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VARIATIONAL ANALYSIS OF A FRICTIONAL CONTACT PROBLEM FOR THE BINGHAM FLUID

BASSAM AWBI*, LYNDA SELMANI**
MIRCEA SOFONEA*

We consider a mathematical model which describes the flow of a Bingham fluid with friction. We assume a stationary flow and we model the contact with damped response and a local version of Coulomb's law of friction. The problem leads to a quasi-variational inequality for the velocity field. We establish the existence of a weak solution and, under additional assumptions, its uniqueness. The proofs are based on a new result obtained in (Motreanu and Sofonea, 1999). We also establish the continuous dependence of the solution with respect to the contact conditions.

Keywords: Bingham fluid, damped response, Coulomb's friction law, quasi-variational inequality, weak solution.

1. Introduction

The constitutive law of the Bingham fluid has been used in various publications in order to model the flow of metals in a die. Such situations abound in industry, for example the wire-drawing process. Because of the importance of this process, a considerable effort has been made in its modeling and numerical simulations, and the engineering literature concerning this topic is rather extensive, see e.g. (Cristescu, 1976; 1980) and references therein.

Variational analysis including the existence of weak solutions to evolution or stationary problems involving the Bingham model can be found in (Duvaut and Lions, 1970; 1972), in the case of adherence boundary conditions and in (Ionescu, 1985; Ionescu and Sofonea, 1993) in the case of contact conditions with non-local friction. The blocking phenomenon in the study of the Bingham model was analyzed in (Ionescu and Sofonea, 1986) while a numerical analysis of problems with the Bingham fluid can be found in (Fortin, 1972; Glowinski *et al.*, 1976) and references therein. A practical application of the Bingham model in the study of wire-drawing through a conical die was considered in (Ionescu and Vernescu, 1988). There, numerical computations were provided, using a velocity variational formulation and the finite-element method.

* Laboratoire de Théorie des Systèmes, Université de Perpignan, 52 Avenue de Villeneuve, 66860 Perpignan Cedex, France, e-mail: {awbi,sofonea}@univ-perp.fr.

** Départment des Mathématiques, Université de Sétif, Route de Scipion, 1900 Sétif, Algeria.

The stationary flow of a Bingham fluid in the presence of a deformable obstacle was recently considered in (Awbi *et al.*, 1999). There, a general relation between the normal stress and the normal velocity on the contact boundary, governed by a non-negative function p_ν, was assumed. The friction was modelled by a non-local version of Coulomb's law, involving a friction bound function p_τ and a smoothing operator. The introduction of this non-local smoothing operator was mode for technical reasons, since the trace of the stress tensor on the boundary is too rough. A velocity variational formulation for the process was derived and the existence of a weak solution was established by using classical results for elliptic variational inequalities and fixed-point arguments.

The present paper parallels (Awbi *et al.*, 1999). Here we consider the same physical setting and we use the same normal contact condition in order to model the reaction of the obstacle to penetration. The novelty consists in the fact that here we model the friction by a local version of Coulomb's law which avoids the use of smoothing operators. We derive a variational formulation of the problem for which we obtain an existence result. We also provide conditions on the contact functions p_ν and p_τ in order to have the uniqueness of the solution and its Lipschitz continuous dependence with respect to the data. The proofs use a different functional method and are based on a recent result on quasi-variational inequalities obtained in (Motreanu and Sofonea, 1999). We complete our results with the study of the dependence of the solution on the contact boundary conditions.

The paper is structured as follows. In Section 2, the mechanical problem is stated and the frictional contact conditions are discussed. In Section 3, we propose a variational formulation of the model and state our main existence and uniqueness result, Theorem 1. The proof is established in Section 5, and it is based on an abstract result that we recall in Section 4. In Section 6, we study the dependence of the weak solution on the contact boundary conditions and we establish a convergence result. Finally, in Section 7, we present some concluding remarks.

2. Problem Statement

In this section, we describe a model for the process and we discuss the boundary conditions. The physical setting is the following. We consider the flow of a Bingham fluid in a domain $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ with a regular boundary Γ. We assume that Γ is divided into three disjoint measurable parts Γ_1, Γ_2 and Γ_3 such that $\operatorname{meas}\Gamma_1 > 0$. The fluid is supposed to be incompressible and the velocity is known on Γ_1. Given body forces of density $\boldsymbol{f}_0$ are acting on Ω and given surface tractions of density $\boldsymbol{f}_2$ are acting on Γ_2. On Γ_3 we impose frictional contact conditions which we now describe. We assume that the normal stress σ_ν satisfies a general damped response condition of the form

$$-\sigma_\nu = p_\nu(u_\nu) \tag{1}$$

where p_ν is a prescribed nonnegative function and u_ν represents the normal velocity. As a concrete example, we may consider

$$p_\nu(r) = k\, r_+ \tag{2}$$

where $k \geq 0$ and $r_+ = \max\{0, r\}$. This condition shows that the contact pressure is proportional to the normal velocity, but only under compression. It was already considered in (Rochdi *et al.*, 1998a), in the study of a quasi-static viscoelastic contact problem with friction.

The tangential stress $\boldsymbol{\sigma}_\tau$ is related to the tangential velocity $\boldsymbol{u}_\tau$ by the relation

$$\begin{cases} |\boldsymbol{\sigma}_\tau| \leq p_\tau(u_\nu) \\ |\boldsymbol{\sigma}_\tau| < p_\tau(u_\nu) \Rightarrow \boldsymbol{u}_\tau = 0 \\ |\boldsymbol{\sigma}_\tau| = p_\tau(u_\nu) \Rightarrow \boldsymbol{\sigma}_\tau = -\lambda \boldsymbol{u}_\tau, \quad \lambda \geq 0 \end{cases} \tag{3}$$

This represents a version of Coulomb's law of friction in which p_τ is a non-negative function, the so-called *friction bound.* The friction law (3) was used in various papers to study quasi-static elastic or viscoelastic problems, see e.g. (Rochdi *et al.*, 1998b). It states that the tangential shear cannot exceed the maximal frictional resistance p_τ. When inequality holds, there is adherence of the fluid on Γ_3 and the fluid is in the so-called stick state. When equality holds, there is relative sliding, the so-called slip state. A choice of the function p_τ is given by

$$p_\tau = \mu p_\nu \tag{4}$$

where $\mu \geq 0$ is a coeficient of friction. Plugging (4) in (3) leads to the classical Coulomb law of friction, used e.g. in (Duvaut and Lions, 1972). Recently, a new version of Coulomb's law was derived in (Strömberg, 1995; Strömberg *et al.*, 1996), based on thermodynamic considerations. It consists in using in (3) the friction bound

$$p_\tau = \mu p_\nu (1 - \alpha p_\nu)_+ \tag{5}$$

where α is a small positive coefficient related to the wear and hardness of the contact surface.

Throughout the paper the indices i and j run from 1 to d, the summation convention over repeated indices is implied and the index that follows a comma indicates a partial derivative. We use S_d to represent the space of second-order symmetric tensors on $\mathbb{R}^d$ or, equivalently, the space of symmetric matrices of order d. We define the inner product and corresponding norm on $\mathbb{R}^d$ and S_d by

$$\boldsymbol{u} \cdot \boldsymbol{v} = u_i v_i, \qquad |\boldsymbol{v}| = (\boldsymbol{v} \cdot \boldsymbol{v})^{\frac{1}{2}} \quad \forall \boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^d$$

$$\boldsymbol{\sigma} \cdot \boldsymbol{\tau} = \sigma_{ij} \tau_{ij}, \qquad |\boldsymbol{\tau}| = (\boldsymbol{\tau} \cdot \boldsymbol{\tau})^{\frac{1}{2}} \quad \forall \boldsymbol{\sigma}, \boldsymbol{\tau} \in S_d$$

We denote by $\boldsymbol{u} = (u_i) : \Omega \longrightarrow \mathbb{R}^d$ the velocity field and by $\boldsymbol{\sigma} = (\sigma_{ij}) : \Omega \longrightarrow S_d$ the stress field. The constitutive law of the Bingham fluid is given by

$$\left.\begin{array}{ll} \boldsymbol{\sigma}' = 2\eta D(\boldsymbol{u}) + g \dfrac{D(\boldsymbol{u})}{|D(\boldsymbol{u})|} & \text{if } |D(\boldsymbol{u})| \neq 0 \\ |\boldsymbol{\sigma}'| \leq g & \text{if } |D(\boldsymbol{u})| = 0 \end{array}\right\} \tag{6}$$

Here $\eta > 0$ is the coefficient of viscosity, $g \geq 0$ is the yield limit, $D(\boldsymbol{u})$ denotes the rate deformation tensor defined by

$$D(\boldsymbol{u}) = \big(D_{ij}(\boldsymbol{u})\big), \quad D_{ij}(\boldsymbol{u}) = \frac{1}{2}(u_{i,j} + u_{j,i})$$

and $\boldsymbol{\sigma}'$ is the deviator of $\boldsymbol{\sigma}$ given by

$$\boldsymbol{\sigma}' = (\sigma'_{ij}), \quad \sigma'_{ij} = \sigma_{ij} - \frac{\sigma_{kk}}{3}\,\delta_{ij}$$

The Bingham model (6) has been considered by many authors in order to model real materials like pastes, metals, etc. Various details and mechanical interpretation on this model can be found e.g. in (Cristescu and Suliciu, 1982).

With these assumption, the classical formulation of the mechanical problem is the following.

Problem $\boldsymbol{P}$: Find a velocity field $\boldsymbol{u} = (u_i) : \Omega \longrightarrow \mathbb{R}^d$ and a stress field $\boldsymbol{\sigma} : \Omega \longrightarrow S_d$ such that

$$\left.\begin{array}{ll} \boldsymbol{\sigma}' = 2\eta D(\boldsymbol{u}) + g\,\dfrac{D(\boldsymbol{u})}{|D(\boldsymbol{u})|} & \text{if } |D(\boldsymbol{u})| \neq 0 \\ |\boldsymbol{\sigma}'| \leq g & \text{if } |D(\boldsymbol{u})| = 0 \end{array}\right\} \text{ in } \Omega \qquad (7)$$

$$\operatorname{Div}\boldsymbol{\sigma} + \boldsymbol{f}_0 = 0 \quad \text{in } \Omega \qquad (8)$$

$$\operatorname{div}\boldsymbol{u} = 0 \quad \text{in } \Omega \qquad (9)$$

$$\boldsymbol{u} = \boldsymbol{U} \quad \text{on } \Gamma_1 \qquad (10)$$

$$\boldsymbol{\sigma\nu} = \boldsymbol{f}_2 \quad \text{on } \Gamma_2 \qquad (11)$$

$$\left.\begin{array}{l} -\sigma_\nu = p_\nu(u_\nu), \quad |\boldsymbol{\sigma}_\tau| \leq p_\tau(u_\nu) \\ |\boldsymbol{\sigma}_\tau| < p_\tau(u_\nu) \Rightarrow \boldsymbol{u}_\tau = 0 \\ |\boldsymbol{\sigma}_\tau| = p_\tau(u_\nu) \Rightarrow \boldsymbol{\sigma}_\tau = -\lambda \boldsymbol{u}_\tau, \quad \lambda \geq 0 \end{array}\right\} \text{ on } \Gamma_3 \qquad (12)$$

Here (8) represents the equilibrium equation in which $\operatorname{Div}\boldsymbol{\sigma} = (\sigma_{ij,j})$ and (9) is the incompressibility condition in which $\operatorname{div}\boldsymbol{u} = u_{i,i}$. Finally, (10) is the velocity condition in which $\boldsymbol{U}$ is given and (11) represents the traction boundary condition.

3. Variational Formulation

In this section, we set the mechanical problem (7)–(12) into a variational formulation, list the assumption on the data and state our main existence and uniqueness result.

To this end, we need some additional notation. We use the classical notation for L^p and Sobolev spaces. If X represents a real Hilbert space endowed with the inner product $\langle\cdot,\cdot\rangle_X$ and norm $|\cdot|_X$, we denote by X^d and $X_s^{d\times d}$ the spaces

$$X^d = \{\boldsymbol{x} = (x_i) \mid x_i \in X, \quad i = 1, 2, \ldots, d\}$$

$$X_s^{d\times d} = \{\boldsymbol{x} = (x_{ij}) \mid x_{ij} = x_{ji} \in X, \quad i, j = 1, 2, \ldots, d\}$$

endowed with the inner products

$$\langle \boldsymbol{x}, \boldsymbol{y}\rangle_{X^d} = \langle x_i, y_i\rangle_X, \quad \langle \boldsymbol{x}, \boldsymbol{y}\rangle_{X_s^{d\times d}} = \langle x_{ij}, y_{ij}\rangle_X$$

respectively. The associated norms on X^d and $X_s^{d\times d}$ will be denoted by $|\cdot|_{X^d}$ and $|\cdot|_{X_s^{d\times d}}$, respectively.

For every vector field $\boldsymbol{v} \in H^1(\Omega)^d$, we write $\boldsymbol{v}$ for the trace of $\boldsymbol{v}$ on Γ and use the symbols v_ν and $\boldsymbol{v}_\tau$ to denote the normal and tangential components of $\boldsymbol{v}$ on the boundary given by

$$v_\nu = \boldsymbol{v}\cdot\boldsymbol{\nu}, \quad \boldsymbol{v}_\tau = \boldsymbol{v} - v_\nu\boldsymbol{\nu}$$

respectively. We use the notation div and D for the divergence and rate deformation operators, respectively, defined by

$$\operatorname{div}\boldsymbol{v} = (v_{i,i})$$

$$D(\boldsymbol{v}) = \big(D_{ij}(\boldsymbol{v})\big), \quad D_{ij}(\boldsymbol{v}) = \frac{1}{2}(v_{i,j} + v_{j,i})$$

for all $\boldsymbol{v} \in H^1(\Omega)^d$.

We consider in the sequel the closed subspace of $H^1(\Omega)^d$ given by

$$V = \{\ \boldsymbol{v} \in H^1(\Omega)^d \mid \operatorname{div}\boldsymbol{v} = 0 \ \text{ in } \ \Omega, \quad \boldsymbol{v} = 0 \text{ on } \ \Gamma_1\ \}$$

Since meas $\Gamma_1 > 0$, Korn's inequality holds:

$$|\boldsymbol{v}|_{H^1(\Omega)^d} \le C_k\big|D(\boldsymbol{v})\big|_{L^2(\Omega)_s^{d\times d}} \quad \forall v \in V \tag{13}$$

where C_k is a strictly positive constant depending on Ω and Γ_1. A proof of Korn's inequality can be found in (Nečas and Hlaváček, 1981, p.79). From (13) we see that

$$\langle \boldsymbol{u}, \boldsymbol{v}\rangle_V = \big\langle D(\boldsymbol{u}), D(\boldsymbol{v})\big\rangle_{L^2(\Omega)_s^{d\times d}} \tag{14}$$

is an inner product on V and the corresponding norm $|\cdot|_V$ is equivalent to the norm $|\cdot|_{H^1(\Omega)^d}$ on V. Moreover, by the Sobolev trace theorem, (13) and (14), we have a constant C_0 depending only on the domain Ω, Γ_1 and Γ_3 such that

$$|\boldsymbol{v}|_{L^2(\Gamma_3)^d} \le C_0|\boldsymbol{v}|_V \quad \forall \boldsymbol{v} \in V \tag{15}$$

We define the set of admissible kinematic velocity fields to be

$$K = \{\ \boldsymbol{v} \in H^1(\Omega)^d \mid \operatorname{div}\boldsymbol{v} = 0 \ \text{ in } \ \Omega, \quad \boldsymbol{v} = \boldsymbol{U} \ \text{ on } \ \Gamma_1\ \}$$

We suppose that the boundary data $\boldsymbol{U}$ is such that

$$K \neq \emptyset \tag{16}$$

and the forces and tractions have respectively the regularity

$$\boldsymbol{f}_0 \in L^2(\Omega)^d, \qquad \boldsymbol{f}_2 \in L^2(\Gamma_3)^d \tag{17}$$

We also assume that the contact functions $p_r : \Gamma_3 \times \mathbb{R} \longrightarrow \mathbb{R}_+$ $(r = \nu, \tau)$ satisfy the following conditions:

$$\left\{\begin{array}{ll} \text{(a)} & \text{there exists } L_r > 0 \text{ such that} \\ & \big|p_r(\boldsymbol{x}, u_1) - p_r(\boldsymbol{x}, u_2)\big| \leq L_r|u_1 - u_2| \ \ \forall u_1, u_2 \in \mathbb{R}, \ \text{ a.e. } \boldsymbol{x} \in \Gamma_3; \\ \text{(b)} & \boldsymbol{x} \mapsto p_r(\boldsymbol{x}, u) \ \text{ is Lebesgue measurable on } \Gamma_3, \text{ for all } u \in \mathbb{R}; \\ \text{(c)} & \boldsymbol{x} \mapsto p_r(\boldsymbol{x}, u) = 0 \quad \text{if } \ u \leq 0, \ \text{ a.e. } \boldsymbol{x} \in \Gamma_3 \end{array}\right. \tag{18}$$

We observe that the assumptions (18) on the functions p_ν and p_τ are pretty general. The most severe restriction comes from condition (a) which requires the functions to grow at most linearly. Certainly, the function defined in (2) satisfies condition (18a). We also observe that if the functions p_ν and p_τ are related by (4) or (5) and p_ν satisfies condition (18a), then p_τ also satisfies condition (18a) with $L_\tau = \mu L_\nu$. Condition (18c) shows that when there is no compression (i.e. $u_\nu \leq 0$), then the tractions vanish ($\sigma_\nu = 0$, $\boldsymbol{\sigma}_\tau = \boldsymbol{0}$). Clearly, this condition is satisfied for the function given by (2). We also observe that if the functions p_ν and p_τ are related by (4) or (5) and p_ν satisfies condition (18c), then p_τ also satisfies condition (18c).

Next, we introduce the following notation:

$$a : H^1(\Omega)^d \times H^1(\Omega)^d \longrightarrow \mathbb{R}\,, \quad a(\boldsymbol{u}, \boldsymbol{v}) = 2\eta \int_\Omega D(\boldsymbol{u}) \cdot D(\boldsymbol{v})\, \mathrm{d}x \tag{19}$$

$$\varphi : H^1(\Omega)^d \longrightarrow \mathbb{R}\,, \quad \varphi(\boldsymbol{v}) = g \int_\Omega \big|D(\boldsymbol{v})\big|\, \mathrm{d}x \tag{20}$$

$$\left\{\begin{array}{l} \phi : H^1(\Omega)^d \times H^1(\Omega)^d \longrightarrow \mathbb{R}\,, \\ \phi(\boldsymbol{u}, \boldsymbol{v}) = \displaystyle\int_{\Gamma_3} p_\nu\,(u_\nu)|v_\nu|\, \mathrm{d}a + \int_{\Gamma_3} p_\tau\,(u_\nu)|\boldsymbol{v}_\tau|\, \mathrm{d}a \end{array}\right. \tag{21}$$

$$F : H^1(\Omega)^d \longrightarrow \mathbb{R}\,, \quad F(\boldsymbol{v}) = \int_\Omega \boldsymbol{f}_0 \cdot \boldsymbol{v}\, \mathrm{d}x + \int_{\Gamma_2} \boldsymbol{f}_2 \cdot \boldsymbol{v}\, \mathrm{d}a \tag{22}$$

where da represents a surface element. Using conditions (18) and the regularities (17), it follows that the integrals in (21) and (22) are well-defined.

Lemma 1. *Let* (16)–(18) *hold. If* $\{u, \sigma\}$ *are suficiently regular functions satisfying* (7)–(12)*, then*

$$u \in K, \quad a(u, v-u) + \varphi(v) - \varphi(u) + \phi(u, v) - \phi(u, u) \geq F(v-u) \quad \forall v \in K \tag{23}$$

Proof. Let $\{u, \sigma\}$ be smooth functions satisfying (7)–(12) and let $v \in K$. Using (9) and (7), we have

$$\int_\Omega \sigma \cdot \Big(D(v) - D(u)\Big)\, dx \leq a(u, v-u) + \varphi(v) - \varphi(u) \tag{24}$$

Moreover, integrating by parts and using (8), (10)–(11) yields

$$\int_\Omega \sigma \cdot \Big(D(v) - D(u)\Big)\, dx = \int_\Omega f_0 \cdot (v-u)\, dx + \int_{\Gamma_2} f_2 \cdot (v-u)\, da + \int_{\Gamma_3} \sigma\nu \cdot (v-u)\, da$$

and by (22) it follows that

$$\int_\Omega \sigma \cdot \Big(D(v) - D(u)\Big)\, dx = F(v-u) + \int_{\Gamma_3} \sigma\nu \cdot (v-u)\, da \tag{25}$$

Using now the contact conditions (12) we obtain

$$\int_{\Gamma_3} \sigma\nu \cdot (v-u)\, da \geq \int_{\Gamma_3} p_\nu(u_\nu)\big(|u_\nu| - |v_\nu|\big)\, da + \int_{\Gamma_3} p_\tau(u_\nu)\big(|u_\tau| - |v_\tau|\big)\, da$$

and by (21) we find

$$\int_{\Gamma_3} \sigma\nu \cdot (v-u)\, da \geq \phi(u, u) - \phi(u, v) \tag{26}$$

Inequality (23) is now a consequence of (24)–(26). ■

Lemma 1 leads to the following variational formulation of the mechanical Problem $\boldsymbol{P}$:

Problem $\boldsymbol{Q}$: Find a velocity field $u : \Omega \longrightarrow \mathbb{R}^d$ such that u is a solution to the variational inequality (23).

Our main existence and uniqueness result which we establish in the next section, is as follows:

Theorem 1. *Let* (16)–(18) *hold. Then:*
(1) Problem $\boldsymbol{Q}$ *has at least a solution* $u \in K$.

(2) There exist $L_0 > 0$ depending only on Ω, Γ_1, Γ_3 and η such that if $L_\nu + L_\tau < L_0$ then Problem $\boldsymbol{Q}$ has a unique solution $\boldsymbol{u} \in K$ and the map $(\boldsymbol{f}_0, \boldsymbol{f}_2) \longmapsto \boldsymbol{u}$ is Lipschitz continuous from $L^2(\Omega)^d \times L^2(\Gamma_3)^d$ to $H^1(\Omega)^d$.

We conclude that, under the assumptions of Theorem 1, there exists a velocity field $\boldsymbol{u}$ which can be interpreted as a weak solution to the mechanical problem (7)–(12). Moreover, by Theorem 1 (2) it follows that this solution is unique if the Lipschitz constants L_ν and L_τ are small enough. As it follows from the proof of this theorem, the critical value L_0 depends neither on the material constant g nor on the external forces $\boldsymbol{f}_0$ and $\boldsymbol{f}_2$, nor on the imposed velocity $\boldsymbol{U}$.

The proof of Theorem 1 can be obtained in much the same way as in (Ionescu, 1985; Ionescu and Sofonea, 1993), based on abstract results on Ky-Fan's inequality. It can also be proved using the Schauder-Tychonoff fixed-point theorem and the method employed in (Awbi *et al.*, 1999). Here we present a different functional argument in solving the variational Problem $\boldsymbol{Q}$, based on a recent abstract existence and uniqueness result for elliptic quasi-variational inequalities that we present in the next section.

4. An Abstract Existence and Uniqueness Result

Throughout this section V will represent a real Hilbert space endowed with the inner product $\langle\cdot,\cdot\rangle_V$ and the associated norm $|\cdot|_V$. We denote by '$\rightharpoonup$' weak convergence in V. Let $A : V \longrightarrow V$ be a non-linear operator, $j : V \times V \longrightarrow \mathbb{R}$ and $f \in V$. With these data we consider the following quasi-variational inequality: Find $\boldsymbol{u} \in V$ such that

$$\langle A\boldsymbol{u}, \boldsymbol{v} - \boldsymbol{u}\rangle_V + j(\boldsymbol{u}, \boldsymbol{v}) - j(\boldsymbol{u}, \boldsymbol{u}) \geq \langle f, \boldsymbol{v} - \boldsymbol{u}\rangle_V \quad \forall \boldsymbol{v} \in V \tag{27}$$

In order to solve (27), we assume that A is strongly monotone and Lipschitz contiunous, i.e.

$$\left\{\begin{array}{ll} \text{(a)} & \text{There exists } m > 0 \text{ such that} \\ & \langle A\boldsymbol{u} - A\boldsymbol{v}, \boldsymbol{u} - \boldsymbol{v}\rangle_V \geq m|\boldsymbol{u} - \boldsymbol{v}|_V^2 \quad \forall \boldsymbol{u}, \boldsymbol{v} \in V \\ \text{(b)} & \text{There exists } M > 0 \text{ such that} \\ & |A\boldsymbol{u} - A\boldsymbol{v}|_V \leq M|\boldsymbol{u} - \boldsymbol{v}|_V \quad \forall \boldsymbol{u}, \boldsymbol{v} \in V \end{array}\right. \tag{28}$$

The function j fulfils the requirement

$$j(\boldsymbol{\xi}, \cdot) : \boldsymbol{v} \longrightarrow \mathbb{R} \text{ is a convex functional on } V, \text{ for all } \boldsymbol{\xi} \in V \tag{29}$$

Keeping in mind (29), it is well-known that the directional derivative

$$j_2'(\boldsymbol{\xi}, \boldsymbol{u}; \boldsymbol{v}) = \lim_{\lambda \downarrow 0} \frac{1}{\lambda}\Big[j(\boldsymbol{\xi}, \boldsymbol{u} + \lambda\boldsymbol{v}) - j(\boldsymbol{\xi}, \boldsymbol{u})\Big] \quad \forall \boldsymbol{\xi}, \, \boldsymbol{u}, \, \boldsymbol{v} \in V \tag{30}$$

exists. We now formulate additional conditions on the function j:

$$\begin{cases} \text{For every sequence } \{\boldsymbol{u}_n\} \subset V \text{ with } |\boldsymbol{u}_n|_V \to \infty \\ \text{and every sequence } \{t_n\} \subset [0,1], \text{ one has} \\ \liminf\limits_{n\to\infty} \left[\dfrac{1}{|\boldsymbol{u}_n|_V^2} j_2'(t_n \boldsymbol{u}_n, \boldsymbol{u}_n; -\boldsymbol{u}_n) \right] < m \end{cases} \tag{31}$$

$$\begin{cases} \text{For every sequence } \{\boldsymbol{u}_n\} \subset V \text{ with } |\boldsymbol{u}_n|_V \to \infty \\ \text{and every bounded sequence } \{\boldsymbol{\xi}_n\} \subset V, \text{ one has} \\ \liminf\limits_{n\to\infty} \left[\dfrac{1}{|\boldsymbol{u}_n|_V^2} j_2'(\boldsymbol{\xi}_n, \boldsymbol{u}_n; -\boldsymbol{u}_n) \right] < m \end{cases} \tag{32}$$

$$\begin{cases} \text{For all sequences } \{\boldsymbol{u}_n\} \subset V \text{ and } \{\boldsymbol{\xi}_n\} \subset V \text{ such that } \boldsymbol{u}_n \rightharpoonup \boldsymbol{u} \in V, \\ \boldsymbol{\xi}_n \rightharpoonup \boldsymbol{\xi} \in V \text{ and for every } \boldsymbol{v} \in V, \text{ one has} \\ \limsup\limits_{n\to\infty} \left[j(\boldsymbol{\xi}_n, \boldsymbol{v}) - j(\boldsymbol{\xi}_n, \boldsymbol{u}_n) \right] \le j(\boldsymbol{\xi}, \boldsymbol{v}) - j(\boldsymbol{\xi}, \boldsymbol{u}) \end{cases} \tag{33}$$

$$\begin{cases} j(\boldsymbol{u},\boldsymbol{v}) - j(\boldsymbol{u},\boldsymbol{u}) + j(\boldsymbol{v},\boldsymbol{u}) - j(\boldsymbol{v},\boldsymbol{v}) \le \beta |\boldsymbol{u} - \boldsymbol{v}|_V^2 \quad \forall \boldsymbol{u}, \boldsymbol{v} \in V, \\ \text{for some } \beta \in \mathbb{R} \text{ with } \beta < m \end{cases} \tag{34}$$

In the study of the quasi-variational inequality (27) we have the following result:

Theorem 2. *Let* (28)–(29) *hold. Then:*

(1) Under assumptions (31)–(33) *there exists a least one element* $\boldsymbol{u} \in V$ *which solves* (27).

(2) Under assumptions (31)–(34), *problem* (27) *has a unique solution* $\boldsymbol{u} = \boldsymbol{u}_f$ *which depends Lipschitz continuously on* f *with the Lipschitz constant* $(m - \beta)^{-1}$.

Theorem 2 has been obtained recently (Motreanu and Sofonea, 1999) and therefore we do not provide here the details of the proof. We just specify that the proof is obtained in several steps and it is based on standard arguments of elliptic variational inequalities and topological degree theory. A trait of novelty of Theorem 2 consists, to the best of our knowledge, in considering conditions (31) and (32), formulated in terms of the directional derivative of the functional j.

5. Proof of Theorem 1

We turn now to the proof of Theorem 1 which will be carried out in several steps. We assume in the sequel that (16)–(18) hold. In the first step we shall obtain homogenuous boundary conditions. To this end, let $\boldsymbol{u}_0 \in K$ (see (16)). Since

$$K = V + \boldsymbol{u}_0 \tag{35}$$

it is easy to see that $\boldsymbol{u}$ is a solution of the variational inequality (23) if and only if $\tilde{\boldsymbol{u}} = \boldsymbol{u} - \boldsymbol{u}_0$ is a solution of the variational inequality

$$\begin{aligned}\tilde{\boldsymbol{u}} \in V, \quad & a(\tilde{\boldsymbol{u}} + \boldsymbol{u}_0, \boldsymbol{v} - \tilde{\boldsymbol{u}}) + \varphi(\boldsymbol{v} + \boldsymbol{u}_0) - \varphi(\tilde{\boldsymbol{u}} + \boldsymbol{u}_0) \\ & + \phi(\tilde{\boldsymbol{u}} + \boldsymbol{u}_0, \boldsymbol{v} + \boldsymbol{u}_0) - \phi(\tilde{\boldsymbol{u}} + \boldsymbol{u}_0, \tilde{\boldsymbol{u}} + \boldsymbol{u}_0) \geq F(\boldsymbol{v} - \tilde{\boldsymbol{u}}) \quad \forall \tilde{\boldsymbol{u}} \in V \end{aligned} \tag{36}$$

Moreover, using the Riesz representation theorem, we may consider the operator $A : V \longrightarrow V$ defined by

$$\langle A\boldsymbol{u}, \boldsymbol{v}\rangle_V = a(\boldsymbol{u} + \boldsymbol{u}_0, \boldsymbol{v}) \quad \forall \boldsymbol{u}, \boldsymbol{v} \in V \tag{37}$$

Let now $j : V \times V \longrightarrow V$ be the functional given by

$$j(\boldsymbol{u}, \boldsymbol{v}) = \varphi(\boldsymbol{v} + \boldsymbol{u}_0) + \phi(\boldsymbol{u} + \boldsymbol{u}_0, \boldsymbol{v} + \boldsymbol{u}_0) \quad \forall \boldsymbol{u}, \boldsymbol{v} \in V \tag{38}$$

Next, keeping in mind (17), it follows that F is a continuous linear functional on V. Therefore, by using again the Riesz representation theorem, there exists $\boldsymbol{f} \in V$ such that

$$\langle \boldsymbol{f}, \boldsymbol{v}\rangle_V = F(\boldsymbol{v}) \quad \forall \boldsymbol{v} \in V \tag{39}$$

Summarizing, from (35)–(39) we find the following result:

Lemma 2. *The element $\boldsymbol{u} \in K$ is a solution to Problem $\boldsymbol{Q}$ if and only if the element $\tilde{\boldsymbol{u}} = \boldsymbol{u} - \boldsymbol{u}_0$ is a solution to the quasi-variational inequality*

$$\tilde{\boldsymbol{u}} \in V, \quad \langle A\tilde{\boldsymbol{u}}, \boldsymbol{v} - \tilde{\boldsymbol{u}}\rangle_V + j(\tilde{\boldsymbol{u}}, \boldsymbol{v}) - j(\tilde{\boldsymbol{u}}, \tilde{\boldsymbol{u}}) \geq \langle \boldsymbol{f}, \boldsymbol{v} - \tilde{\boldsymbol{u}}\rangle_V \quad \forall \boldsymbol{v} \in V \tag{40}$$

In the next step we investigate the properties of the functional j defined by (38). Clearly, j satisfies (29). Moreover, we have the following results:

Lemma 3. *The functional j satisfies assumptions* (31) *and* (32).

Proof. Let $\boldsymbol{\xi}, \boldsymbol{u} \in V$ and $\lambda \in]0,1]$. Using (38), (20) and (21), after some algebra we obtain

$$\begin{aligned} j(\boldsymbol{\xi}, \boldsymbol{u} - \lambda\boldsymbol{u}) - j(\boldsymbol{\xi}, \boldsymbol{u}) \leq\ & \lambda g \int_\Omega |D(\boldsymbol{u}_0)| \mathrm{d}x - \lambda g \int_\Omega |D(\boldsymbol{u} + \boldsymbol{u}_0)| \mathrm{d}x \\ & + \lambda \int_{\Gamma_3} p_\nu(\xi_\nu + u_{0\nu})|u_{0\nu}| \, \mathrm{d}a - \lambda \int_{\Gamma_3} p_\nu(\xi_\nu + u_{0\nu})|u_\nu + u_{0\nu}| \, \mathrm{d}a \\ & + \lambda \int_{\Gamma_3} p_\tau(\xi_\nu + u_{0\nu})|\boldsymbol{u}_{0\tau}| \, \mathrm{d}a - \lambda \int_{\Gamma_3} p_\tau(\xi_\nu + u_{0\nu})|\boldsymbol{u}_\tau + \boldsymbol{u}_{0\tau}| \, \mathrm{d}a \end{aligned}$$

and, since $g \geq 0$, $p_\nu, p_\tau \geq 0$ a.e. on Γ_3, we deduce that

$$j(\boldsymbol{\xi}, \boldsymbol{u} - \lambda \boldsymbol{u}) - j(\boldsymbol{\xi}, \boldsymbol{u})$$
$$\leq \lambda g \int_\Omega |D(\boldsymbol{u}_0)| \mathrm{d}x + \lambda \int_{\Gamma_3} p_\nu(\xi_\nu + u_{0\nu})|u_{0\nu}| \,\mathrm{d}a + \lambda \int_{\Gamma_3} p_\tau(\xi_\nu + u_{0\nu})|\boldsymbol{u}_{0\tau}| \,\mathrm{d}a$$

Using now (30), (18) and (15) in the previous inequality, it follows that

$$j_2'(\boldsymbol{\xi}, \boldsymbol{u}; -\boldsymbol{u}) \leq g \int_\Omega |D(\boldsymbol{u}_0)| \mathrm{d}x + C_0^2(L_\nu + L_\tau)|\boldsymbol{\xi} + \boldsymbol{u}_0|_V |\boldsymbol{u}_0|_V \quad \forall \boldsymbol{\xi},\ \boldsymbol{u} \in V \tag{41}$$

Lemma 3 is now a consequence of (41). ■

Lemma 4. *The functional j satisfies assumption* (33).

Proof. Let $\{\boldsymbol{u}_n\} \subset V$ and $\{\boldsymbol{\xi}_n\} \subset V$ be two sequences such that $\boldsymbol{u}_n \rightharpoonup \boldsymbol{u} \in V$, $\boldsymbol{\xi}_n \rightharpoonup \boldsymbol{\xi} \in V$, and let $\boldsymbol{v} \in V$. From the compactness property of the trace map and (18) it follows that

$$p_r(\xi_{n\nu} + u_{0\nu}) \longrightarrow p_r(\xi_\nu + u_{0\nu}) \ \text{ in } \ L^2(\Gamma_3) \ \ (r = \nu, \tau)$$

$$u_{n\nu} \longrightarrow u_\nu, \quad |\boldsymbol{u}_{n\tau}| \longrightarrow |\boldsymbol{u}_\tau| \ \text{ in } \ L^2(\Gamma_3)$$

Therefore, by (21) we obtain

$$\phi(\boldsymbol{\xi}_n + \boldsymbol{u}_0, \boldsymbol{v} + \boldsymbol{u}_0) \longrightarrow \phi(\boldsymbol{\xi} + \boldsymbol{u}_0, \boldsymbol{v} + \boldsymbol{u}_0) \tag{42}$$

$$\phi(\boldsymbol{\xi}_n + \boldsymbol{u}_0, \boldsymbol{u}_n + \boldsymbol{u}_0) \longrightarrow \phi(\boldsymbol{\xi} + \boldsymbol{u}_0, \boldsymbol{u} + \boldsymbol{u}_0) \tag{43}$$

Moreover, since the functional φ defined by (20) is a convex lower semi-continuous functional, we find

$$\liminf_n \varphi(\boldsymbol{u}_n + \boldsymbol{u}_0) \geq \varphi(\boldsymbol{u} + \boldsymbol{u}_0) \tag{44}$$

Condition (33) results now from (38) and (42)–(44). ■

Lemma 5. *The functional j satisfies the inequality*

$$j(\boldsymbol{u}, \boldsymbol{v}) - j(\boldsymbol{u}, \boldsymbol{u}) + j(\boldsymbol{v}, \boldsymbol{u}) - j(\boldsymbol{v}, \boldsymbol{v}) \leq C_0^2(L_\nu + L_\tau)|\boldsymbol{u} - \boldsymbol{v}|_V^2 \quad \forall \boldsymbol{u}, \boldsymbol{v} \in V \tag{45}$$

Proof. Let $u, v \in V$. Using (38), (20) and (21), we get

$$
\begin{aligned}
j(u,v) &- j(u,u) + j(v,u) - j(v,v) \\
&= \int_{\Gamma_3} \left(p_\nu(u_\nu + u_{0\nu}) - p_\nu(v_\nu + u_{0\nu})\right)\left(|v_\nu + u_{0\nu}| - |u_\nu + u_{0\nu}|\right) da \\
&\quad + \int_{\Gamma_3} \left(p_\tau(u_\nu + u_{0\nu}) - p_\tau(v_\nu + u_{0\nu})\right)\left(|v_\tau + u_{0\nu}| - |u_\tau + u_{0\nu}|\right) da \\
&\le L_\nu \int_{\Gamma_3} |u_\nu - v_\nu|^2\, da + L_\tau \int_{\Gamma_3} |u_\nu - v_\nu|\ |v_\tau - u_\tau|\, da \le (L_\nu + L_\tau) \int_{\Gamma_3} |u - v|^2\, da
\end{aligned}
$$

Using now (15) in the previous inequality, we obtain (45). ■

We have now all the ingredients to prove the theorem.

Proof of Theorem 1.

(1) Using (37), (19) and (14) we find

$$\langle Au - Av, u - v\rangle_V = 2\eta|u - v|_V^2, \quad \langle Au - Av, w\rangle_V = 2\eta\langle u - v, w\rangle_V$$

for all u, v, $w \in V$. From these inequalities it is straightforward to see that the operator A satisfies conditions (28) with $m = M = 2\eta$. We recall that the functional j given by (38) satisfies (29) and by Lemmas 3 and 4 it follows that j also satisfies conditions (31)–(33). Therefore, using Theorem 2 (1) we deduce that the quasi-variational inequality (40) has at least one solution $\tilde{u} \in V$. The existence part of Theorem 1 follows now from Lemma 2.

(2) Let $L_0 = 2\eta/C_0^2$. Clearly, L_0 depends only on Ω, Γ_1, Γ_3 and η. Let now assume that $L_\nu + L_\tau < L_0$. Then there exists $\beta \in \mathbb{R}$ such that $C_0^2(L_\nu + L_\tau) < \beta < 2\eta$. Using (45), we obtain

$$j(u,v) - j(u,u) + j(v,u) - j(v,v) \le \beta|u - v|_V^2 \quad \forall u, v \in V$$

We conclude that the functional j satisfies condition (34) and, using Theorem 2 (2), we deduce that (40) has a unique solution $\tilde{u} \in V$ which depends Lipschitz continuously on f. The uniqueness part of Theorem 1 follows now from Lemma 2. Using now (39) and (22) we see that the map $(f_0, f_2) \longmapsto f$ is Lipschitz continuous from $L^2(\Omega)^d \times L^2(\Gamma_3)^d$ to V. Therefore, since the maps $f \longmapsto \tilde{u} : V \longrightarrow V$ and $\tilde{u} \longmapsto u = \tilde{u} - u_0 : V \longrightarrow H^1(\Omega)^d$ are Lipschitz continuous, we find that $(f_0, f_2) \longmapsto u : L^2(\Omega)^d \times L^2(\Gamma_3)^d \longrightarrow H^1(\Omega)^d$ is also a Lipschitz continuous map, which concludes the proof. ■

6. A Continuous Dependence Result

In this section, we study the dependence of the solution to Problem Q on the perturbations of the contact functions p_ν and p_τ. To this end, let us suppose in the sequel that (16)–(18) hold and $L_\nu + L_\tau < L_0$. Using Theorem 1 we deduce that Problem $\boldsymbol{Q}$ has a unique solution $\boldsymbol{u} \in K$. Next, for every $\alpha \geq 0$, let p_r^α be a perturbation of p_r which satisfies (18) with Lipschitz constant L_r^α $(r = \nu, \tau)$. Let also introduce the functionals ϕ^α which are obtained by replacing p_ν and p_τ by p_ν^α and p_τ^α in (21).

We consider now the following problem:

Problem $\boldsymbol{Q}^\alpha$: For $\alpha \geq 0$, find a velocity field $\boldsymbol{u}^\alpha : \Omega \longrightarrow \mathbb{R}^d$ such that

$$\boldsymbol{u}^\alpha \in K, \qquad a(\boldsymbol{u}^\alpha, \boldsymbol{v} - \boldsymbol{u}^\alpha) + \varphi(\boldsymbol{v}) - \varphi(\boldsymbol{u}^\alpha) + \phi^\alpha(\boldsymbol{u}^\alpha, \boldsymbol{v}) - \phi^\alpha(\boldsymbol{u}^\alpha, \boldsymbol{u}^\alpha) \geq F(\boldsymbol{v} - \boldsymbol{u}^\alpha) \quad \forall \boldsymbol{v} \in K \tag{46}$$

Clearly, $\boldsymbol{Q}^\alpha$ represents the variational formulation of the problem (7)–(12) in which p_ν and p_τ are replaced by p_ν^α and p_τ^α, respectively.

We suppose now that the contact functions satisfy the following assumption: There exists $\delta > 0$ and $\theta_r : \mathbb{R}_+ \longrightarrow \mathbb{R}$ $(r = \nu, \tau)$ such that

$$\begin{cases} \text{(a)} & \left|p_r^\alpha(\boldsymbol{x}, u) - p_r(\boldsymbol{x}, u)\right| \leq \theta_r(\alpha)|u| \quad \forall u \in \mathbb{R}, \text{ a.e. } \boldsymbol{x} \in \Gamma_3 \\ \text{(b)} & \lim\limits_{\alpha \to 0} \theta_r(\alpha) = 0 \\ \text{(c)} & L_\nu^\alpha + L_\tau^\alpha + \delta < L_0 \end{cases} \tag{47}$$

for all $\alpha \geq 0$. Under this assumption, using again Theorem 1 we deduce that for each $\alpha \geq 0$, Problem $\boldsymbol{Q}^\alpha$ has a unique solution $\boldsymbol{u}^\alpha \in K$. Moreover, we have the following convergence result:

Theorem 3. *The solution $\boldsymbol{u}^\alpha$ to Problem $\boldsymbol{Q}^\alpha$ converges in $H^1(\Omega)^d$ to the solution $\boldsymbol{u}$ to Problem $\boldsymbol{Q}$ as $\alpha \to 0$.*

In addition to the mathematical interest in this result, it is of importance in applications, as it indicates that small inaccuracies in the contact conditions lead to small inaccuracies in the solutions.

Proof of Theorem 3. In what follows, we denote by C a strictly positive constant which may depend on Ω, Γ_1, Γ_3, η and $\boldsymbol{u}$, but does not depend on α, and whose value may change from line to line. Let $\alpha \geq 0$. Taking $\boldsymbol{v} = \boldsymbol{u}$ in (46), $\boldsymbol{v} = \boldsymbol{u}^\alpha$ in (23) and adding the resulting inequalities yields

$$a(\boldsymbol{u}^\alpha - \boldsymbol{u}, \boldsymbol{u}^\alpha - \boldsymbol{u}) \leq \phi^\alpha(\boldsymbol{u}^\alpha, \boldsymbol{u}) - \phi^\alpha(\boldsymbol{u}^\alpha, \boldsymbol{u}^\alpha) + \phi(\boldsymbol{u}, \boldsymbol{u}^\alpha) - \phi(\boldsymbol{u}, \boldsymbol{u}) \tag{48}$$

Using (21) we have

$$\phi^\alpha(\boldsymbol{u}^\alpha, \boldsymbol{u}) - \phi^\alpha(\boldsymbol{u}^\alpha, \boldsymbol{u}^\alpha) + \phi(\boldsymbol{u}, \boldsymbol{u}^\alpha) - \phi(\boldsymbol{u}, \boldsymbol{u}) \leq \Big(\left|p_\nu^\alpha(u_\nu^\alpha) - p_\nu(u_\nu)\right|_{L^2(\Gamma_3)} + \left|p_\tau^\alpha(u_\nu^\alpha) - p_\tau(u_\nu)\right|_{L^2(\Gamma_3)}\Big)|\boldsymbol{u}^\alpha - \boldsymbol{u}|_{L^2(\Gamma_3)} \tag{49}$$

Now, (47a) implies

$$\left|p_r^\alpha(u_\nu^\alpha) - p_r(u_\nu)\right|_{L^2(\Gamma_3)} \le L_r^\alpha |\boldsymbol{u}^\alpha - \boldsymbol{u}|_{L^2(\Gamma_3)} + \theta_r(\alpha)|\boldsymbol{u}|_{L^2(\Gamma_3)} \quad (r = \nu, \tau)$$

and plugging this inequality in (49) yields

$$\begin{aligned}\phi^\alpha(\boldsymbol{u}^\alpha, \boldsymbol{u}) - \phi^\alpha(\boldsymbol{u}^\alpha, \boldsymbol{u}^\alpha) + \phi(\boldsymbol{u}, \boldsymbol{u}^\alpha) - \phi(\boldsymbol{u}, \boldsymbol{u})& \\ \le \left(L_\nu^\alpha + L_\tau^\alpha\right)|\boldsymbol{u}^\alpha - \boldsymbol{u}|_{L^2(\Gamma_3)}^2 + \left(\theta_\nu(\alpha) + \theta_\tau(\alpha)\right)|\boldsymbol{u}|_{L^2(\Gamma_3)}|\boldsymbol{u}^\alpha - \boldsymbol{u}|_{L^2(\Gamma_3)}&\end{aligned}$$

Using (35), we remark that $\boldsymbol{u}^\alpha - \boldsymbol{u} \in V$. Thus, plugging (47c) and (15) in the previous inequality, we obtain

$$\begin{aligned}\phi^\alpha(\boldsymbol{u}^\alpha, \boldsymbol{u}) - \phi^\alpha(\boldsymbol{u}^\alpha, \boldsymbol{u}^\alpha) + \phi(\boldsymbol{u}, \boldsymbol{u}^\alpha) - \phi(\boldsymbol{u}, \boldsymbol{u})& \\ \le C_0^2(L_0 - \delta)|\boldsymbol{u}^\alpha - \boldsymbol{u}|_V^2 + C_0^2\left(\theta_\nu(\alpha) + \theta_\tau(\alpha)\right)|\boldsymbol{u}|_V|\boldsymbol{u}^\alpha - \boldsymbol{u}|_V&\end{aligned} \tag{50}$$

Moreover, by (19) and (14) we deduce that

$$a(\boldsymbol{u}^\alpha - \boldsymbol{u}, \boldsymbol{u}^\alpha - \boldsymbol{u}) = 2\eta|\boldsymbol{u}^\alpha - \boldsymbol{u}|_V^2 \tag{51}$$

Therefore, from (51), (48) and (50), keeping in mind that $L_0 = 2\eta/C_0^2$, we obtain

$$\delta\,|\boldsymbol{u}^\alpha - \boldsymbol{u}|_V \le \left(\theta_\nu(\alpha) + \theta_\tau(\alpha)\right)|\boldsymbol{u}|_V \tag{52}$$

Theorem 3 follows now from (52), (47b) and Korn's inequality (13).

7. Conclusion

In this paper, we have studied a mathematical model which describes the flow of a Bingham fluid with friction. Such types of problems arise in metal forming, e.g. in the wire-drawing process. The purpose of this paper was to present the variational analysis of the model in order to lay the necessary groundwork for numerical approximations and practical applications. The main novel contribution of this paper consists in considering new contact boundary conditions which lead to a nonstandard mathematical model. Thus, rather than the adherence condition and nonlocal friction laws, we described the contact with a general relation between the normal stress and the normal velocity on the contact boundary and a version of the local Coulomb's law of friction. We presented a velocity variational formulation of the mechanical problem and we obtained a new existence and uniqueness result, Theorem 1. We proved Theorem 1 using as a key argument an abstract result on quasi-variational inequalities obtained recently in (Motreanu and Sofonea, 1999). We also studied the continuous dependence of the solution with respect to the contact boundary conditions and we proved a convergence result, Theorem 3.

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