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A STABILIZING CONTROL LAW FOR INVARIANT SYSTEMS ON LIE GROUPS

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This paper deals with the stabilizability of invariant control systems defined on Lie groups. A stabilization technique is presented which, under certain hypotheses, can lead to a criterion assuring the existence of a feedback controller which steers every initial condition to a specified target point of the state space of these systems.

Keywords: invariant systems, Lie groups, stabilization, feedback controller

1. Introduction

This paper deals with the stabilization of invariant control systems on Lie groups. By stabilizability we mean that a feedback controller which steers every initial state to a specified target point exists. Our aim is to present a stabilization technique, based on an appropriate decomposition of the state space, which can lead to stabilizability criteria. The technique leads to a piecewise constant feedback control law using only a finite set of values of the control parameter. The work is organized as follows. In Section 2, the case where the state space is solvable is examined. In turn, Section 3 deals with the case where the state space is semi-simple. Finally, in Section 4 a general case is treated.

Let G be a real analytic and simply connected Lie group of dimension n and $\text{Lie}(G) = L$ be the corresponding Lie algebra of left invariant vector fields on G. Here G stands for the state space of the systems occuring in the sequel. Consider also the following control system on G:

$$\dot{x} = f(x, u) = f^u(x),$$

where $x \in G$ is the state of the system, and u is the control parameter taking values in a subset U_a of the control space which is an analytic manifold U. Finally, $f\colon G \times U \to TG$ (where TG is the tangent bundle of G) is an analytic mapping. It is noted that the set U_a of the acceptable control values can be much 'smaller' than the control space U, e.g. a discrete submanifold. The system described above is called invariant if the vector fields f^u on G are left invariant for every constant $u \in U_a$. An invariant system on G will be identified with the subset $\Gamma = \{f^u, u \in U_a\}$

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of L, i.e. with the set of the acceptable control directions of the system. It is always assumed that Γ contains the zero vector field. We write $\exp tX$ for the integral curve of $X \in L$ passing through e (the identity element of G) at $t = 0$. The integral curve of X through $x \in G$ is then $x \exp tX$. Furthermore, if $W \subseteq \mathbb{R}$, we set $\exp WX = \bigcup\{\exp tX\colon\ t \in W\}$. A point $y \in G$ is accessible from $x \in G$ for Γ if

$$y = x \exp(t_1 X_1) \cdots \exp(t_k X_k)$$

with $t_i > 0,\ X_i \in \Gamma,\ i = 1, \dots, k$ (the reader can find more details about these Lie theoretic preliminaries in (Helgason, 1978; Varadarajan, 1984).

The Lie algebra of Γ is the smallest Lie subalgebra of L which contains Γ. It is denoted by $\mathrm{Lie}(\Gamma)$. As is well-known, $\mathrm{Lie}(\Gamma)$ is closely related to the controllability properties of the system and, in fact, $\mathrm{Lie}(\Gamma) = L$ is a necessary condition for controllability.

Let us now proceed to the stabilization problem (treatments of various aspects of this problem can be found in (Isidori, 1989; Sontag, 1990; Tsinias, 1989). An invariant system on G is called stabilizable if there exists a feedback control law $u = \varphi(x)$ such that the emerging closed-loop vector field V with

$$V_x = f(x, \varphi(x))$$

steers every initial state to a specified target point (in the sequel, this target point will be the identity element, but a modification of the results that will be presented later covers the general case). Thus stabilization means the construction of a vector field V on G such that for every $x \in G$ there exists some $X \in \Gamma$ with $V_x = X_x$. In the following sections, suitable decompositions of the state space will be incorporated in order to present a construction technique which results in a discontinuous, piecewise-constant stabilizing vector field.

2. The Solvable Case

In this section, a stabilization technique is presented for the case where the state space G is a solvable Lie group. This technique is based on an appropriate decomposition of the state space. However, before the statement of the main result some technical notions and facts, which can also be found in (Varadarajan, 1984), will be briefly stated. Let us first give the definition of a solvable Lie algebra. This definition requires the concept of derived subalgebras. The derived subalgebra DL of a Lie algebra L is defined as $DL = [L, L]$. The k-th derived subalgebra of L is then

$$D^k L = D(D^{k-1} L), \quad D^0 L = L.$$

Definition 1. A Lie algebra L is called solvable if $D^k L = 0$ for some $k \geq 1$.

Observe that, if L is solvable and $D^m L \neq 0$, then $D^{m+1} L \subset D^m L$. Thus $D^k L$ strictly decrease until they become zero. A Lie group G is called solvable if its Lie algebra is solvable. Next, a technical key lemma is stated whose proof is omitted here, since it can be found in (Varadarajan, 1984, Lemma 3.18.5).

Lemma 1. *Let G be a real analytic and simply-connected Lie group. Suppose that $L_1, L_2, \dots, L_r$ are subalgebras of L such that:*

(i) $L = L_1 \oplus \cdots \oplus L_r$, and

(ii) if $W_k = L_1 + \cdots + L_k$, then W_k is a subalgebra of L and an ideal of W_{k+1} for every k.

Let also $G_1, \dots, G_r$ be the respective analytic subgroups corresponding to $L_1, \dots, L_r$. Then the G_i's are all closed and simply connected and the mapping $(x_1, \dots, x_r) \to x_1 \cdots x_r$, $x_i \in G_i$, $i = 1, \dots, r$ is an analytic diffeomorphism of $G_1 \times \cdots \times G_r$ onto G.

From Lemma 1 it follows immediately that every $x \in G$ has a unique expression of the form $x = x_1 \cdots x_r$ where $x_i \in G_i$.

Now the main result, which is essentially a stabilizability criterion for a control system Γ, can be stated and proved. It is shown that under certain hypotheses a stabilizing feedback controller can be constructed for the system Γ. The main condition on the system is that its subsystems $\Gamma \cap D^k L$ defined on the Lie subgroups of G corresponding to the $D^k L$'s have the accessibility property. The exact statement is the following:

Theorem 1. *Let G be a real analytic, simply-connected and solvable Lie group with Lie algebra L and Γ a control system on G. If $\Gamma = -\Gamma$ (symmetry) and* $\mathrm{Lie}(\Gamma \cap D^k L) = D^k L$ *for $k = 0, 1, \dots$, then Γ is stabilizable.*

In the proof of Theorem 1 a preliminary technical lemma is used (see also a similar well-known result in (Varadarajan, 1984, Corollary 3.7.5)). Let L be solvable and $d_k = \dim(D^k L)$. Since the derived subalgebras strictly decrease, there exists a basis $\{X_1, \dots, X_n\}$ of L such that the first d_k vectors constitute a basis of $D^k L$ for every k. For this kind of basis one has the following:

Lemma 2. *Let L be a solvable Lie algebra. Assume that $\{X_1, \dots, X_n\}$ is a basis of L of the form described previously. Then $M_i = \mathrm{span}\{X_1, \dots, X_i\}$ is a subalgebra of L and an ideal of M_{i+1} for $i = 1, \dots, n-1$.*

Proof. By construction of the basis, for every i there exists a maximal k such that $M_i \subseteq D^k L$. M_i is a subalgebra of L since $[M_i, M_i] \subseteq D^{k+1} L \subset M_i$. If $M_i \neq D^k L$, then $M_{i+1} \subseteq D^k L$ and $[M_{i+1}, M_i] \subseteq D^{k+1} L \subset M_i$. If $M_i = D^k L$, then $M_{i+1} \subseteq D^{k-1} L$ ($= L$ if $k = 0$) and $[M_{i+1}, M_i] \subseteq D^k L = M_i$. In both the cases M_i is an ideal of M_{i+1} and the proof is complete. ■

Proof of Theorem 1. We shall first prove that Γ contains a basis of L of the form previously described. Indeed, Γ contains a basis of $D^k L$ for every $k = 0, 1, \dots$, such that $D^k L$ is non-trivial. To see this, let m be the maximal integer such that $D^m L \neq 0$. Since $D^m L$ is abelian and $\mathrm{Lie}(\Gamma \cap D^m L) = D^m L$, it is clear that Γ contains a basis of $D^m L$. Now, if Γ contains a basis of $D^k L$ for some $1 \leq k \leq m$, then Γ also contains a basis of $D^{k-1}L$. If this is not the case, then $\operatorname{span}(\Gamma \cap D^{k-1}L) \neq D^{k-1}L$ and

$$\left[\Gamma \cap D^{k-1}L, \Gamma \cap D^{k-1}L\right] \subseteq D^k L \subseteq \operatorname{span}(\Gamma \cap D^{k-1}L) \neq D^{k-1}L,$$

which contradict the hypothesis $\operatorname{Lie}(\Gamma \cap D^{k-1}L) = D^{k-1}L$. Thus one can choose a basis of $\operatorname{Lie}(G)$ contained in Γ which is of the desired form. Let $\{X_1, \ldots, X_n\}$ be such a basis. Now we are going to construct a feedback controller for Γ which is a piecewise left-invariant vector field V. It will then be proved that V steers every initial state x to e. From Lemma 2 it is clear that Lemma 1 applies. Thus

$$G = G_1 \cdots G_n, \quad G_i = \exp \mathbb{R} X_i.$$

For $k = 1, \ldots, n$ define the following subsets of G:

$$S_k = G_1 \cdots G_k,$$

$$S_k^+ = G_1 \cdots G_{k-1} \exp \mathbb{R}^+ X_k,$$

$$S_k^- = G_1 \cdots G_{k-1} \exp \mathbb{R}^- X_k,$$

$$S_k^0 = G_1 \cdots G_{k-1} = S_{k-1}.$$

Observe that $S_n = G$, $S_1^0 = \{e\}$. Furthermore, the sets $S_1^0, S_1^\pm, S_2^\pm, \ldots, S_n^\pm$ are pairwise disjoint and cover G. On each of these sets define a vector field V as follows:

$$V = \begin{cases} -X_k & \text{on} \quad S_k^+, \\ X_k & \text{on} \quad S_k^-. \end{cases}$$

Since (from symmetry) $-X_i \in \Gamma$ for $i = 1, \ldots, n$. On $S_1^0 = \{e\}$ we naturally define $V = 0$. Also observe that for $x \in S_k$ we have $x \exp tV \in S_k$ for every t such that $-\epsilon < t < \epsilon$ for some $\epsilon > 0$. This fact ensures that V is well-defined as far as the existence and uniqueness of integral curves are concerned. It is now easy to see that V steers every initial state x to e.

Consider any initial $x \in G$. For example, let $x \in S_k^+$. There exists some $t_k > 0$ such that

$$x = x_1 \cdots x_{k-1} \exp t_k X_k$$

for some x_i's in G_i, $i = 1, \ldots, k-1$. Thus the application of $-X_k$ leads the state to the subset S_{k-1} within the finite time t_k, since

$$x \exp t_k(-X_k) = x_1 \cdots x_{k-1} \exp(t_k X_k) \exp(-t_k X_k) = x_1 \cdots x_{k-1} \in S_{k-1}.$$

Inductively, one can see that the state eventually reaches $S_1^0 = \{e\}$ and remains there under the application of the zero vector field. Thus V is a stabilizing feedback controller for Γ and the proof is complete. ■

Remark 1. If a feedback controller which steers every initial state to a point $x \neq e$ is desired, then the stabilizing vector field $V = \mp X_k$ on $K_k^{\pm} = xS_k^{\pm}$.

Remark 2. The stabilizing vector field constructed as before depends on the basis of L which is contained in Γ. Thus the stabilizing feedback controller is not unique and depends on the choice of a particular basis of L contained in Γ.

Remark 3. Since the proposed feedback control law is piecewise constant and incorporates only a finite number of values of the control parameter, it can be used in the case where the control parameter is restricted to belong to a discrete subset of the control space.

Remark 4. As shown in the proof of Theorem 1, the conditions of this theorem imply that $\operatorname{span}(\Gamma) = L$. This is a stronger assumption than the accessibility condition $\operatorname{Lie}(\Gamma) = L$. However, in order to construct the presented stabilizing control law, this assumption is necessary.

Example 1. The following simple and illustrative example will clarify the technique. Let G be the Lie group consisting of the 2×2 real matrices of the form:

$$\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}, \quad a > 0, \quad b \in \mathbb{R}.$$

This is the connected component of $\operatorname{Aff}(1)$, the affine group of the line, containing the identity element. The Lie algebra of G is then

$$L = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}, \quad a, b \in \mathbb{R} \right\}.$$

It is well-known that G is solvable. Consider the following control system on G:

$$\Gamma = \left\{ 0, \pm \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \pm \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right\}.$$

It is clear that Γ is symmetric and contains the basis

$$\left\{ \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right\} = \{X, Y\}$$

of L. Then we have

$$\exp \mathbb{R}X = \left\{ \begin{pmatrix} a & a-1 \\ 0 & 1 \end{pmatrix}, \quad a > 0 \right\},$$

$$\exp \mathbb{R}Y = \left\{ \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}, \quad a \in \mathbb{R} \right\},$$

$$S_2^+ = \left\{ \begin{pmatrix} a & a(b+1)-1 \\ 0 & 1 \end{pmatrix}, \quad a, b > 0 \right\},$$

$$S_2^- = \left\{ \begin{pmatrix} a & a(b+1)-1 \\ 0 & 1 \end{pmatrix}, \quad a > 0, \quad b < 0 \right\},$$

$$S_1^+ = \left\{ \begin{pmatrix} a & a-1, \\ 0 & 1 \end{pmatrix}, \quad a > 1 \right\},$$

$$S_1^- = \left\{ \begin{pmatrix} a & a-1 \\ 0 & 1 \end{pmatrix}, \quad a \in (0,1) \right\}.$$

Thus

$$V = \begin{cases} -\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} & \text{on} \quad S_2^+, \\ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} & \text{on} \quad S_2^-, \\ -\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} & \text{on} \quad S_1^+, \\ \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} & \text{on} \quad S_1^-. \end{cases}$$

3. The Semi-Simple Case

In the previous section it was shown that if the state space G is solvable, a general stabilizability criterion is valid. In this section the case where G is semi-simple and non-compact is examined. Another stabilizability criterion for systems on such a state space will be proved. Some mathematical preliminaries which are also treated in much greater detail in (Helgason, 1978) are given below.

Let L be a Lie algebra. The radical of L, $\mathrm{rad}(L)$, is the maximal solvable ideal of L.

Definition 2. A Lie algebra L is called semi-simple if $\mathrm{rad}(L) = 0$.

A Lie group G is called semi-simple if the corresponding Lie algebra is semi-simple. Let L be a semi-simple Lie algebra and consider the Cartan decomposition $L = L_1 \oplus P$, where L_1 is a compact subalgebra (i.e. there exists a compact Lie group with Lie algebra isomorphic to L_1) of L and P denotes a subspace of L. Assume that A is a maximal Abelian subspace of P (all the subspaces of this kind have the same dimension). An element α belonging to the dual space A^* of A is called a restricted root of (L, A) if

$$L(\alpha) = \{X \in L : \ [Y, X] = \alpha(Y)X \ \text{ for every } \ Y \in A\} \neq 0.$$

Fixing a Weyl chamber, we can order the roots and determine the set of positive roots Σ^+. Then we get the following decomposition of L:

$$L = L_1 \oplus A \oplus L^+,$$

where $L^+ = \text{span}\{L(\alpha), \alpha \in \Sigma^+\}$ and $L_2 = A \oplus L^+$ is solvable. This decomposition of a semi-simple Lie algebra is known as the Iwasawa decomposition.

Theorem 2. *Let G be a real analytic, connected and semi-simple Lie group with Lie algebra L. Consider the Iwasawa decomposition $L = L_1 \oplus A \oplus L^+$. Let H_1, H_A, H^+ be the Lie subgroups of G corresponding to L_1, A, L^+. Then the mapping $(x_1, x_A, x^+) \to x_1 x_A x^+$ is an analytic diffeomorphism of $H_1 \times H_A \times H^+$ onto G. Furthermore, H_A and H^+ are simply connected.*

A proof of this theorem can be found in (Helgason, 1978, Chapter VI, Theorem 5.1). One can express the Iwasawa decomposition in the form

$$L = L_1 \oplus L_2, \quad G = H_1 H_2,$$

where $L_2 = A \oplus L^+$, $H_2 = H_A H^+$. Next we impose the following hypothesis on G:

(H) Let G be a Lie group with Lie algebra L. If $L = L_1 \oplus L_2$ is an Iwasawa decomposition of L, then L_1 is solvable.

Hypothesis (H) implies that L_1 is Abelian. Since L_1 is compact, it follows that (cf. Helgason, 1978; Chapter II, Proposition 6.6) $L_1 = \text{centre}(L_1) \oplus DL_1$ with DL_1 semi-simple. But DL_1 has also to be solvable as a subalgebra of the solvable algebra L_1. This means that $DL_1 = 0$ and L_1 is Abelian. The presented stabilization technique can be applied in the case of Lie groups satisfying (H). Taking into account the classification of semi-simple Lie algebras (cf. Sagle and Walde, 1973), it follows that L satisfies (H) iff it is of the form

$$L = \underbrace{\text{sl}(2, \mathbb{R}) \oplus \cdots \oplus \text{sl}(2, \mathbb{R})}_{\mu \text{ times}},$$

where $\text{sl}(2, \mathbb{R})$ consists of the 2×2 real matrices of zero trace. In the following theorem a slightly modified decomposition is used in order to examine the stabilizability of a control system on a semi-simple Lie group.

Theorem 3. *Assume that G satisfies (H) and let Γ be a control system on G. If $\Gamma = -\Gamma$, $\text{Lie}(\Gamma \cap L_1) = L_1$ and $\text{Lie}(\Gamma \cap D^k L_2) = D^k L_2$ for $k = 0, 1, 2, \ldots$, then Γ is stabilizable.*

Proof. Since G is simply connected, so is H_1. Furthermore, Γ contains a basis $\{X_1, \ldots, X_\mu, Y_1, \ldots, Y_\nu\}$ of L such that $\{X_1, \ldots, X_\mu\}, \{Y_1, \ldots, Y_\nu\}$ are bases of L_1, L_2 of the form described in Lemma 2. This is true because, as observed before, L_1 is Abelian and L_2 is solvable. Now we can write

$$G = \exp(\mathbb{R}X_1) \cdots \exp(\mathbb{R}X_\mu) \exp(\mathbb{R}Y_1) \cdots \exp(\mathbb{R}Y_\nu).$$

In a manner similar to that of the proof of Theorem 1 we can construct a vector field V which steers every initial condition to e. Thus Γ is stabilizable and the proof is complete. ■

Example 2. Consider the semi-simple Lie algebra $\mathrm{sl}(2, \mathbb{R})$. Then $\mathrm{sl}(2, \mathbb{R}) = L_1 \oplus P$, where

$$L_1 = \mathbb{R}\begin{pmatrix} 1 & 1 \\ -2 & -1 \end{pmatrix}, \quad P = \mathbb{R}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \mathbb{R}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

is a Cartan decomposition. Set

$$A = \mathbb{R}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Then a simple calculation shows that

$$L^+ = \mathbb{R}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Thus

$$\mathrm{sl}(2, \mathbb{R}) = \mathbb{R}\begin{pmatrix} 1 & 1 \\ -2 & -1 \end{pmatrix} \oplus \mathbb{R}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \oplus \mathbb{R}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

4. The General Case

In the previous sections, the cases where the state space was solvable or semi-simple were treated. These special cases are very important because every Lie algebra can be decomposed into a solvable and a semi-simple subalgebra. In this section, the general case is examined. Let G be any real analytic and simply-connected Lie group with Lie algebra L. Let us now remind what the Levi decomposition of a Lie algebra is (for a complete treatment the reader is referred to (Varadarajan, 1984)). If $L_r = \mathrm{rad}(L)$, then the quotient algebra L/L_r is semi-simple. A Lie subalgebra L_m of L is called a Levi subalgebra if

$$L = L_m \oplus L_r.$$

A Levi subalgebra of L is isomorphic to L/L_r, so it is semi-simple. Now we have the following theorem (cf. Varadarajan, 1984, Theorem 3.14.1).

Theorem 4. *Any Lie algebra admits Levi subalgebras.*

Thus every Lie algebra can be decomposed into a solvable subalgebra and a semi-simple subalgebra. For the corresponding Lie groups we have the following (Varadarajan, 1984, Theorem 3.18.13).

Theorem 5. *Let G be a real analytic and simply-connected Lie group with Lie algebra L. Assume that $L = L_m \oplus L_r$ is a Levi decomposition of L and G_m, G_r denote the Lie subgroups corresponding to L_m, L_r. Then G_m, G_r are closed and the map $(x_m, x_r) \to x_m x_r$ is an analytic diffeomorphism of $G_m \oplus G_r$ onto G.*

From this theorem it follows immediately that G_m and G_r are simply connected. Since G_m is semi-simple, it follows that G_m admits an Iwasawa decomposition. Hence we can write

$$L_m = L_1 \oplus L_2, \quad G_m = G_1 \oplus G_2,$$

where L_1, L_2, G_1, G_2 are defined as in Section 3. For notational simplicity, we let $L_3 = L_r, G_3 = G_r$. Now the stabilizability of a control system on G is examined by stating and proving the following theorem.

Theorem 6. *Let G, L be as before. Suppose that L_m is any Levi subalgebra satisfying the hypothesis (H). Let Γ be a control system on G. If $\Gamma = -\Gamma$, $\mathrm{Lie}(\Gamma \cap L_1) = L_1$ and $\mathrm{Lie}(\Gamma \cap D^k L_i) = D^k L_i$ for $i = 2,3$ and $k = 0,1,\ldots,$ then Γ is stabilizable.*

Proof. It is evident that $G = G_1 G_2 G_3$. Using the same arguments as in Theorems 1 and 3, it can be concluded that Γ contains a basis $\{X_1, \ldots, X_\lambda, Y_1, \ldots, Y_\mu, Z_1, \ldots, Z_\nu\}$ of L such that $\{X_1, \ldots, X_\lambda\}$, $\{Y_1, \ldots, Y_\mu\}$, $\{Z_1, \ldots, Z_\nu\}$ are bases of L_1, L_2, L_3, respectively, of the form described in Lemma 2. Hence

$$G = \exp(\mathbb{R}X_1) \cdots \exp(\mathbb{R}X_\lambda) \exp(\mathbb{R}Y_1) \cdots \exp(\mathbb{R}Y_\mu) \exp(\mathbb{R}Z_1) \cdots \exp(\mathbb{R}Z_\nu).$$

In a manner similar to that introduced in the proof of Theorem 1 we can construct a vector field which steers every initial condition to the identity element and the proof is complete. ■

Example 3. Consider the Lie algebra $\mathrm{gl}(2,\mathbb{R})$ consisting of all 2×2 real matrices. The radical of $\mathrm{gl}(2,\mathbb{R})$ is

$$L_r = \mathbb{R}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and $\mathrm{sl}(2,\mathbb{R})$ is isomorphic to $\mathrm{gl}(2,\mathbb{R})/L_r$. Thus the following decomposition emerges:

$$\mathrm{gl}(2,\mathbb{R}) = \mathbb{R}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \oplus \mathrm{sl}(2,\mathbb{R}).$$

Taking into account Example 2, it follows that

$$\mathrm{gl}(2,\mathbb{R}) = \mathbb{R}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \oplus \mathbb{R}\begin{pmatrix} 1 & 1 \\ -2 & -1 \end{pmatrix} \oplus \mathbb{R}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \oplus \mathbb{R}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

5. Conclusion

In this paper, the problem of the stabilizability of invariant control systems on Lie groups has been treated. A stabilization technique, based on an appropriate decomposition of the state space, has been introduced. As has been shown, under certain hypotheses concerning the structure of the state space and the system itself, this technique can lead to general criteria assuring the existence of a constructed stabilizing feedback controller. In order to extend the application field of these criteria, it is necessary to relax some of the conditions or to replace them by milder ones. This problem is currently under investigation.

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