathcal{M}_x(A, C_4), & \bar{D}_{42} &= M_I \mathcal{M}_u(A, B_2, C_4, D_{42}) \\
\bar{D}_{43} &= M_I \mathcal{M}_e(A, B_3, C_4, D_{43}), & \bar{D}_{44} &= M_I \mathcal{M}_u(A, B_4, C_4, D_{44})
\end{aligned}$$

Then the minimum variance prediction vectors $\tilde{z}$, $\tilde{\phi}_E$ *and* $\tilde{\phi}_I$ *based on the certainty-equivalence principle are given by the extended system*

$$\begin{aligned}
\hat{x}(k+1) &= A\hat{x}(k) + \bar{B}_2\tilde{w}(k) + B_3\hat{e}(k) + \bar{B}_4\tilde{u}(k) \\
\tilde{z}(k) &= \bar{C}_2\hat{x}(k) + \bar{D}_{22}\tilde{w}(k) + \bar{D}_{23}\hat{e}(k) + \bar{D}_{24}\tilde{u}(k) \\
\tilde{\phi}_E(k) &= \bar{C}_3\hat{x}(k) + \bar{D}_{32}\tilde{w}(k) + \bar{D}_{33}\hat{e}(k) + \bar{D}_{34}\tilde{u}(k) \\
\tilde{\phi}_I(k) &= \bar{C}_4\hat{x}(k) + \bar{D}_{42}\tilde{w}(k) + \bar{D}_{43}\hat{e}(k) + \bar{D}_{44}\tilde{u}(k)
\end{aligned}$$

where $\hat{e}(k)$ *is given by (4).*

The proof can be constructed easily by successive substitution of the nominal state equation (3).

4. Controller Parametrization

The presented controller is a variation on the well-known Youla parametrization, where the time-invariant parameter $Q(q)$ is replaced by a time-varying parameter $\mathcal{Q}$. Further, the feedforward action $\widetilde{w} \mapsto \Delta u$ and the feedback action $y \mapsto \Delta u$ are split to guarantee that the feedforward action is not influenced, and from that destabilized, by the model error. Section 7 will go further into this matter.

Theorem 1. *Consider a nominal feasible SPCP as given in Definition 3. Let F_w, F_e, L_w and L_e be matrices such that $A - B_4F_w$, $A - B_4F_e$ and $A - L_eC_5$ have all eigenvalues strictly inside the unit circle. Further, let $E_w = [I\ 0\ \dots\ 0]$ be a selection matrix such that $w(k) = E_w\,\widetilde{w}(k)$. Finally, define the mapping $(\Delta u, \xi) = K_1(\widetilde{w}, y, v)$ as*

$$\hat{x}_e(k+1) = A\hat{x}_e(k) + L_e\xi(k) + L_w\widetilde{w}(k) + B_4\Delta u(k) \tag{5}$$

$$\Delta u(k) = -F_e\hat{x}_e(k) + v(k) \tag{6}$$

$$\xi(k) = -C_5\hat{x}_e(k) - D_{52}E_w\widetilde{w}(k) + y(k) \tag{7}$$

where $v(k) \in \ell_\infty^{n_4}$ is an additional input-signal and $\xi(k) \in S^{m_5}$ is an additional output signal.

Then the mapping $(\delta_2, z, \phi_E, \phi_I, \xi) = \Lambda(\delta_1, \widetilde{w}, e, v)$ is a mapping from ℓ_∞ to ℓ_∞ and given by the state-space description

$$\begin{bmatrix} x_{T1}(k+1) \\ x_{T2}(k+1) \\ \delta_2(k) \\ z(k) \\ \phi_E(k) \\ \phi_I(k) \\ \xi(k) \end{bmatrix} = \left[\begin{array}{cc|cccc} A-B_4F_e & B_4F_e & B_1 & B_2E_w & B_3 & B_4 \\ 0 & A-L_eC_5 & B_1-L_eD_{51} & B_2E_w-L_w & B_3-L_eD_{53} & 0 \\ \hline C_1-D_{14}F_e & D_{14}F_e & D_{11} & D_{12}E_w & D_{13} & D_{14} \\ C_2-D_{24}F_e & D_{24}F_e & D_{21} & D_{22}E_w & D_{23} & D_{24} \\ C_3-D_{34}F_e & D_{34}F_e & D_{31} & D_{32}E_w & D_{33} & D_{34} \\ C_4-D_{44}F_e & D_{44}F_e & D_{41} & D_{42}E_w & D_{43} & D_{44} \\ 0 & C_5 & D_{51} & 0 & D_{53} & 0 \end{array}\right]$$

$$\times \begin{bmatrix} x_{T1}(k) \\ x_{T2}(k) \\ \delta_1(k) \\ \tilde{w}(k) \\ e(k) \\ v(k) \end{bmatrix} \tag{8}$$

Furthermore, let a mapping $v = K_2(\xi, \tilde{w})$ *be given by*

$$\hat{x}_w(k+1) = \big(A - B_4 F_w\big)\hat{x}_w(k) + L_w \tilde{w}(k) + B_4(Q_w \tilde{w})(k) \tag{9}$$

$$v(k) = \big(F_e - F_w\big)\hat{x}_w(k) + (Q_w \tilde{w})(k) + (Q_e \xi)(k) \tag{10}$$

where Q_w *and* Q_e *are mappings from* ℓ_∞ *to* ℓ_∞.

Then the resulting mapping $(\delta_2, z, \phi_E, \phi_I) = \Psi(\delta_1, \tilde{w}, e)$ *is a mapping from* ℓ_∞ *to* ℓ_∞ *and the closed-loop with controller* $\Delta u = K(y, \tilde{w})$ *will be Bounded-Input Bounded-Output (BIBO) stable.*

The proof is given in Appendix A.

Interpretation and design strategy. The configuration of the controller K is given in Fig. 2 and will be designed as follows: First, an arbitrary LTI nominally stabilizing feedback controller is designed ($\Omega(q) = 0$, $\tilde{w} = 0$). This part of the controller is completely determined by F_e and L_e. Any F_e and L_e that satisfy the assumptions in Theorem 1 are allowed. As usual, we will choose L_e as the optimal Kalman gain. The effect of F_e on the proposed control strategy and guidelines for its choice are discussed in Section 8. Secondly, a nominal feedforward controller is designed ($\xi = 0$). This part of the controller is completely determined by F_w, L_w and Q_w.

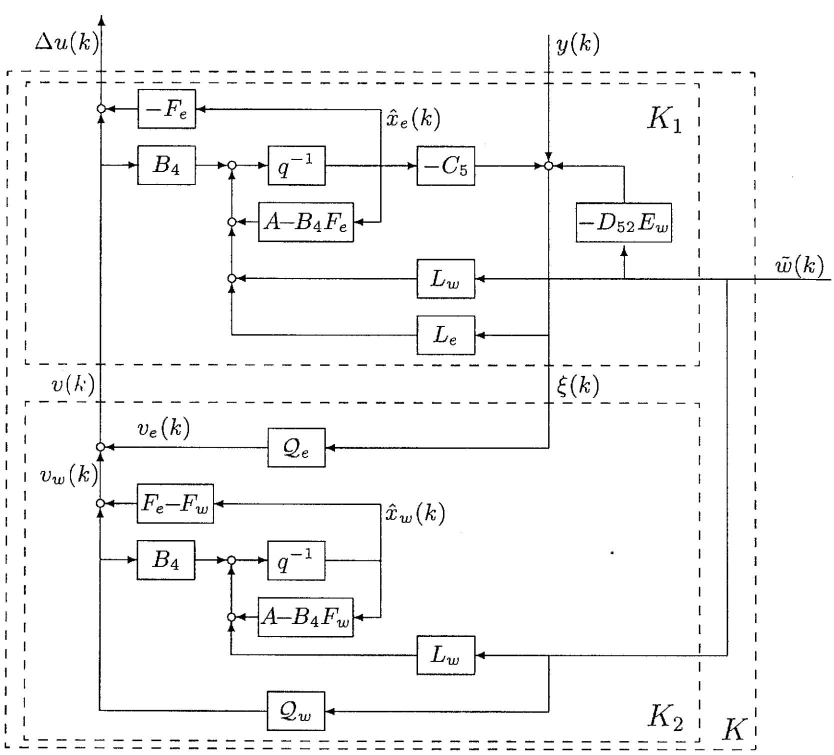

Fig. 2. SPCP controller configuration.

In this paper, F_w, L_w and $\mathcal{Q}_w = Q_w$ are chosen such that $-F_w\,\hat{x}_w(k) + Q_w\widetilde{w}(k)$ is equal to the optimal (guaranteed stabilizing, LTI) predictive controller output in the nominal, disturbance-free case without inequality constraints. To handle the inequality constraints, the operator $\mathcal{Q}_w$ becomes time-varying and is optimized at every sample time. This feedforward controller is incorporated in K in such a way that $\Delta u = -F_w\,\hat{x}_w(k) + \mathcal{Q}_w\widetilde{w}(k)$ when $\xi = 0$. The design of the feedforward controller is given in Section 5. Thirdly, a feedback controller $\mathcal{Q}_e$ is designed for optimal disturbance rejection and model-error compensation. The time-varying operator $\mathcal{Q}_e$ is optimized at every sample time such that Δu is the optimal predictive controller output. If there are no inequality constraints, the optimal $\mathcal{Q}_e$ becomes time-invariant. To obtain a simple (convex) optimization problem, $\mathcal{Q}_e$ will be parametrized as a (time-varying) finite-impulse-response operator. The stability of this operator (and the nominal stability of the closed loop) will be guaranteed by defining some constraint on the magnitude of its impulse-response parameters. Robust stability of the entire closed loop will be guaranteed by an additional constraint on $\mathcal{Q}_e$ which guarantees that the small-gain theorem is satisfied. The design of $\mathcal{Q}_e$ is given in Sections 6 and 7.

5. Reference Signal Tracking

In this section, the unperturbed case will be considered, so there is no noise ($e(k) = 0$), no model error ($\Omega(q) = 0$) and we have an equal initial state ($x(0) = x_e(0) = x_w(0)$). The SPCP problem is then solved by setting $\mathcal{Q}_e = 0$ and only tuning the feedforward part of controller K_2.

Theorem 2. *Consider a nominal feasible SPCP, where $e(k) = 0$ and $\Omega(q) = 0$, and the controller of Theorem 1, where F_e and L_e are matrices such that $A - B_4F_e$ and $A - L_eC_5$ have all eigenvalues strictly inside the unit circle. Define the matrices*

$$\begin{aligned} L_w &= B_2E_w \\ \widetilde{F}_w &= [I\ \ 0]\begin{bmatrix} 2\bar{D}_{24}^T\bar{D}_{24} & \bar{D}_{34}^T \\ \bar{D}_{34} & 0 \end{bmatrix}^{-1}\begin{bmatrix} 2\bar{D}_{24}^T\bar{C}_2 \\ \bar{C}_3 \end{bmatrix}, \qquad F_w = E_u\widetilde{F}_w \\ \widetilde{Q}_w &= -[I\ \ 0]\begin{bmatrix} 2\bar{D}_{24}^T\bar{D}_{24} & \bar{D}_{34}^T \\ \bar{D}_{34} & 0 \end{bmatrix}^{-1}\begin{bmatrix} 2\bar{D}_{24}^T\bar{D}_{22} \\ \bar{D}_{32} \end{bmatrix}, \qquad Q_w = E_u\widetilde{Q}_w \end{aligned}$$

and let $\widetilde{\nu}(k) \in \ell_\infty^{(N+1)n_4}$ be the the solution to the following optimization problem:

$$\min J(k) = \min_{\widetilde{\nu}(k)} \widetilde{\nu}(k)^T\bar{D}_{24}^T\bar{D}_{24}\widetilde{\nu}(k) \tag{11}$$

subject to the constraints

$$(\bar{C}_4 - \bar{D}_{44}\widetilde{F}_w)\hat{x}_w(k) + (\bar{D}_{42} - \bar{D}_{44}\widetilde{Q}_w)\widetilde{w}(k) + \bar{D}_{44}\widetilde{\nu}(k) < 1 \tag{12}$$

$$\bar{D}_{34}\widetilde{\nu}(k) = 0 \tag{13}$$

$$\|\widetilde{\nu}(k)\| \leq \nu_{\max} \tag{14}$$

The SPCP for $e(k) = 0$, $\Omega(q) = 0$ and equal initial state ($x(0) = \hat{x}_e(0) = \hat{x}_w(0)$) is solved by the controller given in Theorem 1, where the mappings $\mathcal{Q}_w$ and $\mathcal{Q}_e$ are given by

$$\begin{aligned}
\big(\mathcal{Q}_w\widetilde{w}\big)(k) &= Q_w\widetilde{w}(k) + \nu(k)\\
\big(\mathcal{Q}_e\xi\big)(k) &= 0\\
\nu(k) &= E_u\widetilde{\nu}(k)
\end{aligned}$$

and $\widetilde{\nu}(k)$ is the optimal solution to the optimization problem (11)–(14).

The proof of this theorem is given in Appendix B.

Remark 4. The extra condition $\|\widetilde{\nu}(k)\| \leq \nu_{\max}$ guarantees $\nu(k) \in l_{\text{rms}}$. This means that the mapping $\widetilde{w} \mapsto v$ given in the above theorem is a mapping from ℓ_∞ to ℓ_∞ and so for the case where $e(k) = 0$ and $\Omega(q) = 0$ the closed loop will be BIBO stable.

Remark 5. Note that if $\widetilde{\phi}_I(k) \leq 1$ for $\widetilde{\nu}(k) = 0$ it follows that $\nu(k) = 0$ is the optimal input signal. This corresponds to the fact that the optimal SPCP controller is LTI if no inequality constraints are active.

Remark 6. Let us note that the optimal feedback can be expressed as

$$\Delta u(k) = -F_w\hat{x}_w(k) + \big(\mathcal{Q}_w\widetilde{w}\big)(k)$$

By construction of the controller we have $\hat{x}_e \longrightarrow \hat{x}_w$ for $k \longrightarrow \infty$. By choosing $\hat{x}_w(0) = \hat{x}_e(0) = x(0)$ we guarantee that $\hat{x}_w(k) = \hat{x}_e(k)$ for all k, and

$$\Delta u(k) = F_w\hat{x}_w(k) + \big(\mathcal{Q}_w\widetilde{w}\big)(k) = \big(F_e - F_w\big)\hat{x}_w(k) - F_e\hat{x}_e(k) + \big(\mathcal{Q}_w\widetilde{w}\big)(k)$$

6. Nominal Performance

In this section, we consider the standard predictive control problem for the model error-free case (so $\Omega(q) = 0$ and thus $\delta_1(k) = 0$). This is done by tuning the feedback part $\mathcal{Q}_e$ of the mapping K_2. In this paper, the mapping $v_e = \mathcal{Q}_e\xi$ is based on a time-varying finite-impulse-response (TV-FIR)

$$v_e(k) = \sum_{i=0}^{n_M} M_i(k)\xi(k-i) \tag{15}$$

where $v_e(k) \in \ell_\infty^{n_4}$ and $M_i(k)$ are the time-varying finite-impulse-response parameters. We choose the number of parameters $n_M > N$. A state-space representation for this TV-FIR is given by

$$x_Q(k+1) = A_Q x_Q(k) + B_Q\xi(k) \tag{16}$$

$$v_e(k) = C_Q(k)x_Q(k) + D_Q(k)\xi(k) \tag{17}$$

where A_Q, B_Q, $C_Q(k)$ and $D_Q(k)$ are given by

$$A_Q = \begin{bmatrix} 0 & & & 0 \\ I & 0 & & \\ & \ddots & \ddots & \\ 0 & & I & 0 \end{bmatrix}, \qquad B_Q = \begin{bmatrix} I \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{18}$$

$$C_Q(k) = \begin{bmatrix} M_1(k) \; M_2(k) \; \dots \; M_{n_M}(k) \end{bmatrix}, \qquad D_Q(k) = M_0(k) \tag{19}$$

Lemma 1. *Consider the mapping $\mathcal{Q}_e$ given by (16)–(19). Further, define the parameter vector*

$$\theta(k) = \mathrm{col} \begin{bmatrix} D_Q^T(k) \\ C_Q^T(k) \end{bmatrix} \in \mathbb{R}^{n_\theta}$$

and let the parameter vector be bounded by $|\theta(j)| \leq \theta_{\max}$ for all $j \in \mathbb{Z}$. Then $\mathcal{Q}_e$ is a mapping from ℓ_∞ to ℓ_∞.

The proof of Lemma 1 is omitted. Since the filter in (15) is FIR, though time-varying, the uniform boundedness of its coefficients makes the statement of Lemma 1 trivial. From now on we will use the notation $C_Q(\theta, k)$, $D_Q(\theta, k)$ and $M_i(\theta, k)$ to emphasize the dependence on the parameter vector θ that will be optimized later.

Making a prediction based on the actual parameters $\theta(k)$, we obtain an extended vector $\widetilde{v}_e(k) \in \ell_\infty^{(N+1)n_4}$ given by

$$\widetilde{v}_e(k) = \bar{C}_Q(\theta, k) x_Q(k) + \bar{D}_Q(\theta, k) \xi(k) \tag{20}$$

where

$$\bar{C}_Q(\theta, k) = \begin{bmatrix} C_Q \\ C_Q A_Q \\ C_Q A_Q^2 \\ \vdots \\ C_Q A_Q^{N-1} \end{bmatrix}$$

$$= \begin{bmatrix} M_1(\theta,k) & M_2(\theta,k) & \dots & M_{1+n_M-N}(\theta,k) & \dots & M_{n_M}(\theta,k) \\ M_2(\theta,k) & M_3(\theta,k) & \dots & M_{2+n_M-N}(\theta,k) & \dots & 0 \\ \vdots & \dots & & \dots & & \vdots \\ M_N(\theta,k) & M_{N+1}(\theta,k) & \dots & M_{n_M}(\theta,k) & \dots & 0 \end{bmatrix} \tag{21}$$

and

$$\bar{D}_Q(\theta,k) = \begin{bmatrix} D_Q(\theta,k) \\ C_Q(\theta,k)B_Q \\ C_Q(\theta,k)A_Q B_Q \\ \vdots \\ C_Q(\theta,k)A_Q^{N-2}B_Q \end{bmatrix} = \begin{bmatrix} M_0(\theta,k) \\ M_1(\theta,k) \\ \vdots \\ M_{N-1}(\theta,k) \end{bmatrix} \tag{22}$$

Define

$$\Gamma_e(k) = \begin{bmatrix} V_{x,1}\, x_Q(k) + V_{\xi,1}\,\xi(k) & \cdots & V_{x,n_\theta}\, x_Q(k) + V_{\xi,n_\theta}\,\xi(k) \end{bmatrix} \tag{23}$$

where $V_{x,i}$ and $V_{\xi,i}$ are such that

$$\bar{C}_Q(\theta,k) = \sum_{i=1}^{n_\theta} V_{x,i}\theta_i(k), \qquad \bar{D}_Q(\theta,k) = \sum_{i=1}^{n_\theta} V_{\xi,i}\theta_i(k)$$

Then

$$\widetilde{v}_e(k) = \bar{C}_Q(\theta,k)x_Q(k) + \bar{D}_Q\,\xi(k) = \Gamma_e(k)\,\theta(k)$$

For the nominal case we give the following theorem which is an extension of Theorem 2.

Theorem 3. *Consider a nominal feasible SPCP with $\Omega(q) = 0$ and a controller in the configuration of Theorem 1, where $\widetilde{F}_w$, L_w, $\widetilde{Q}_w$ and $\widetilde{\nu}$ are given by Theorem 2. Further, let $L_e = B_3 D_{53}^{-1}$ and let $\widetilde{F}_e$ be given such that $E_u\widetilde{F}_e = F_e$ and $A - \bar{B}_4\widetilde{F}_e = A - B_4 F_e$ has all eigenvalues strictly inside the unit circle. Define the signals*

$$\begin{aligned} \widetilde{u}_e(k) &= -\widetilde{F}_e\hat{x}_e(k) + \big(\widetilde{F}_e - \widetilde{F}_w\big)\,\hat{x}_w(k) + \widetilde{Q}_w\,\widetilde{w}(k) + \widetilde{\nu}(k) \\ \widetilde{z}_e(k) &= \bar{C}_2\,\hat{x}_e(k) + \bar{D}_{22}\,\widetilde{w}(k) + \bar{D}_{23}\,e(k) + \bar{D}_{24}\,\widetilde{u}_e(k) \\ \widetilde{\phi}_{Ee}(k) &= \bar{C}_3\,\hat{x}_e(k) + \bar{D}_{32}\,\widetilde{w}(k) + \bar{D}_{33}\,e(k) + \bar{D}_{34}\,\widetilde{u}_e(k) \\ \widetilde{\phi}_{Ie}(k) &= \bar{C}_4\,\hat{x}_e(k) + \bar{D}_{42}\,\widetilde{w}(k) + \bar{D}_{43}\,e(k) + \bar{D}_{44}\,\widetilde{u}_e(k) \end{aligned}$$

Finally, let the mapping $\mathcal{Q}_e$ be given by

$$(\mathcal{Q}_e\xi)(k) = E_u\Gamma_e(k)\,\theta(k)$$

where $\Gamma_e(k)$ is given by (23) and $\theta(k)$ is the solution to the quadratic-programming (QP) problem

$$\min_{\theta(k)}\ \theta^T(k)\,\Gamma_e^T(k)\,\bar{D}_{24}^T\,\bar{D}_{24}\,\Gamma_e(k)\,\theta(k) + \widetilde{z}_e^T(k)\,\bar{D}_{24}\,\Gamma_e(k)\,\theta(k) \tag{24}$$

subject to the constraints

$$\widetilde{\phi}_E(k) = \widetilde{\phi}_{Ee}(k) + \bar{D}_{34}\,\Gamma_e(k)\,\theta(k) = 0 \tag{25}$$

$$\widetilde{\phi}_I(k) = \widetilde{\phi}_{Ie}(k) + \bar{D}_{44}\,\Gamma_e(k)\theta(k) \leq 1 \tag{26}$$

$$|\theta(k)| \leq \theta_{\max} \tag{27}$$

In these settings, the controller $\Delta u(k) = K(\widetilde{w}, y)$ *is the solution to the SPCP for* $\Omega(q) = 0$.

The proof is given in Appendix C.

7. Robust Stability

In this section, we will solve the robust SPCP. $\mathcal{Q}_e$ is parametrized by a TV-FIR as in Theorem 3 and bounds on the parameters θ will be derived to guarantee robust stability for all $||\Omega||_{(1)} \leq \epsilon_\Omega$.

Theorem 4. *Consider the SPCP as in Definition 1 and a controller in the configuration of Theorem 3, with the TV-FIR Youla parameter* $\mathcal{Q}_e$*. Let* $\delta_1(k) \in \ell_\infty^{n_1}$*,* $\delta_2(k) \in \ell_\infty^{m_1}$*, and let the functions* $h(k) : \mathbb{Z} \to \ell_\infty^{m_1 \times n_1}$*,* $g(k) : \mathbb{Z} \to \ell_\infty^{m_1}$ *and* $f(k) : \mathbb{Z} \to \ell_\infty^{n_1}$ *be impulse responses given by*

$$h(0) = D_{11}, \quad h(k) = 0 \quad \textit{for} \quad k < 0$$

$$h(k) = \begin{bmatrix} C_1 - D_{14}F_e & D_{14}F_e \end{bmatrix} \begin{bmatrix} A - B_4F_e & B_4F_e \\ 0 & A - L_eC_5 \end{bmatrix}^{k-1} \times \begin{bmatrix} B_1 \\ B_1 - L_e\,D_{51} \end{bmatrix} \quad \textit{for} \quad k > 0$$

$$g(0) = D_{14}, \quad g(k) = 0 \quad \textit{for} \quad k < 0$$

$$g(k) = \begin{bmatrix} C_1 - D_{14}F_e \end{bmatrix} \begin{bmatrix} A - B_4F_e \end{bmatrix}^{k-1} [B_4] \quad \textit{for} \quad k > 0$$

$$f(0) = [D_{51}], \quad f(k) = 0 \quad \textit{for} \quad k < 0$$

$$f(k) = [C_5] \begin{bmatrix} A - L_eC_5 \end{bmatrix}^{k-1} \begin{bmatrix} B_1 - L_e\,D_{51} \end{bmatrix} \quad \textit{for} \quad k > 0$$

Define the truncation-tail functions

$$\epsilon_h(k) = \begin{cases} 0 & \textit{for} \quad k \leq \mu \\ h(k) & \textit{for} \quad k > \mu \end{cases}$$

$$\epsilon_g(k) = \begin{cases} 0 & \textit{for} \quad k \leq \mu \\ g(k) & \textit{for} \quad k > \mu \end{cases}$$

$$\epsilon_f(k) = \begin{cases} 0 & \textit{for} \quad k \leq \mu \\ f(k) & \textit{for} \quad k > \mu \end{cases}$$

for some integer $\mu > 0$ *and define*

$$\eta_{\text{trunc}} = \|\epsilon_h\|_1 + \|\epsilon_g\|_1 \bar{M}_\theta \|f\|_1 + \|g\|_1 \bar{M}_\theta \|\epsilon_f\|_1$$

where $\bar{M}_\theta = \max_{j \geq 0} \| M_i(\theta, k-j) \|$ *for* M_i *given in (15).*

Finally, define a kernel-function $H(k+j, k+j-m)$ *for* $j \geq 0,\ m \geq 0$*:*

$$H\big(k+j, k+j-m\big) = h(m) + \sum_{i=j}^{\mu} \sum_{l=i}^{\min(n,m+i)} g(i) M_{l-i}(\theta, k-i) f(m-l)$$

Then a sufficient condition for robust stability is as follows:

$$\max_a \sum_{b=1}^{n_1} \sum_{m=0}^{\mu} \Big| H_{ab}\big(k+j,\ k+j-m\big) + \sum_{i=0}^{j} \sum_{l=i}^{\min(m,n+i)} g_a(i)\, M_{l-i}(\theta, k)\, f_b(m-l) \Big| \leq \epsilon_\Omega - \eta_{\text{trunc}} \quad \forall j \geq 0 \qquad (28)$$

where H_{ab} *denotes the* (a,b)*-th element of the matrix* H*,* g_a *denotes the* a*-th element of the vector* g *and* f_b *denotes the* b*-th element of the vector* f*.*

The proof is given in Appendix D.

Because the TV-FIR function $M(\theta, k)$ is linear in the parameters $\theta(k)$, the derived robustness constraint is convex in $\theta(k)$. Convex optimization algorithms can now be used to compute the optimal robust standard preditive controller. How to choose the parameter μ will be discussed in the next section about tuning.

8. Tuning

8.1. Tuning of Feedback Controller Parameters F_e and L_e

It is desirable that $h(k)$ (as defined in Theorem 4) has a 1-norm less than $1/\epsilon_\Omega$, or

$$\max_i \sum_{j=1}^{n_1} \sum_{k=0}^{\infty} \big| h_{ij}(k) \big| \leq 1/\epsilon_\Omega \qquad (29)$$

This is because the nominal controller will then satisfy the robustness constraint which increases the likelihood of a feasible solution. Therefore F_e and L_e have to be tuned such that $A - B_4 F_e$ and $A - L_e C_5$ have all eigenvalues inside the unit circle and condition (29) is satisfied. If A allready has all eigenvalues inside the unit disc and the system $\Sigma_{11}(q) = C_1(qI - A)^{-1}B_1 + D_{11}$ has a 1-norm less than $1/\epsilon_\Omega$, (e.g. for an additive model error we have $\Sigma_{11} = 0$), the choice $\widetilde{F}_e = 0$ and $L_e = B_3 D_{53}^{-1}$ is obvious, thereby $h(k) = 0$. Otherwise, select F_e such that $A - B_4 F_e$ has all eigenvalues inside the unit circle and the 1-norm condition (29) is satisfied. $\widetilde{F}_e$ can now be defined as

$$\widetilde{F}_e = \begin{bmatrix} F_e \\ F_e\big(A - B_4 F_e\big)^{-1} \\ \vdots \\ F_e\big(A - B_4 F_e\big)^{-N+1} \end{bmatrix}$$

If for a given L_e there is no stabilizing F_e such that $h(k)$ has a 1-norm less than $1/\epsilon_\Omega$, there are two possibilities. Either we detune L_e or we choose a stabilizing F_e that does not give a $h(k)$ that has a 1-norm less than $1/\epsilon_\Omega$ with a higher risk that infeasibility occurs. Note that if condition (29) is satisfied and there are no signal constraints, feasibility is guaranteed. If there are only equality constraints, feasibility at all future times is guaranteed if there exists a solution at the initial time sample, independent of the condition (29).

8.2. Tuning of Prediction Horizon N and Control Horizon N_u

The tuning of N and N_u is as usual in predictive control. The main difference is that we assume the SPCP to be well-defined and nominal feasible, so nominal stability is required. This means that either an end-point constraint is introduced, or that we first compute the feedback matrix F_w as in Theorem 2 and check if $A - B_4 F_w$ has all eigenvalues inside the unit circle. The increase in N has a strong influence on the computational burden due to the robustness constraint.

8.3. Tuning of TV-FIR Parameters n_M

If $n_M \geq N$ we do not loose any degree of freedom in choosing the optimal $\widetilde{u}(k)$ in comparison with conventional predictive control techniques as long as $\mathrm{rank}(\Gamma_e) \geq N$. By choosing $n_M > N$ the chance that a feasible solution exists increases and the robustness constraint may become less conservative because of the extra degrees of freedom. However, increasing n_M increases the computational load.

8.4. Tuning of Truncation Parameter μ

The parameter μ should be chosen such that $\eta_{\text{trunc}} \ll \epsilon_\Omega$. The values $||\epsilon_h||_1$, $||\epsilon_g||_1$, $||\epsilon_f||_1$, $||g||_1$ and $||h||_1$ can be computed beforehand. The value $\bar{M}_\theta$ depends on the actual optimization. The value η_{trunc} is convex in $\theta(k)$ and can be included in the optimization. Another option is to put an upper bound on $\bar{M}_\theta$ during the optimization. A third possibility is to compute the value $\eta_{\rm trunc}$ after the optimization and to adapt the value μ if the value $\bar{M}_\theta$ becomes too large.

9. Simulation Example

In this section, the results of a predictive control problem will be shown using the state-space controller, as derived in the previous sections.

Consider the process

$$G(q) = \frac{0.5\,q^{-2}(1-0.4q^{-1})}{(1-0.5q^{-1})(1-0.9q^{-1})(1-q^{-1})} \tag{30}$$

with the noise model

$$H(q) = \frac{1}{1-q^{-1}}$$

A constrained GPC problem (Clarke *et al.*, 1987) with additive model error is considered and the following parameters are chosen:

- prediction horizon $H_p = N = 8$,
- control horizon $H_c = N_c = 4$,
- control weighting: $\lambda = 0.1$,
- number of TV-FIR parameters: $n = 8$.

The reference signal $r(k)$ is given by

$$r(k) = \begin{cases} 0 & \text{for} \quad k < 10 \\ 1 & \text{for} \quad k \geq 10 \end{cases}$$

The input increment signal is bounded by the rate constraint

$$|\Delta u(k)| \leq 0.25$$

A perturbed process is given by

$$\bar{G}(q) = \frac{1.4\,q^{-1}(1-0.41q^{-1})}{(1+0.15q^{-1})(1-0.75q^{-1})(1-q^{-1})} \tag{31}$$

which results in a bound on the additive model error by

$$||\Omega||_{(1)} = ||G-\bar{G}||_{(1)} \leq 5.0$$

The disturbance signal $e(k)$ is a zero-mean white noise sequence with an additional peak at time $k = 50$, so

$$e(k) = e_1(k) + e_2(k) \tag{32}$$

where $e_1(k)$ is a zero-mean white noise sequence with variance 4×10^{-4}, and

$$e_2(k) = \begin{cases} 0 & \text{for} \quad k \neq 50 \\ 0.3 & \text{for} \quad k = 50 \end{cases} \tag{33}$$

The output-disturbance, denoted by $d(q)$, is equal to

$$d(k) = H(q)\, e(k)$$

The matrices for the SPCP are given by

$$\left[\begin{array}{c|cccc} A & B_1 & B_2 & B_3 & B_4 \\ \hline C_1 & D_{11} & D_{12} & D_{13} & D_{14} \\ C_2 & D_{21} & D_{22} & D_{23} & D_{24} \\ C_3 & D_{31} & D_{32} & D_{33} & D_{34} \\ C_4 & D_{41} & D_{42} & D_{43} & D_{44} \\ C_5 & D_{51} & D_{52} & D_{53} & D_{54} \end{array}\right] = \left[\begin{array}{ccc|ccc c} 0.5 & 0 & 0 & 0 & 0 & 0 & 1.25 \\ 0 & 0.9 & 0 & 0 & 0 & 0 & -5.625 \\ 0 & 0 & 1 & 0 & 0 & 1 & 6 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 1 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & -4 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 \end{array}\right]$$

$$M_z = \begin{bmatrix} 0_{16\times1} & I_{16\times16} & 0_{16\times1} \end{bmatrix}, \qquad M_E = \begin{bmatrix} 0_{4\times5} & I_{4\times4} \end{bmatrix}, \qquad M_I = \begin{bmatrix} I_{8\times8} & 0_{8\times10} \end{bmatrix}$$

The simulation results (with $0 \leq k \leq 100$) are given in Figs. 3–5. The left panels give the results when the true process is equal to the nominal model (30). The right panels give the results when the true process is equal to the perturbed model (31). The solid lines in all figures show the results if the robustness bound $\epsilon_\Omega = 5$ is taken into account, the dash-dot lines correspond to the case where the robustness bound is not taken into account, i.e. $\epsilon_\Omega = \infty$. In Fig. 3 the reference signal $r(k)$ is given (dashed line) together with the optimal output $y(k)$ (solid line/dash-dot line) and the output disturbance signal $d(k)$ (dotted line). In Fig. 4 the control increment signal $\Delta u(k)$ is given. Clearly, the rate-constraint $|\Delta u(k)| \leq 0.25$ is not violated. In Fig. 5 the magnitudes of the parameters $\theta_j(k)$ are given.

It is clear that the SPCP results in a robust controller for the nominal, as well as for the perturbed case, where the design with $\epsilon_\Omega = \infty$ gives a good performance for the nominal case, but becomes unstable for the perturbed case.

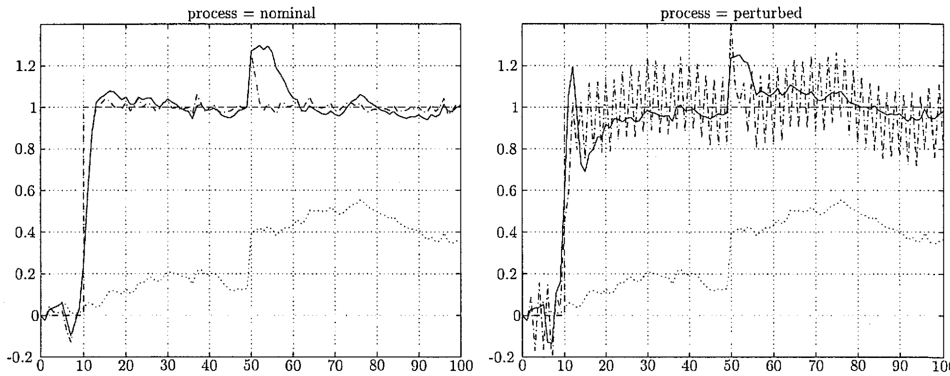

Fig. 3. Signals $r(k)$ (dashed line), $d(k)$ (dotted line) and $y(k)$, nominal design (dash-dot line), robust design (solid line).

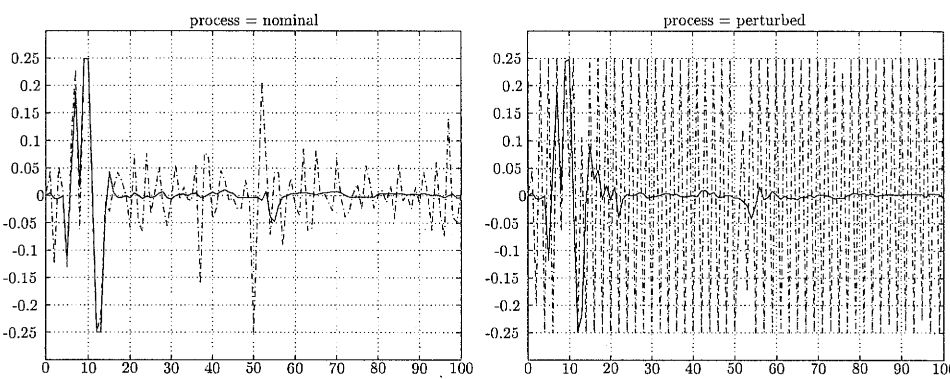

Fig. 4. Signals $\Delta u(k)$, nominal design (dash-dot line), robust design (solid line).

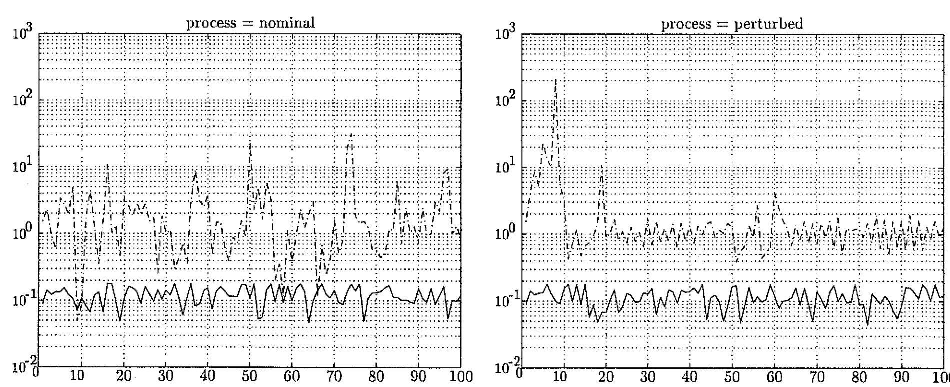

Fig. 5. Magnitude of $\theta(k)$, nominal design (dash-dot line), robust design (solid line).

10. Conclusions and Future Research

A standard predictive control problem (SPCP) has been defined which unifies most of many existing predictive control problems. The SPCP is solved for the unperturbed case, the nominal case and the robust case. The internal structure of the SPCP controller consists of a feedforward controller and a feedback controller using a Youla parametrization, where a time-varying Youla-parameter $\mathcal{Q}_e$ is optimized. It has been shown that using this structure conditions guaranteeing nominal and robust (BIBO) stability can easily be derived. The resulting optimization problem is convex in the parameters of $\mathcal{Q}_e$ and therefore easy to solve. Finally, a straightforward procedure has been given for the tuning of the degrees of freedom in the SPCP controller. An example shows that by using the derived algorithm a perturbed process can be controlled, while handling constraints and preserving robust stability.

The proposed scheme provides a firm basis for robust adaptive control. Identification algorithms that provide models with 1-norm model error bounds are available (Hakvoort and Van den Hof, 1997) and can be included in the presented framework. To limit the computational burden, involving the robustness constraint, a dual Youla parametrization scheme (Wams and Van den Boom, 1997) can be used. In this approach, the functions in Theorem 4 simplify to $h(k) = 0$, $g(k) = \delta(k)\,I$ and $f(k) = \delta(k)\,I$. Thus the optimization and the tuning become easier.

Appendices

A. Proof of Theorem 1

For a controller $(y, \widetilde{w}, v) \mapsto (\Delta u, \xi)$, given by eqns. 5–7, the closed-loop mapping $(\delta_2, z, \phi_E, \phi_I, \xi) = \Lambda(\delta_1, \widetilde{w}, e, v)$ can be found by choosing $x_{T1}(k) = x(k)$, $x_{T2}(k) = x(k) - \hat{x}_e(k)$ and elimination of the variables $\Delta u(k)$ and $y(k)$:

$$\begin{aligned}
\xi(k) &= -C_5\hat{x}_e(k) - D_{52}E_w\widetilde{w}(k) + y(k)\\
&= -C_5\hat{x}_e(k) - D_{52}E_w\widetilde{w}(k) + C_5x(k) + D_{51}\delta_1(k) + D_{52}E_w\widetilde{w}(k) + D_{53}e(k)\\
&= C_5x_{T2}(k) + D_{51}\delta_1(k) + D_{53}e(k)
\end{aligned}$$

$$\begin{aligned}
x(k+1) &= Ax(k) + B_1\delta_1(k) + B_2E_w\widetilde{w}(k) + B_3e(k) + B_4\Delta u(k)\\
\hat{x}_e(k+1) &= A\hat{x}_e(k) + L_e\xi(k) + L_w\widetilde{w}(k) + B_4\Delta u(k)\\
&= \big(A - L_eC_5\big)\hat{x}_e(k) + L_eC_5x(k) + L_eD_{51}\delta_1(k)\\
&\quad + L_w\widetilde{w}(k) + L_eD_{53}e(k) + B_4\Delta u(k)\\
x_{T2}(k+1) &= \big(A - L_eC_5\big)x_{T2}(k) + \big(B_1 - L_eD_{51}\big)\delta_1(k)\\
&\quad + \big(B_2E_w - L_w\big)\widetilde{w}(k) + \big(B_3 - L_eD_{53}\big)e(k)
\end{aligned}$$

$$x(k+1) = Ax(k) + B_1\delta_1(k) + B_2E_w\widetilde{w}(k) + B_3e(k) + B_4\Delta u(k)$$

$$\begin{aligned} x_{T1}(k+1) = & \big(A - B_4F_e\big)x_{T1}(k) + B_4F_e x_{T2}(k) + B_1\delta_1(k) \\ & + B_2E_w\widetilde{w}(k) + B_3e(k) + B_4v(k) \end{aligned}$$

$$\begin{bmatrix} \delta_2(k) \\ z(k) \\ \phi_E(k) \\ \phi_I(k) \end{bmatrix} = \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix} x(k) + \begin{bmatrix} D_{11} \\ D_{21} \\ D_{31} \\ D_{41} \end{bmatrix} \delta_1(k) + \begin{bmatrix} D_{12} \\ D_{22} \\ D_{32} \\ D_{42} \end{bmatrix} w(k) + \begin{bmatrix} D_{13} \\ D_{23} \\ D_{33} \\ D_{43} \end{bmatrix} e(k)$$

$$+ \begin{bmatrix} D_{14} \\ D_{24} \\ D_{34} \\ D_{44} \end{bmatrix} \Delta u(k) = \begin{bmatrix} C_1 - D_{14}F_e \\ C_2 - D_{24}F_e \\ C_3 - D_{34}F_e \\ C_4 - D_{44}F_e \end{bmatrix} x_{T1}(k) + \begin{bmatrix} D_{14}F_e \\ D_{24}F_e \\ D_{34}F_e \\ D_{44}F_e \end{bmatrix} x_{T2}(k)$$

$$+ \begin{bmatrix} D_{11} \\ D_{21} \\ D_{31} \\ D_{41} \end{bmatrix} \delta_1(k) + \begin{bmatrix} D_{12} \\ D_{22} \\ D_{32} \\ D_{42} \end{bmatrix} w(k) + \begin{bmatrix} D_{13} \\ D_{23} \\ D_{33} \\ D_{43} \end{bmatrix} e(k) + \begin{bmatrix} D_{14} \\ D_{24} \\ D_{34} \\ D_{44} \end{bmatrix} v(k)$$

This results in the state-space description (8) which has all system eigenvalues strictly inside the unit circle. So the closed-loop mapping $(\delta_2, z, \phi_E, \phi_I, \xi) = \Lambda(\delta_1, \widetilde{w}, e, v)$ is stable.

Note that the signal $v(k)$ only affects the state $x_{T1}(k)$ and not state $x_{T2}(k)$, and that the signal $\xi(k)$ is not affected by the state $x_{T1}(k)$. Thus the mapping $v(k) \mapsto \xi(k)$ is zero and

$$\delta_1 \in \ell_\infty,\ e \in \ell_\infty \Rightarrow \xi \in \ell_\infty$$

Because $\mathcal{Q}_w$ and $\mathcal{Q}_e$ are mappings from ℓ_∞ to ℓ_∞, it follows that

$$\widetilde{w} \in \ell_\infty,\ \xi \in \ell_\infty \Rightarrow v \in \ell_\infty$$

Finally,

$$\delta_1 \in \ell_\infty,\ \widetilde{w} \in \ell_\infty,\ e \in \ell_\infty \Rightarrow \delta_2 \in \ell_\infty,\ z \in \ell_\infty,\ \phi_E \in \ell_\infty,\ \phi_I \in \ell_\infty$$

B. Proof of Theorem 2

For given L_w, L_e, $e(k)=0$ and $\delta_1(k)=0$ we have from the closed-loop equation (8)

$$\begin{aligned} x_{T2}(k+1) &= \big(A - L_eC_5\big)x_{T2}(k) + \big(B_1 - L_eD_{51}\big)\delta_1(k) \\ &\quad + \big(B_2E_w - L_w\big)\widetilde{w}(k) + \big(B_3 - L_eD_{53}\big)e(k) \\ &= \big(A - L_eC_5\big)x_{T2}(k) = 0 \end{aligned}$$

because $x_{T2}(0) = x(0) - \hat{x}_e(0) = 0$. As a result, $\xi(k) = 0$ and so $v_e(k) = (\mathcal{Q}_e\xi)(k) = 0$. Now it follows immediately from (1), (5) and (9) that

$$\begin{aligned} x(k) = \hat{x}_e(k) = \hat{x}_w(k) &= \big(A - B_4F_e\big)\hat{x}_w(k) + L_w\widetilde{w}(k) + B_4v(k) \\ &= \big(A - B_4F_w\big)\hat{x}_w(k) + L_w\widetilde{w}(k) + \nu(k) \end{aligned}$$

First consider the case without inequality constraints: Define two additional signals:

$$\widetilde{z}'(k) = \bar{C}_2\,\hat{x}_w(k) + \bar{D}_{22}\,\widetilde{w}(k) \tag{B1}$$

$$\widetilde{\phi}'_E(k) = \bar{C}_3\,\hat{x}_w(k) + \bar{D}_{32}\,\widetilde{w}(k) \tag{B2}$$

From Proposition 1 we know that

$$\widetilde{z}(k) = \bar{D}_{24}\,\widetilde{u}(k) + \widetilde{z}'(k) \quad \text{and} \quad \widetilde{\phi}_E(k) = \bar{D}_{34}\,\widetilde{u}(k) + \widetilde{\phi}'_E(k)$$

First consider the problem of minimizing $\min_{\widetilde{u}(k)} \widetilde{z}^T(k)\widetilde{z}(k)$ subject to the constraint $\widetilde{\phi}_E(k) = \bar{D}_{34}\,\widetilde{u}(k) + \widetilde{\phi}'_E(k) = 0$. Using the Lagrange multiplier method, the following solution is found:

$$\begin{aligned} \widetilde{u}_0(k) &= -[I\ ,\ 0]\begin{bmatrix} 2\bar{D}_{24}^T\bar{D}_{24} & \bar{D}_{34}^T \\ \bar{D}_{34} & 0 \end{bmatrix}^{-1}\begin{bmatrix} 2\bar{D}_{24}^T\ z(k)' \\ \widetilde{\phi}'_E(k) \end{bmatrix} \\ &= -\widetilde{F}_w\,\hat{x}_w(k) + \widetilde{Q}_w\,\widetilde{w}(k) \\ &= -\widetilde{F}_w\,\hat{x}_w(k) + \widetilde{Q}_w\,\widetilde{w}(k) \\ \widetilde{z}_0(k) &= \bar{C}_2\,\hat{x}_w(k) + \bar{D}_{22}\,\widetilde{w}(k) + \bar{D}_{24}\,\widetilde{u}_0(k) \end{aligned}$$

Now we adapt the value $\tilde{u}_0$ such that the inequality constraint

$$\widetilde{\phi}_I(k) \le 1$$

is not violated. Define the signals

$$\widetilde{u}(k) = \widetilde{u}_0(k) + \widetilde{\nu}(k)$$

and so

$$\tilde{z}(k) = \tilde{z}_0(k) + \bar{D}_{24}\,\tilde{\nu}(k)$$

The criterion $J(k)$ becomes

$$J(k) = \tilde{z}^T(k)\tilde{z}(k) = \tilde{z}_0^T(k)\tilde{z}_0(k) + \tilde{\nu}(k)^T\bar{D}_{24}^T\bar{D}_{24}\,\tilde{\nu}(k)$$

where we used the fact that $\tilde{u}_0(k)$ is the optimal solution and so $\tilde{z}_0^T(k)\bar{D}_{24}\,\tilde{\nu}(k) = 0$ for all $\tilde{\nu}(k)$ satisfying $\bar{D}_{34}\,\tilde{\nu}(k) = 0$. Further the equality constraint becomes

$$\tilde{\phi}_I(k) = \bar{D}_{34}\,\tilde{v}(k) = \bar{D}_{34}\,\tilde{\nu}(k) = 0$$

and the inequality constraint

$$\tilde{\phi}_I(k) = \tilde{\phi}_{IO}(k) + \bar{D}_{44}\,\tilde{\nu}(k) < 1$$

where

$$\begin{aligned}\tilde{\phi}_{IO}(k) &= \bar{C}_4\,\hat{x}_w(k) + \bar{D}_{42}\,\tilde{w}(k) + \bar{D}_{44}\,\tilde{u}_0(k)\\ &= \left(\bar{C}_4 - \bar{D}_{44}\tilde{F}_w\right)\hat{x}_w(k) + (\bar{D}_{42} + \bar{D}_{44}\tilde{Q}_w)\,\tilde{w}(k)\end{aligned}$$

The optimal value of $\tilde{\nu}(k)$ is now given by the optimization problem (11)–(14). It is clear that the optimal input signal is given by

$$\begin{aligned}\Delta u(k) &= -F_w\,\hat{x}_w(k) + Q_w\tilde{w}(k) + \nu(k)\\ &= -F_e\,\hat{x}_e(k) + (F_e - F_w)\,\hat{x}_w(k) + (Q_w\tilde{w})(k)\end{aligned}$$

C. Proof of Theorem 8

Theorem 1 gives

$$\begin{aligned}x_{T2}(k+1) &= \big(A - L_eC_5\big)x_{T2}(k) + \big(B_1 - L_eD_{51}\big)\delta_1(k)\\ &\quad + \big(B_2E_w - L_w - L_eD_{52}E_w\big)\tilde{w}(k) + \big(B_3 - L_eD_{53}\big)e(k)\\ &= \big(A - L_eC_5\big)x_{T2}(k)\end{aligned}$$

for $\delta_1(k) = 0$, $L_w = B_2E_w - L_eD_{52}E_w$ and $L_e = B_3D_{53}^{-1}$. The eigenvalues of $A - L_eC_5 = A - B_3D_{53}^{-1}C_5$ are strictly inside the unit circle by Definition 1. This means that $x_{T2}(k)$ will converge to zero for any initial condition and, because $x_{T2}(k) = x(k) - \hat{x}_e(k)$, we find that $x_e(k)$ will converge to $x(k)$. From Proposition 1 we have

$$\begin{aligned}\tilde{u}(k) &= \tilde{u}_e(k) + \Gamma_e(k)\theta(k)\\ \tilde{z}(k) &= \tilde{z}_e(k) + \bar{D}_{24}\,\Gamma_e(k)\theta(k)\\ \tilde{\phi}_E(k) &= \tilde{\phi}_{Ee}(k) + \bar{D}_{34}\,\Gamma_e(k)\theta(k)\\ \tilde{\phi}_I(k) &= \tilde{\phi}_{Ie}(k) + \bar{D}_{44}\,\Gamma_e(k)\theta(k)\end{aligned}$$

The problem $\min_{\widetilde{u}(k)} \widetilde{z}^T(k)\widetilde{z}(k)$ subject to the constraints $\widetilde{\phi}_E(k) = 0$ and $\widetilde{\phi}_I(k) \leq 1$ has now become an optimization one over $\theta(k)$. The optimal value of $\theta(k)$ is given by the solution to the optimization problem (24)–(27). Condition (27) guarantees that $\mathcal{Q}_e$ is a mapping from ℓ_∞ to ℓ_∞ and the controller is stabilizing.

D. Proof of Theorem 4

Consider the system Λ in Theorem 1 for $r = 0$ and $e = 0$. Then

$$\begin{bmatrix} \delta_2(k) \\ \xi(k) \end{bmatrix} = \begin{bmatrix} \Lambda_{11}(q) & \Lambda_{14}(q) \\ \Lambda_{51}(q) & 0 \end{bmatrix} \begin{bmatrix} \delta_1(k) \\ v(k) \end{bmatrix}$$

$$v(k) = \mathcal{Q}_e(\theta)\, \xi(k)$$

$$\delta_1(k) = \Omega\, \delta_2(k)$$

The impulse-response parameters corresponding to $\Lambda_{11}(q)$, $\Lambda_{14}(q)$ and $\Lambda_{51}(q)$ are given by $h(k)$, $g(k)$ and $f(k)$, respectively. Further, the Kernel parameters corresponding to the time-varying mapping $\mathcal{Q}_e(\theta)$ are given by $M_m(k)$. Now it follows that

$$\delta_2(k) = \sum_{m=0}^{\infty} h(m)\, \delta_1(k-m) + \sum_{m=0}^{\infty} g(m)\, v(k-m) \tag{D1}$$

$$\xi(k) = \sum_{m=0}^{\infty} f(m)\, \delta_1(k-m) \tag{D2}$$

$$v(k) = \sum_{m=0}^{\infty} M_m(k)\, \xi(k-m) \tag{D3}$$

Substitution of equations (D2) and (D3) in (D1) gives (for $j \geq 0$)

$$\begin{aligned} \delta_2(k+j) &= \sum_{m=0}^{\infty} h(m)\, \delta_1\big(k+j-m\big) \\ &\quad + \sum_{m=0}^{\infty}\sum_{i=0}^{\infty}\sum_{l=i}^{\min(m,n+i)} g(i)\, M_{l-i}\big(k+j-i\big)\, f(m-l)\, \delta_1\big(k+j-m\big) \\ &= \sum_{m=0}^{\infty} \bar{H}\big(k+j, k+j-m\big)\, \delta_1\big(k+j-m\big) \end{aligned}$$

where

$$\bar{H}(k+j, k+j-m) = h(m) + \sum_{i=0}^{\infty}\sum_{l=i}^{\min(m,n+i)} g(i)\, M_{l-i}\big(k+j-i\big)\, f(m-l), \quad j \geq 0$$

To obtain robust stability for all $||\Omega||_{(1)} \leq \epsilon_\Omega$, the mapping $\delta_2(k) = \mathcal{T}(\theta)\delta_1(k)$ must satisfy

$$\sup_{||\delta_1(k)||_\infty \leq 1} ||\, \delta_2(k)\, ||_\infty = \sup_{||\delta_1(k)||_\infty \leq 1} \left\| \sum_{m=0}^{\infty} \bar{H}(k, k-m)\, \delta_1(k-m) \right\|_\infty \leq 1/\epsilon_\Omega$$

and a necessary and sufficient condition is

$$\max_a \sum_{b=1}^{n_1} \sum_{m=0}^{\infty} \left| \bar{H}_{ab}(k+j, k+j-m) \right| \leq 1/\epsilon_\Omega \quad \text{for all} \quad j > 0$$

Now for all $j > 0$ an upper bound can be derived:

$$\max_a \sum_{b=1}^{n_1} \sum_{m=0}^{\infty} \left| \bar{H}_{ab}(k+j, k+j-m) \right|$$

$$= \max_a \sum_{b=1}^{n_1} \sum_{m=0}^{\infty} \left| h_{ab}(m) + \sum_{i=0}^{\infty} \sum_{l=i}^{\min(m,n+i)} g_a(i)\, M_{l-i}(k+j-i)\, f_b(m-l) \right|$$

$$\leq \max_a \sum_{b=1}^{n_1} \sum_{m=0}^{\mu} \left| h_{ab}(m) + \sum_{i=0}^{\mu} \sum_{l=i}^{\min(m,n+i)} g_a(i)\, M_{l-i}(k+j-i)\, f_b(m-l) \right|$$

$$+ \max_a \sum_{b=1}^{n_1} \sum_{m=\mu+1}^{\infty} \left| h_{ab}(m) \right|$$

$$+ \max_a \sum_{b=1}^{n_1} \sum_{m=0}^{\infty} \sum_{i=\mu+1}^{\infty} \sum_{l=i}^{\min(m,n+i)} |g_a(i)| \left| M_{l-i}(k+j-i) \right| \left| f_b(m-l) \right|$$

$$+ \max_a \sum_{b=1}^{n_1} \sum_{m=\mu+1}^{\infty} \sum_{i=0}^{\mu} \sum_{l=i}^{\min(m,n+i)} |g_a(i)| \left| M_{l-i}(k+j-i) \right| \left| f_b(m-l) \right|$$

$$\leq \max_a \sum_{b=1}^{n_1} \sum_{m=0}^{\mu} \left| h_{ab}(m) + \sum_{i=0}^{\mu} \sum_{l=i}^{\min(m,n+i)} g_a(i)\, M_{l-i}(k+j-i)\, f_b(m-l) \right| + \eta_{\text{trunc}}$$

where

$$\eta_{\text{trunc}} = ||\epsilon_h||_1 + ||\epsilon_g||_1\, \bar{M}_\theta ||f||_1 + ||g||_1\, \bar{M}_\theta ||\epsilon_f||_1$$

To guarantee a feasible solution in the future, we can choose $\theta(k+j) = \theta(k)$ for $j \geq 0$, and so $M_l(k+j) = M_l(k)$ for $j \geq 0$. Hence

$$\Big|h_{ab}(m) + \sum_{i=0}^{\mu} \sum_{l=i}^{\min(m,n+i)} g_a(i)\, M_{l-i}\big(k+j-i\big)\, f_b(m-l)\Big|$$

$$= \Big|h_{ab}(m) + \sum_{i=j+1}^{\mu} \sum_{l=i}^{\min(m,n+i)} g_a(i)\, M_{l-i}\big(k+j-i\big)\, f_b(m-l)$$

$$+ \sum_{i=0}^{j} \sum_{l=i}^{\min(m,n+i)} g_a(i)\, M_{l-i}\big(k+j-i\big)\, f_b(m-l)\Big|$$

$$= \Big|H_{ab}\big(k+j, k+j-m\big) + \sum_{i=0}^{j} \sum_{l=i}^{\min(m,n+i)} g_a(i)\, M_{l-i}(k)\, f_b(m-l)\Big|$$

and a sufficient condition for robust stability is given by

$$\max_a \sum_{b=1}^{n_1} \sum_{m=0}^{\mu} \Big|H_{ab}\big(k+j, k+j-m\big) + \sum_{i=0}^{j} \sum_{l=i}^{\min(m,n+i)} g_a(i)\, M_{l-i}(k)\, f_b(m-l)\Big| \leq \epsilon_\Omega - \eta_{\text{trunc}}$$

References

Clarke D.W., Mohtadi C. and Tuffs P.S. (1987): *Generalized predictive control — Part 1 and 2.* — Automatica, Vol.23, No.2, pp.137–160.

Clarke D.W. and Mohtadi C. (1989): *Properties of generalized predictive control.* — Automatica, Vol.25, No.6, pp.859–875.

Clarke D.W. and Scattolini R. (1991): *Constrained receding horizon predictive control.* — IEE Proc.-D, Vol.138, No.4.

Cutler C.R. and Ramaker B.L. (1979): *Dynamic matrix control — A computer control algorithm.* — AIChE Nat. Meeting, Houston, Texas, USA, also: Proc. Joint American Contr. Conf., San Francisco, California, USA, 1980, paper WP5-B.

De Vries R.A.J. and Van den Boom T.J.J. (1994): *Constrained predictive control with guaranteed stability and convex optimization.* — Proc. American Control Conference, Baltimore, Maryland, USA, pp.2842–2846.

De Vries R.A.J. and van den Boom T.J.J. (1996): *Predictive control with guaranteed nominal stability, A generalization of the end-point constraint.* — Proc. CESA-IMACS Conf., Lille, France, pp.467–472.

De Vries R.A.J. and Van den Boom T.J.J. (1997): *Robust stability constraints for predictive control.* — Proc. *European Control Conference*, Brussels, Belgium, paper FR-A B2.

Garcia C.E., Prett D.M. and Morari M. (1989): *Model predictive control: Theory and practice — A survey.* — Automatica, Vol.25, No.3, pp.335–348.

Gencelli H. and Nikolaou M. (1993): *Robust stability analysis of l_1-norm model predictive control.* — AIChE J., Vol.39, No.12, pp.1954–1965.

Hakvoort R.G. and Van den Hof P.M.J. (1997): *Identification of probabilistic system uncertainty regions by explicit evaluation of bias and variance errors.* — IEEE-AC, Vol.42, No.11, pp.1516–1528.

Kinnaert M. (1989): *Adaptive generalized predictive controller for MIMO systems.* — Int. J. Contr., Vol.50, No.1, pp.161–172.

Kothare M.V., Balakrishnan V. and Morari M. (1996): *Robust contrained predictive control using linear matrix inequalities.* — Automatica, Vol.32, No.10, pp.1361–1379.

Kouvaritakis B., Rossiter J.A. and Chang A.O.T. (1992): *Stable generalized predictive control: an algorithm with guaranteed stability.* — Proc. IEE, Vol.139, No.4, pp.349–362.

Lee J.H., Morari M. and Garcia C.E. (1994): *State-space interpretation of model predictive control.* — Automatica, Vol.30, No.4, pp.707–717.

Lee J.H. and Yu Z.H. (1994): *Tuning of model predictive controllers for robust performance.* — Comp. Chem. Eng., Vol.18, No.1, pp.15–37.

Mosca E. and Zhang J. (1992): *Stable redesign of predictive control.* — Automatica, Vol.28, No.6, pp.1229–1233.

Rawlings J.B. and Muske K.R. (1993): *The stability of constrained receding horizon control.* — IEEE Trans. Automat. Contr., Vol.AC-38, No.10, pp.1512–1516.

Richalet J. (1993): *Industrial applications of model based predictive control.* — Automatica, Vol.29, No.5, pp.1251–1274.

Rossiter J.A., Kouvaritakis B. and Rice M.J. (1998): *A numerically robust state-space approach to stable-predictive control strategies.* — Automatica, Vol.34, No.1, pp.65–73.

Soeterboek A.R.M. (1992): *Predictive Control — A Unified Approach.* — New York: Prentice Hall.

Vidyasagar M. (1993): *Nonlinear System Analysis.* — New York: Prentice-Hall.

Wams B. and Van den Boom T.J.J. (1997): *Youla-like parametrizations for predictive control of a class of linear time-varying systems.* — Proc. American Control Conference, Albequerque, New Mexico, USA, pp.2051–2052.

Zafiriou E. (1990): *Robust model predictive control of processes with hard constraints.* — Comp. Chem. Eng., Vol.14, No.4/5, pp.359–371.

Zheng Z.Q. and Morari M. (1993): *Robust stability of contrained model predictive control.* — Proc. *American Control Conference*, San Francisco, California, USA, pp.379–383.

Zheng A. and Morari M. (1994): *Global stabilization of linear discrete-time systems with bounded controls — A model predictive control approach.* — Proc. *American Control Conference*, Baltimore, Maryland, USA, pp.2847–2851.

Zheng A. (1995): *Robust control of systems subject to constraints.* — Ph.D. Thesis, California Institute of Technology, Pasadena.