Int. J. Appl. Math. and Comp. Sci., 1999, Vol.9, No.2, 421–434

LIMIT DISTRIBUTIONS OF THE CAPACITY OF LARGE REGULAR CHANNEL GRAPHS

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The paper is devoted to the investigation of reliability of large channel graphs having links with low reliabilities. Under some regularity assumptions regarding such graphs, we derive bounds and limit distributions of their capacities. The main goal of the paper is to prove a Poisson convergence of the capacity.

Keywords: channel graph, reliability.

1. Introduction

Consider a network to be a system involving the movement of some commodity such as information, products, or people. Computer networks, electronic circuits, communications networks are a few common examples. Colbourn (1987) gives a wide review of the definitions of such networks and their reliabilities from a combinatorial point of view. Assume that components of the network are failed or occupied independently and with a prescribed probability. We can consider this network as a stochastic one. The most common model of such a network is a probabilistic graph.

Lee (1955) introduces a simple model for interconnection networks with switches or intermediate node partitions into stages, in which links are failed whenever they are unavailable, either due to a component failure or to the occupancy with other traffic. As a suitable model, Lee proposes the channel (probabilistic) graphs defined in Section 2.

A fundamental problem that arises in stochastic networks is determiniation of appriopriate measures of the network performance. In general, network reliability problems are at least as difficult as NP-complete problems. Ball (1980) shows that for channel graphs this problem is #P-complete. Thus, there is little hope of efficiently calculating the exact reliability of channel graphs. Instead, one can try to obtain bounds or limit results for the reliability.

In the paper, we concentrate on the second problem, i.e. limit results, when the probabilities that the links are not occupied are small, i.e. the reliabilities of components are low. Moreover, we assume that all the components operate independently. The above assumptions can be fulfilled when a large network is shared with many users operating independently, each of them using the network with high intensity.

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Thus from the point of view of a particular user, all links are occupied with a high probability, so they are idle with a low probability.

Our paper is organized as follows. In the next section we introduce basic definitions of channel graphs. In Section 3 we consider results regarding known estimations and bounds for the reliability of general systems and their applications to channel graphs. Section 4 is devoted to algorithms based on the well-known Ford-Fulkerson theorem (Ford and Fulkerson, 1962) of a maximum flow applied to a channel graph. In Section 5 a limit distribution of the capacity of channel graphs is investigated, and in Section 6 some numerical results, exact and simulated, are analyzed.

2. Preliminaries

Let $V = V_0 \cup V_1 \cup \cdots \cup V_n \cup V_{n+1}$ be a disjoint union of $n+2$ sets, each set being a stage of nodes, $|V_j| = m_j$, $V_0 = \{s\}$ the source, $V_{n+1} = \{t\}$ the terminal. For $0 \le i \le n$, E_i is a set of directed links; each link goes from a node of V_i to a node of V_{i+1}, and $E = E_0 \cup E_1 \cup \cdots \cup E_n$. $G = (V, E)$ is a directed graph, called the channel graph with n stages. A channel graph is regular if s is connected with all nodes from V_1, all nodes from V_n are connected with t and all nodes from V_j have the same in-degree k''_j and out-degree k'_j for $2 \le j \le n-1$, see Fig. 1. A channel graph is completely regular if $m_j = m$ and $k'_j = k''_j = k$ for all j.

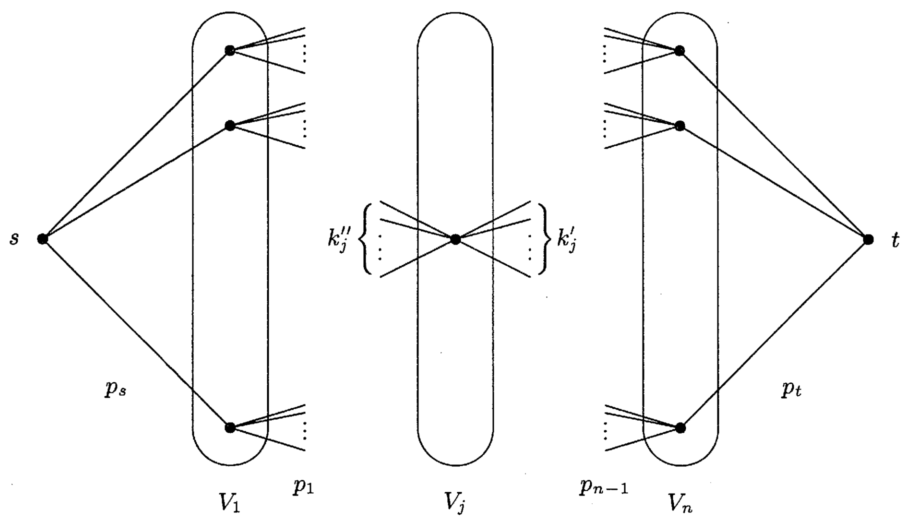

Fig. 1. Channel graph.

For bistate links, each link of a channel graph is either occupied (state 0) or idle (state 1). The probability of the occupancy of a link e is known and equal to p_e. Assume that the occupancy probabilities are independent. In regular channel graphs we assume that p_e are the same for links going from V_i to V_{i+1} — such a probability will be denoted by p_i and $p_0 = p_s$, $p_n = p_t$. In completely regular channel graphs we frequently assume that $p_j = p$ for $j \neq 0$ and $j \neq n$. The blocking probability of a channel graph is the probability that every path joining s and t contains at least one occupied link. A well-known problem is to calculate or estimate the blocking probability. In this paper, we state a more general question: How many disjoint paths with all idle links do exist in a channel graph? Our main goal is to find a limit distribution of the capacity C, i.e. the number of such paths as $m \to \infty$, $p \to 0$ and possibly $n \to \infty$.

The first part of the book of Harms *et al.* (1995) is devoted to problems of channel graphs. Most definitions and properties of such graphs are adopted from this book.

3. Estimation and Bounds for the Reliability of Channel Graphs

The following definitions are mainly from Fu and Koutras (1995), Koutras *et al.* (1995; 1996). In those papers, the authors give several new bounds for the reliability of a coherent system in a case of independent but not necessarily identical components. In this section, we apply those results to a system determined by channel graphs. It is obvious that the system described by a channel graph functions if and only if its capacity C is positive.

Write $R = \Pr(C > 0)$. Let $\mathcal{P}$ be the set of all paths from the source s to the terminal t. The set $\mathcal{P}$ is also the set of all minimal paths in the sense of reliability theory (Barlow and Proshan, 1975).

Koutras and Papastavridis (1993) and Koutras *et al.* (1995) present the following definitions and theorems. Let

$$\mu(\mathcal{P}) = \min \{|P_j|, \ 1 \le j \le M\}$$

and

$$v(\mathcal{P}) = \max_{1 \le i \le M} \left|\{P_j \in \mathcal{P} : P_j \cap P_i \neq \emptyset\}\right|$$

where $|P_j|$ denotes the cardinality of P_j. Let $p = \max p_i$ and $p_A = \prod_{i \in A} p_i$.

Theorem 1. *If*

$$\lambda = \sum_{j=1}^{M} p_{P_j}$$

then

$$\left|\Pr(C = 0) - e^{-\lambda}\right| \le \left(1 - e^{-\lambda}\right) \left(v(\mathcal{P}) p^{\mu(\mathcal{P})} + (v(\mathcal{P}) - 1)p\right) \tag{1}$$

For some ordering $\mathcal{P} = (P_1, \ldots, P_M)$ Fu and Koutras (1995) define sets K_j^* in the following way:

$$K_1^* = \emptyset, \quad K_j^* = \{i : P_i \cap P_j \neq \emptyset, \ 1 \le i < j\}, \quad j = 2, 3, \ldots, M \tag{2}$$

For every nonempty K_j^*, let $K_j \subseteq E$ be an index set such that

$$K_j \cap P_i \neq \emptyset \text{ for every } i \in K_j^*, \quad K_j \cap P_j = \emptyset$$

and set $K_j = \emptyset$ if $K_j^* = \emptyset$.

Consider a sequence of systems $(E_r, \mathcal{P}_r)$ with reliabilities R_r and $P_j^{(r)}$, $K_j^{(r)}$, $p_e^{(r)}$, $M^{(r)}$, respectively. Koutras *et al.* (1995) proved the following limit theorem.

Theorem 2. *If the sequence of* $(E_r, \mathcal{P}_r)$ *is not a parallel-series and*

$$\lim_{r\to\infty} \sum_{j=1}^{M^{(r)}} \prod_{e\in P_j^{(r)}} p_e^{(r)} = \lambda^{(r)}\big(1+o(1)\big) \tag{3}$$

$$\lim_{r\to\infty} \sum_{j=1}^{M^{(r)}} \prod_{e\in P_j^{(r)}} p_e^{(r)} \prod_{e\in K_j^{(r)}} q_e^{(r)} = \lambda^{(r)}\big(1+o(1)\big) \tag{4}$$

then $R_r = (1-e^{-\lambda})(1+o(1))$ *as* $r\to\infty$.

Immediately from Theorem 2 one can obtain the following easy observation.

Proposition 1. *If the conditions* (3) *and* (4) *are both fulfilled, then a.s. (i.e. with probability tending to 1), idle elements form parallel paths, i.e. paths without common elements.*

Remark 1. Fu and Koutras (1995) assume that $\lambda^{(r)} \to \lambda = \text{const}$, but it is easy to see that their proof remains true in the case stated in Theorem 2.

Now, we apply the above theorems to channel graphs. At first we determine the quantities needed in Theorem 1. It is obvious that $\mu(\mathcal{P}) = n+1$, $p = \max_{0\le i\le n} p_i$ and

$$p_{P_j} = \prod_{i=0}^{n} p_i$$

for every path P_j. Then write $p_{st} = p_{P_j}$.

The exact computation of $v(\mathcal{P})$ is more complicated, but in fact, an upper bound on this quantity is sufficient. Denoting by n_{ij} the number of paths which match the path P_i on the link e_j going from V_j to V_{j+1}, we have the inequality

$$v(\mathcal{P}) \le \sum_{j=1}^{n} n_{ij} \tag{5}$$

By induction it is easy to check that the number of all paths from s to t is equal to

$$M = m_1 k_1' k_2' \cdots k_{n-1}' = k_2'' k_3'' \cdots k_n'' m_n \tag{6}$$

In the case of completely regular channel graphs we have that the number of all such paths is equal to mk^{n-1}.

Let e_s be a source link (a link from s to a node in V_1). Then using (6) there are $k_1' \cdots k_{n-1}'$ paths matched on the link e_s. Similarly, if e_t is a terminal link (a link from a node in V_n to t), then there are $k_2'' \cdots k_n''$ such paths. For any other link e_j joining stage V_j with V_{j+1}, there are $k_2'' \cdots k_j'' k_{j+1}' \cdots k_{n-1}''$ such paths. Hence, from (5) we obtain

$$v(\mathcal{P}) \leq k_1' \cdots k_{n-1}' + k_2'' \cdots k_n'' + \sum_{j=2}^{n-2} k_2'' \cdots k_j'' k_{j+1}' \cdots k_{n-1}'' = \bar{v}(\mathcal{P}) \qquad (7)$$

and for a completely regular channel graph

$$v(\mathcal{P}) \leq 2k^{n-1} + (n-3)k^{n-2} \leq nk^{n-1}$$

From Theorem 1 and (7) we immediately obtain

Corollary 1. *If M is given by* (6), $p = \max_{0 \leq i \leq n} p_i$,

$$p_{st} = \prod_{i=0}^{n} p_i$$

and $\lambda = Mp_{st}$, *then*

$$\left|\Pr(C=0) - e^{-\lambda}\right| \leq 2\left(1-e^{-\lambda}\right)\left(k_1' \cdots k_{n-1}' + k_2'' \cdots k_n'' + \sum_{j=2}^{n-2} k_2'' \cdots k_j'' k_{j+1}' \cdots k_{n-1}''\right) p$$

If a channel graph is completely regular, then from Corollary 1 we have

Corollary 2. *If* $\lambda = mk^{n-1}p^{n+1}$, *then*

$$\left|\Pr(C=0) - e^{-\lambda}\right| \leq 2\left(1-e^{-\lambda}\right) npk^{n-1} \qquad (8)$$

In the sequel we need an estimation of the size of K_j. Note, however, (Koutras *et al.*, 1995) that it is possible to choose K_j so that

$$|K_j| \leq \left|\{i \,:\, 1 \leq i \leq j-1 \text{ and } P_i \cap P_j \neq \emptyset\}\right| \leq v(\mathcal{P}) \qquad (9)$$

Comparing Theorem 2, Corollary 2 and (9), we obtain

Corollary 3. *For completely regular channel graphs, if* $np^{(r)}\left(k^{(r)}\right)^{n-1} \to 0$, *then*

$$\Pr(C=0) = e^{-\lambda^{(r)}}\left(1+o(1)\right)$$

as $r \to \infty$.

4. Maximal Capacity in Regular Channel Graphs

In this section we investigate an algorithm finding a lower bound on the capacity in a channel graph being some subgraph of a regular channel graph, and next, the capacity of such a subgraph. This algorithm is a specialization of the well-known one based on the max-flow, min-cut theorem, originally stated by Ford and Fulkerson (see e.g. Gibbons, 1985, Ch.4). The capacity of a channel graph is a maximal flow in a network on a channel graph with the capacity of idle links equal to 1. In our case the considered subgraph of the regular channel graph is the subgraph $G_I = (V, E_I)$ formed by idle links.

Algorithm:

Input: channel graph G_I,
Output: flow f, lower bound C^* on capacity C and capacity C.

1. **Init.** Set $f(e) = 0$ for all $e \in E_I$. Set $C^* = 0$.

2. **Find.** Find a path from s to t which does not contain links e such that $f(e) = 1$, set $f(e) = 1$ for all links in this path and increase C^* by 1. If such a path does not exist, go to procedure **Enlarge** else repeat the procedure.

3. **Enlarge.** Set $C = C^*$. Find a chain from s to t such that for a forward e in the chain, $f(e) = 0$ and $f(e) = 1$ for a reverse e. Exchange 0 with 1 in the chain. Increment C by 1. If such a chain does not exist, return C and f else repeat the procedure.

The quantity C^* obtained as a result of the procedure **Find** is a lower bound on the capacity C.

The procedures of finding a path in the step **Find** and a chain in the step **Enlarge** are based on the well-known DFS algorithm (Gibbons, 1985, p.20). In both these procedures we assume that a new node is always chosen equally likely from possible ones.

Let $f(P) = 1$ if and only if $f(e) = 1$ for all $e \in P$, where $P \in \mathcal{P}$. Define $I_P = f(P)$, where f is determined by Step 2. Obviously,

$$C^* = \sum_{P \in \mathcal{P}} I_P$$

Hence $C^* \leq C$.

Let A_{ij} be an event such that the path P_i was chosen as the j-th one in Step 2. Then we have the following simple observation:

Lemma 1. *For regular channel graphs* $\Pr(A_{ij}) = \Pr(A_{ik})$ *for every* i *and all pairs* (i, k).

5. Limit Distribution of Capacity

The method used in the section is widely presented in the book of Barbour *et al.* (1992). Denote by $\mathcal{L}(X)$ the distribution of a random variable X (a random vector X or a family of random variables X). Let $\mathrm{Po}(\lambda)$ denote the Poisson distribution with the expectation λ. Write

$$d_{TV}(\mathcal{L}(X), \mathrm{Po}(\lambda)) = \sup_{A \subseteq \{0,1,\dots\}} \left| \Pr(X \in A) - \Pr(Y \in A) \right| \tag{10}$$

where $\mathcal{L}(Y) = \mathrm{Po}(\lambda)$. If $d_{TV}(\mathcal{L}(X), \mathrm{Po}(\lambda)) \to 0$, we say that X is Poisson convergent.

Barbour *et al.* (1992) give as Theorem 2.A the following result:

Theorem 3. *Suppose that*

$$W = \sum_{\alpha \in \Gamma} I_\alpha$$

where I_α's are indicator random variables with expectations π_α, and suppose that, for each $\alpha \in \Gamma$, random variables U_α and V_α can be constructed on a common probability space, in such a way that

$$\mathcal{L}(U_\alpha) = \mathcal{L}(W), \quad \mathcal{L}(1 + V_\alpha) = \mathcal{L}(W | I_\alpha = 1) \tag{11}$$

Then

$$d_{TV}\big(\mathcal{L}(W), \mathrm{Po}(\lambda)\big) \le \frac{1 - e^{-\lambda}}{\lambda} \sum_{\alpha \in \Gamma} \pi_\alpha \,\mathrm{E}\,|U_\alpha - V_\alpha| \tag{12}$$

where $\lambda = \sum_{\alpha \in \Gamma} \pi_\alpha$.

We apply the above theorem to show that C has approximately the Poisson distribution.

Theorem 4. *The capacity C of a channel graph is Poisson convergent if $\min_{1 \le i \le n} m_i \to \infty$ in such a way that*

$$M^{(r)} p_{st}^{(r)} = \lambda^{(r)} \big(1 + o(1)\big) \tag{13}$$

$$M^{(r)} p_{st}^{(r)} \left(1 - p^{(r)}\right)^{\bar{v}(\mathcal{P})} = \lambda^{(r)} \big(1 + o(1)\big) \tag{14}$$

$$p_{st}^{(r)} \lambda^{(r)} \to 0 \tag{15}$$

where $p^{(r)} = \max_{0 \le i \le n} p_i$ and $\bar{v}(\mathcal{P})$ is given by the right-hand side of (7).

Proof. In the case of regular channel graphs, let $\mathcal{P}$ be a set of indices. Then

$$C^* = \sum_{P \in \mathcal{P}} I_P$$

is the number of disjoint paths, i.e. a capacity. Set $f(e) = 1$ for each e in a fixed path P and suppose that the next paths are chosen by Step 2 in the Algorithm. Let V_P be the number of such paths (without path P). Therefore (11) is fulfilled and the formula (12) has the form

$$d_{TV}\big(\mathcal{L}(C^*), \mathrm{Po}(\lambda)\big) \le \frac{1-e^{-\lambda}}{\lambda} \sum_{P \in \mathcal{P}} \pi_P \, \mathrm{E}\,|C^* - V_P| \tag{16}$$

So we have to approximate the quantities $\lambda = \mathrm{E}\,C^*$, $\mathrm{E}\,|C^* - V_P|$ and $\pi_P = \sum_{e \in P} p_e$.

Since $V_P \ge C^*$ and $V_P - C^* \le 1$, it follows that $\mathrm{E}\,|C^* - V_P| = \pi_P = p_{st}$. From (16) we have

$$d_{TV}\big(\mathcal{L}(C^*), \mathrm{Po}(\lambda)\big) \le \frac{1-e^{-\lambda}}{\lambda} \sum_{P \in \mathcal{P}} \pi_P^2 = \frac{1-e^{-\lambda}}{\lambda} M \pi_P^2 \sim \frac{1-e^{-\lambda}}{\lambda} p_{st} \lambda \to 0$$

Furthermore, from Proposition 1 we have $C = C^*$, a.s. which completes the proof. ■

6. Numerical Results

In this section we give some exact and simulated results. We restrict ourselves to complete regular channel graphs and $n = 2$. Simulations were performed using the algorithm of Section 4. All computer programs were written in Turbo Pascal.[1]

The exact results are obtained from the formula given in (Harms *et al.*, 1995, p.10). We give this formula in a slightly different form. Let S denote any state in the stage of edges E containing $|S|$ idle links. If we denote by $x_j(S)$ the number of the nodes in V_1 connected to the j-th node in V_2, and connected to the source by idle links in S, then we have

$$R = \sum_S p_s^{|S|} (1-p_s)^{|V_1| - |S|} \left(1 - \prod_j \left(1 - (1-p)^{x_j(S)} \right) p_t \right)$$

where $p = p_1$.

In Tables 1 and 2 the reliabilities R for some triples (p_s, p, p_t), successive k_i, $m = 8$ and $m = 16$ are given. For comparison, the values of probabilities $p(\lambda) = 1 - e^{-\lambda}$ are added to the tables, for small p_s, p and p_t, $\lambda = mkp_s pp_t$.

In Tables 3 and 4 we give results for a channel graph with a random structure, i.e. every channel graph with fixed sizes of stages $m = 20$ and $m = 50$ and fixed

[1]Source codes of the programs may be obtained via e-mail.

Table 1. Values of R and $p(\lambda)$ for $m = 8$.

p_s	p	p_t	$k=2$		$k=3$		$k=4$	
			R	$p(\lambda)$	R	$p(\lambda)$	R	$p(\lambda)$
0.1	0.1	0.1	0.0157	0.0159	0.0233	0.0237	0.0307	0.0315
0.1	0.1	0.2	0.0311	0.0315	0.0457	0.0469	0.0598	0.0620
0.1	0.1	0.3	0.0462	0.0469	0.0673	0.0695	0.0874	0.0915
0.1	0.2	0.1	0.0310	0.0315	0.0453	0.0469	0.0590	0.0620
0.1	0.2	0.2	0.0606	0.0620	0.0874	0.0915	0.1122	0.1201
0.1	0.2	0.3	0.0889	0.0915	0.1263	0.1341	0.1601	0.1747
0.1	0.3	0.1	0.0457	0.0469	0.0661	0.0695	0.0852	0.0915
0.1	0.3	0.2	0.0885	0.0915	0.1253	0.1341	0.1583	0.1747
0.1	0.3	0.3	0.1284	0.1341	0.1782	0.1943	0.2211	0.2502
0.2	0.1	0.1	0.0311	0.0315	0.0457	0.0469	0.0598	0.0620
0.2	0.1	0.2	0.0612	0.0620	0.0890	0.0915	0.1152	0.1201
0.2	0.1	0.3	0.0902	0.0915	0.1299	0.1341	0.1665	0.1747
0.2	0.2	0.1	0.0606	0.0620	0.0874	0.0915	0.1122	0.1201
0.2	0.2	0.2	0.1170	0.1201	0.1657	0.1747	0.2091	0.2259
0.2	0.2	0.3	0.1697	0.1747	0.2359	0.2502	0.2928	0.3189
0.2	0.3	0.1	0.0885	0.0915	0.1253	0.1341	0.1583	0.1747
0.2	0.3	0.2	0.1682	0.1747	0.2322	0.2502	0.2866	0.3189
0.2	0.3	0.3	0.2400	0.2502	0.3232	0.3508	0.3906	0.4379
0.3	0.1	0.1	0.0462	0.0469	0.0673	0.0695	0.0874	0.0915
0.3	0.1	0.2	0.0902	0.0915	0.1299	0.1341	0.1665	0.1747
0.3	0.1	0.3	0.1321	0.1341	0.1879	0.1943	0.2382	0.2502
0.3	0.2	0.1	0.0889	0.0915	0.1263	0.1341	0.1601	0.1747
0.3	0.2	0.2	0.1697	0.1747	0.2359	0.2502	0.2928	0.3189
0.3	0.2	0.3	0.2432	0.2502	0.3307	0.3508	0.4028	0.4379
0.3	0.3	0.1	0.1284	0.1341	0.1782	0.1943	0.2211	0.2502
0.3	0.3	0.2	0.2400	0.2502	0.3232	0.3508	0.3906	0.4379
0.3	0.3	0.3	0.3370	0.3508	0.4410	0.4769	0.5203	0.5785

Table 2. Values of R and $p(\lambda)$ for $m = 16$.

p_s	p	p_t	$k=2$		$k=3$		$k=4$	
			R	$p(\lambda)$	R	$p(\lambda)$	R	$p(\lambda)$
0.1	0.1	0.1	0.0312	0.0315	0.0461	0.0469	0.0605	0.0620
0.1	0.1	0.2	0.0613	0.0620	0.0894	0.0915	0.1160	0.1201
0.1	0.1	0.3	0.0902	0.0915	0.1301	0.1341	0.1671	0.1747
0.1	0.2	0.1	0.0610	0.0620	0.0886	0.0915	0.1145	0.1201
0.1	0.2	0.2	0.1175	0.1201	0.1671	0.1747	0.2117	0.2259
0.1	0.2	0.3	0.1698	0.1747	0.2367	0.2502	0.2945	0.3189
0.1	0.3	0.1	0.0894	0.0915	0.1279	0.1341	0.1631	0.1747
0.1	0.3	0.2	0.1691	0.1747	0.2349	0.2502	0.2915	0.3189
0.1	0.3	0.3	0.2403	0.2502	0.3247	0.3508	0.3933	0.4379
0.2	0.1	0.1	0.0613	0.0620	0.0894	0.0915	0.1160	0.1201
0.2	0.1	0.2	0.1186	0.1201	0.1701	0.1747	0.2171	0.2259
0.2	0.1	0.3	0.1722	0.1747	0.2429	0.2502	0.3053	0.3189
0.2	0.2	0.1	0.1175	0.1201	0.1671	0.1747	0.2117	0.2259
0.2	0.2	0.2	0.2204	0.2259	0.3039	0.3189	0.3744	0.4007
0.2	0.2	0.3	0.3106	0.3189	0.4161	0.4379	0.4999	0.5361
0.2	0.3	0.1	0.1691	0.1747	0.2349	0.2502	0.2915	0.3189
0.2	0.3	0.2	0.3081	0.3189	0.4104	0.4379	0.4911	0.5361
0.2	0.3	0.3	0.4225	0.4379	0.5420	0.5785	0.6287	0.6840
0.3	0.1	0.1	0.0902	0.0915	0.1301	0.1341	0.1671	0.1747
0.3	0.1	0.2	0.1722	0.1747	0.2429	0.2502	0.3053	0.3189
0.3	0.1	0.3	0.2467	0.2502	0.3405	0.3508	0.4197	0.4379
0.3	0.2	0.1	0.1698	0.1747	0.2367	0.2502	0.2945	0.3189
0.3	0.2	0.2	0.3106	0.3189	0.4161	0.4379	0.4999	0.5361
0.3	0.2	0.3	0.4272	0.4379	0.5521	0.5785	0.6434	0.6840
0.3	0.3	0.1	0.2403	0.2502	0.3247	0.3508	0.3933	0.4379
0.3	0.3	0.2	0.4225	0.4379	0.5420	0.5785	0.6287	0.6840
0.3	0.3	0.3	0.5605	0.5785	0.6875	0.7264	0.7698	0.8224

Table 3. Values of $\overline{C^*}$, $\overline{C}$ and $\overline{N}$ for $m = 20$, $k = 4$ and 8.

p_s	p	p_t	$k = 4$				$k = 8$			
			$\overline{C^*}$	$\overline{C}$	$\overline{N}$	λ	$\overline{C^*}$	$\overline{C}$	$\overline{N}$	λ
0.1	0.1	0.1	0.09	0.09	0.00	0.08	0.15	0.15	0.00	0.16
0.1	0.1	0.2	0.18	0.18	0.00	0.16	0.28	0.29	0.01	0.32
0.1	0.1	0.3	0.21	0.21	0.00	0.24	0.41	0.42	0.01	0.48
0.1	0.2	0.1	0.16	0.16	0.01	0.16	0.33	0.35	0.02	0.32
0.1	0.2	0.2	0.28	0.28	0.00	0.32	0.61	0.67	0.06	0.64
0.1	0.2	0.3	0.41	0.43	0.02	0.48	0.77	0.87	0.10	0.96
0.1	0.3	0.1	0.21	0.22	0.02	0.24	0.44	0.48	0.04	0.48
0.1	0.3	0.2	0.44	0.48	0.04	0.48	0.80	0.93	0.13	0.96
0.1	0.3	0.3	0.64	0.68	0.05	0.72	1.14	1.29	0.15	1.44
0.2	0.1	0.1	0.16	0.16	0.00	0.16	0.29	0.30	0.01	0.32
0.2	0.1	0.2	0.31	0.31	0.00	0.32	0.58	0.60	0.02	0.64
0.2	0.1	0.3	0.48	0.49	0.01	0.48	0.81	0.84	0.03	0.96
0.2	0.2	0.1	0.30	0.30	0.00	0.32	0.60	0.65	0.05	0.64
0.2	0.2	0.2	0.63	0.66	0.03	0.64	1.13	1.23	0.10	1.28
0.2	0.2	0.3	0.89	0.95	0.05	0.96	1.59	1.75	0.17	1.92
0.2	0.3	0.1	0.46	0.48	0.02	0.48	0.90	0.99	0.09	0.96
0.2	0.3	0.2	0.93	1.01	0.07	0.96	1.48	1.74	0.26	1.92
0.2	0.3	0.3	1.37	1.51	0.14	1.44	2.10	2.41	0.30	2.88
0.3	0.1	0.1	0.25	0.25	0.00	0.24	0.47	0.48	0.01	0.48
0.3	0.1	0.2	0.42	0.42	0.01	0.48	0.88	0.92	0.04	0.96
0.3	0.1	0.3	0.68	0.69	0.01	0.72	1.33	1.38	0.04	1.44
0.3	0.2	0.1	0.48	0.49	0.01	0.48	0.86	0.94	0.08	0.96
0.3	0.2	0.2	0.87	0.90	0.03	0.96	1.62	1.78	0.16	1.92
0.3	0.2	0.3	1.26	1.32	0.06	1.44	2.41	2.59	0.18	2.88
0.3	0.3	0.1	0.72	0.75	0.03	0.72	1.23	1.32	0.09	1.44
0.3	0.3	0.2	1.24	1.36	0.12	1.44	2.33	2.55	0.22	2.88
0.3	0.3	0.3	1.87	2.03	0.17	2.16	3.13	3.50	0.37	4.32

Table 4. Values of $\overline{C^*}$, $\overline{C}$ and $\overline{N}$ for $m = 50$, $k = 4$ and 8.

p_s	p	p_t	$k = 4$				$k = 8$			
			$\overline{C^*}$	$\overline{C}$	$\overline{N}$	λ	$\overline{C^*}$	$\overline{C}$	$\overline{N}$	λ
0.1	0.1	0.1	0.19	0.19	0.00	0.20	0.37	0.39	0.02	0.40
0.1	0.1	0.2	0.42	0.43	0.01	0.40	0.79	0.86	0.07	0.80
0.1	0.1	0.3	0.60	0.62	0.02	0.60	1.07	1.15	0.08	1.20
0.1	0.2	0.1	0.40	0.43	0.03	0.40	0.78	0.90	0.12	0.80
0.1	0.2	0.2	0.79	0.85	0.07	0.80	1.36	1.67	0.31	1.60
0.1	0.2	0.3	1.15	1.24	0.09	1.20	1.91	2.34	0.43	2.40
0.1	0.3	0.1	0.58	0.67	0.09	0.60	1.11	1.38	0.27	1.20
0.1	0.3	0.2	1.07	1.32	0.25	1.20	1.91	2.52	0.61	2.40
0.1	0.3	0.3	1.55	1.89	0.34	1.80	2.64	3.37	0.73	3.60
0.2	0.1	0.1	0.37	0.38	0.01	0.40	0.81	0.86	0.05	0.80
0.2	0.1	0.2	0.74	0.75	0.02	0.80	1.44	1.57	0.13	1.60
0.2	0.1	0.3	1.19	1.22	0.03	1.20	2.16	2.34	0.17	2.40
0.2	0.2	0.1	0.78	0.85	0.07	0.80	1.45	1.71	0.26	1.60
0.2	0.2	0.2	1.55	1.69	0.14	1.60	2.92	3.43	0.51	3.20
0.2	0.2	0.3	2.25	2.44	0.19	2.40	4.06	4.79	0.72	4.80
0.2	0.3	0.1	1.10	1.25	0.14	1.20	2.06	2.51	0.44	2.40
0.2	0.3	0.2	2.09	2.47	0.38	2.40	3.92	4.77	0.85	4.80
0.2	0.3	0.3	3.16	3.72	0.56	3.60	5.38	6.40	1.02	7.20
0.3	0.1	0.1	0.54	0.57	0.03	0.60	1.06	1.12	0.07	1.20
0.3	0.1	0.2	1.17	1.21	0.04	1.20	2.24	2.40	0.16	2.40
0.3	0.1	0.3	1.64	1.68	0.04	1.80	3.14	3.38	0.23	3.60
0.3	0.2	0.1	1.20	1.26	0.06	1.20	2.26	2.58	0.32	2.40
0.3	0.2	0.2	2.26	2.45	0.19	2.40	4.07	4.65	0.58	4.80
0.3	0.2	0.3	3.29	3.55	0.27	3.60	5.90	6.62	0.71	7.20
0.3	0.3	0.1	1.65	1.90	0.25	1.80	3.15	3.57	0.42	3.60
0.3	0.3	0.2	3.37	3.80	0.44	3.60	5.74	6.53	0.79	7.20
0.3	0.3	0.3	4.86	5.43	0.57	5.40	8.05	9.07	1.02	10.80

degrees $k = 4$ and $k = 8$ are chosen with the same probability. Next, in such a channel graph we set the links as idle independently, with probabilities p_s, p and p_t. For every triple (p_s, p, p_t) we obtain empirical expectations of $\overline{C^*}$, $\overline{C}$, average $\overline{N}$ and λ of the number of steps in Procedure 3, $\lambda = mkp_s pp_t$.

Table 5 contains the frequencies f_k of the capacity C obtained from computer simulations and the respective probabilities

$$p(k, \lambda) = e^{-\lambda} \frac{\lambda^k}{k!}$$

in the Poisson distribution.

Table 5. Empirical distribution of C and probabilities in Poisson distribution with $\lambda = 1$.

k	$k = 5$, $p_s = 0.2$, $p = 0.1$, $p_t = 0.2$		$k = 20$, $p_s = 0.1$, $p = 0.1$, $p_t = 0.1$		$k = 20$, $p_s = 0.2$, $p = 0.025$, $p_t = 0.2$	
	f_k	$p(\lambda, k)$	f_k	$p(\lambda, k)$	f_k	$p(\lambda, k)$
0	0.37	0.37	0.42	0.37	0.38	0.37
1	0.36	0.37	0.37	0.37	0.17	0.18
2	0.19	0.18	0.15	0.18	0.02	0.02
3	0.05	0.06	0.05	0.06	0.00	0.00
4	0.02	0.02	0.02	0.02	0.00	0.00
5	0.00	0.00	0.00	0.00	0.00	0.00

We choose the parameters of channel graphs such that $\lambda = 1$ for $m = 50$. If

$$k = 5, \quad p_s = 0.2, \quad p = 0.1, \quad p_t = 0.2$$

then the conformity between the simulated and theoretical probabilities is good. Note that in this case the relation (14) is almost fulfilled. If

$$k = 20, \quad p_s = 0.2, \quad p = 0.025, \quad p_t = 0.2$$

then the conformity between the simulated and theoretical probabilities is significantly worse. In this case the relation (14) is far to be fulfilled. Finally, note that if

$$k = 20, \quad p_s = 0.1, \quad p = 0.1, \quad p_t = 0.1$$

then the conformity is again very good. However, this conformity does not follow from Theorem 4.

In all the simulations we have used a random number generator based on the method given in (Marsaglia *et al.*, 1990). Our implementation is based on the program originally written in the C language (Wieczorkowski and Zieliński, 1997).

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Received: 18 August 1998
Revised: 4 Februar 1999
Re-revised: 2 April 1999