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ZEROS IN DISCRETE-TIME MIMO LTI SYSTEMS AND THE OUTPUT-ZEROING PROBLEM

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A geometric interpretation of invariant zeros of MIMO LTI discrete-time systems is provided. The zeros are treated as the triples: complex number, state zero direction, input zero direction. Such a treatment is strictly connected with the output zeroing problem and in that spirit the zeros can be easily interpreted even in the degenerate case (i.e. when each complex number constitutes an invariant zero). Simply, in the degenerate case, to each complex number we can assign an appropriate real initial condition and an appropriate real input sequence which produce a non-trivial solution to the state equation and a zero system response. Clearly, when zeros are treated merely as complex numbers, such an interpretation is impossible. The proposed definition of invariant zeros is compared with other commonly known definitions. It is shown that each Smith zero of the system matrix is also an invariant zero in the sense of the definition adopted in the paper. On the other hand, simple numerical examples show that the considered definition of invariant zeros and the Davison-Wang definition are not comparable. The output-zeroing problem for systems decouplable by state feedback is also described.

Keywords: linear multivariable systems, discrete-time systems, invariant zeros, Davison-Wang definition, output-zeroing problem

1. Introduction

The determination of zeros has received considerable attention in recent years (Amin and Hassan, 1988; El-Ghezawi *et al.*, 1982; Emami-Naeini and Van Dooren, 1982; Hewer and Martin, 1984; Latawiec, 1998; Latawiec *et al.*, 1999; MacFarlane and Karcanias, 1976; Misra *et al.*, 1994; Owens, 1977; Sannuti and Saberi, 1987; Tokarzewski, 1996; 1998; Wolovich, 1973). The zeros are defined in many (not necessarily equivalent) ways (for a survey of these definitions see MacFarlane and Karcanias, 1976; Schrader and Sain, 1989) so that the term 'zero' has become ambiguous. There are three main groups of definitions:

(a) those originating from Rosenbrock's approach (Amin and Hassan, 1988; Emami-Naeini and Van Dooren, 1982; MacFarlane and Karcanias, 1976; Misra *et al.*, 1994; Sannuti and Saberi, 1987; Wolovich, 1973, Rosenbrock, 1970) and related to the Smith or Smith-McMillan form,

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(b) those connected with the concept of state-zero and input-zero directions introduced in (MacFarlane and Karcanias, 1976) cf. (El-Ghezawi *et al.*, 1982; Owens, 1977; Sannuti and Saberi, 1987; Tokarzewski, 1996; 1998; Tokarzewski *et al.*, 1999), and

(c) those employing the notions of inverse systems (Latawiec, 1998; Latawiec *et al.*, 1999).

Although many authors before 1970 alluded to the concept of zeros of multi-input multi-output (MIMO) systems, Rosenbrock (1970) is credited with the first definition of zeros of MIMO systems (multi-variable zeros). Rosenbrock's first multi-variable zeros, termed the decoupling zeros, were related to the notions of controllability and observability. The most common definition of invariant zeros (Rosenbrock, Wolovich, MacFarlane-Karcanias) employs the Smith canonical form of the system (Rosenbrock) matrix and determines the zeros as the roots of the diagonal (invariant) polynomials of the Smith form. In the sequel, these zeros will be called the Smith invariant zeros. They may be defined equivalently as the points of the complex plane where the rank of the system matrix falls below its normal rank (recall that the term 'normal rank' means 'rank' in the field of rational functions). The zeros of a proper transfer matrix are defined (Wolovich, 1973; Emami-Naeini and Van Dooren, 1982; Misra *et al.*, 1994) from its arbitrary minimal (i.e. reachable and observable) standard state-space realization as the points where the system matrix (formed from matrices of a minimal realization) loses its normal rank. In the sequel, these zeros will be called the Smith zeros of a transfer matrix. When the transfer matrix has full normal rank, the above definition is equivalent to the Desoer-Schulman one (Chen, 1984, Appendix H, Theorem H-6, p.631).

Smith invariant zeros, decoupling zeros and Smith zeros of a transfer matrix are involved in several problems of control theory, such as zeroing the output, tracking the reference output, disturbance decoupling, non-interacting control or output regulation (Isidori, 1995, Chapters 3, 4, 5 and 8).

A different definition of zeros, also based on the system matrix rank test, was given by Davison and Wang (Hewer and Martin, 1984; MacFarlane and Karcanias, 1976; Schrader and Sain, 1989).

All the above definitions of multi-variable zeros, although deceivingly simple, consider zeros merely as complex numbers and for this reason create some difficulties in their dynamical interpretation. In order to overcome these difficulties, MacFarlane and Karcanias (1976) introduced the notions of state-zero and input-zero directions and formulated the so-called output-zeroing problem. Unfortunately, in (MacFarlane and Karcanias, 1976) the Smith invariant zeros were not directly related to the output-zeroing problem.

Another definition of invariant zeros, employing the system matrix and state-zero and input-zero directions, was introduced in (Tokarzewski, 1996; 1998) and applied to their algebraic characterization and calculation. However, also in (Tokarzewski, 1996; 1998) the question of interpreting zeros in the context of the output-zeroing problem, especially in the degenerate case, as well as the question of relating zeros with other definitions were not discussed. In this paper, we try to bridge this gap.

To this end, we employ throughout the paper the formulation of the output-zeroing problem given by Isidori (1995). In Section 2, the results of (Tokarzewski *et al.*, 1999) are interpreted in the context of this problem formulation. In Section 3, we describe the problem in detail for the class of all decouplable systems.

Consider a discrete-time system with m inputs and r outputs

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k), \\ y(k) &= Cx(k) + Du(k), \end{aligned} \tag{1}$$

where $k = 0, 1, 2, \ldots$ and $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{r\times n}$, $D \in \mathbb{R}^{r\times m}$, $x(k) \in \mathbb{R}^n$, $y(k) \in \mathbb{R}^r$, $u(k) \in \mathbb{R}^m$. We assume that $B \neq 0$ and if $D = 0$, then also $C \neq 0$.

Definition 1. (Tokarzewski, 1996; 1998)

(i) A number $\lambda \in \mathbb{C}$ is an invariant zero if and only if there exist vectors $0 \neq x^0 \in \mathbb{C}^n$ (state-zero direction) and $g \in \mathbb{C}^m$ (input-zero direction) such that the triple $(\lambda,\, x^0,\, g)$ satisfies

$$\begin{bmatrix} \lambda I - A & -B \\ C & D \end{bmatrix} \begin{bmatrix} x^0 \\ g \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \tag{2}$$

where

$$P(z) = \begin{bmatrix} zI - A & -B \\ C & D \end{bmatrix}$$

is the system (Rosenbrock) matrix.

The system is called degenerate if it has an infinite number of invariant zeros.

(ii) The transmission zeros are defined as the invariant zeros of a reachable and observable subsystem of (1).

(iii) A number $\lambda \in \mathbb{C}$ is an output decoupling (o.d.) zero if and only if there exists a vector $0 \neq x^0 \in \mathbb{C}^n$ such that

$$(\lambda I - A)x^0 = 0, \quad Cx^0 = 0, \tag{3}$$

or equivalently, if and only if λ is an unobservable eigenvalue (mode) of A.

(iv) A number $\lambda \in \mathbb{C}$ is an input decoupling (i.d.) zero if and only if there exists a vector $0 \neq w^0 \in \mathbb{C}^n$ such that

$$(w^0)^*(\bar{\lambda} I - A) = 0, \quad (w^0)^* B = 0, \tag{4}$$

or equivalently, if and only if λ is an unreachable eigenvalue (mode) of A.

(v) A number $\lambda \in \mathbb{C}$ is an input-output decoupling (i.o.d.) zero if and only if λ is an unreachable and unobservable eigenvalue (mode) of A.

From Definition 1(i) it is easily seen that invariant zeros are invariant under similarity transformations of the state space and under constant state feedback (if $D = 0$, then also under constant output feedback). They do not change after introducing a nonsingular pre- or postcompensator. The number of Smith invariant zeros is always finite, while the number of invariant zeros may be infinite (see Examples 1 and 2). In the sense of Definition 1 each transmission zero of the system is its invariant zero. In the sense of Definition 1 each o.d. zero of (1) is also its invariant zero, which is not the case when the Smith invariant zeros are considered. When the system (1) is transformed to its Kalman canonical form, then individual kinds of decoupling zeros are displayed by appropriate block matrices on the diagonal of the A-matrix (Tokarzewski, 1996; 1998).

2. Invariant Zeros and the Output-Zeroing Problem

The output-zeroing problem can be formulated as follows (Isidori, 1995, p.163): Find all pairs $(x^0, u_0(k))$, consisting of an initial state $x^0 \in \mathbb{R}^n$ and a real-valued input vector sequence $u_0(k)$, $k = 0, 1, 2, \ldots$, such that the corresponding output $y(k)$ of (1) is identically zero for all $k = 0, 1, 2, \ldots$. Any non-trivial pair of this kind (i.e. such that $x^0 \neq 0$ or the input sequence $u_0(k)$ is not identically zero) will be called the output-zeroing input.

In each output-zeroing input $(x^0, u_0(k))$, $u_0(k)$ should be understood simply as an open-loop real-valued control signal which, when applied to (1) exactly at the initial state $x^0 \in \mathbb{R}^n$, yields $y(k) = 0$ for all $k = 0, 1, 2, \ldots$. If in an output-zeroing input $(x^0, u_0(k))$ the input sequence $u_0(k)$ is not identically zero, then we say that the transmission of the signal $u_0(k)$ (applied to (1) at the initial state $x^0 \in \mathbb{R}^n$) has been blocked by the system. Thus the transmission blocking property of a system is a particular case of the output-zeroing property.

Of course, the set of all output-zeroing inputs complemented with the trivial pair $(x^0 = 0,\ u_0(k) \equiv 0)$ forms a linear space over $\mathbb{R}$. In fact, if $(x_1^0, u_0^1(k))$ and $(x_2^0, u_0^2(k))$ are output-zeroing and give respectively solutions to the state equation $x_0^1(k)$ and $x_0^2(k)$, then from the linearity of (1) and from the uniqueness of solutions it follows that each pair of the form $(\alpha x_1^0 + \beta x_2^0, \alpha u_0^1(k) + \beta u_0^2(k))$, with arbitrarily fixed $\alpha, \beta \in \mathbb{R}$, is output-zeroing and yields the solution $\alpha x_0^1(k) + \beta x_0^2(k)$. It is also easy to observe that if $(x^0, u_0(k))$ is an output-zeroing input and $x_0(k)$ denotes the correspoding solution, then the input sequence $u_0(k)$ applied to (1) at an arbitrary initial condition (state) $x(0) \in \mathbb{R}^n$ gives the solution to the state equation (1) of the form $A^k(x(0) - x^0) + x_0(k)$ and the system response $y(k) = CA^k(x(0) - x^0)$.

The above discussion shows in particular that if (1) is asymptotically stable, then the input signal $\alpha u_0^1(k) + \beta u_0^2(k)$ applied to the system at an arbitrary initial condition produces an asymptotically vanishing system response, i.e. $y(k) \to 0$ as $k \to \infty$. In this way, in an asymptotically stable system (1), the set of all output-zeroing inputs enables us to generate a class of input sequences which are asymptotically attenuated by the system.

Unfortunately, the same symbol (x^0) is used to denote the *state-zero direction* in the definition of invariant zeros (Definition 1(i)) and to denote the *initial state* in the definition of output-zeroing inputs.

According to Definition 1(i), a *state-zero direction* x^0 must be a non-zero vector (real of complex). Otherwise, this definition becomes senseless (for every system (1) each complex number may serve as an invariant zero). In other words, in the equation

$$P(\lambda)\begin{bmatrix} x \\ u \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

(see (2)) the solutions of the form $\left[\begin{smallmatrix} 0 \\ u \end{smallmatrix}\right]$ are not taken into account in the process of defining invariant zeros.

According to the formulation of the output-zeroing problem, the *initial state* x^0 must be a real vector (but not necessarily non-zero, cf. Example 2). If in a triple (λ, x^0, g) satisfying (2) a *state-zero direction* x^0 is a complex vector, then in the process of asignining to the invariant zero λ output-zeroing inputs we consider two *initial states* $\operatorname{Re} x^0$ and $\operatorname{Im} x^0$ (and, of course, at least one of these *initial states* must be a non-zero vector).

As we show below, Definition 1(i) clearly relates invariant zeros (even in the degenerate case) to the output-zeroing problem. In order to discuss the output-zeroing inputs corresponding to invariant zeros, it is convenient to treat (1) as a complex system, i.e. the one admitting complex inputs, solutions and outputs which are denoted respectively by $\tilde{u}$, $\tilde{x}$ and $\tilde{y}$.

Lemma 1. (Tokarzewski *et al.*, 1999) *If $\lambda \in \mathbb{C}$ is an invariant zero of (1) (i.e. a triple λ, $x^0 \neq 0$, g satisfies (2)), then the input*

$$\tilde{u}(k) = \begin{cases} g & \text{for} \quad k = 0, \\ g\lambda^k & \text{for} \quad k = 1, 2, \dots, \end{cases} \tag{5}$$

applied to (1) at the initial condition $x(0) = x^0$ yields the solution to the state equation in (1) of the form

$$\tilde{x}(k) = \begin{cases} x^0 & \text{for} \quad k = 0, \\ x^0\lambda^k & \text{for} \quad k = 1, 2, \dots \end{cases} \tag{6}$$

and the system response $\tilde{y}(k) \equiv 0, \;\; k = 0, 1, 2, \dots$.

Proof. The simple proof by inspection is omitted here. ■

The above lemma enables us to give a desired dynamical interpretation of invariant zeros in the context of the output-zeroing problem.

Remark 1. If a triple $\lambda = \sigma + j\omega$, $x^0 = \operatorname{Re} x^0 + j \operatorname{Im} x^0$, g satisfies (2), then (2) is also satisfied for the triple $\bar{\lambda} = \sigma - j\omega$, $\bar{x}^0 = \operatorname{Re} x^0 - j \operatorname{Im} x^0$, $\bar{g}$. Moreover, these triples generate two real initial conditions and two real input signals which produce identically zero system responses. More precisely, the real-valued input sequence

$$\begin{aligned} u(k, \operatorname{Re} x^0) &= \frac{1}{2} g \lambda^k + \frac{1}{2} \bar{g} \bar{\lambda}^k \\ &= |\lambda|^k (\operatorname{Re} g \cos k\varphi - \operatorname{Im} g \sin k\varphi), \quad k = 0, 1, 2, \dots \end{aligned} \tag{7}$$

(where the notation $\lambda = |\lambda| e^{j\varphi}$ has been used) and the real initial state $\operatorname{Re} x^0$ yield the solution to the state equation of (1) of the form

$$\begin{aligned} x(k, \operatorname{Re} x^0) &= \frac{1}{2} x^0 \lambda^k + \frac{1}{2} \bar{x}^0 \bar{\lambda}^k \\ &= |\lambda|^k (\operatorname{Re} x^0 \cos k\varphi - \operatorname{Im} x^0 \sin k\varphi), \quad k = 0, 1, 2, \dots \end{aligned} \tag{8}$$

and the system response

$$y(k) = Cx(k, \operatorname{Re} x^0) + Du(k, \operatorname{Re} x^0) \equiv 0, \quad k = 0, 1, 2, \dots .$$

This means that the pair $(\operatorname{Re} x^0, u(k, \operatorname{Re} x^0))$, with $u(k, \operatorname{Re} x^0)$ given by (7), is an output-zeroing input corresponding to the triples considered. Similarly, the pair $(\operatorname{Im} x^0, u(k, \operatorname{Im} x^0))$, with the real input

$$\begin{aligned} u(k, \operatorname{Im} x^0) &= -j\frac{1}{2} g \lambda^k + j\frac{1}{2} \bar{g} \bar{\lambda}^k \\ &= |\lambda|^k (\operatorname{Re} g \sin k\varphi + \operatorname{Im} g \cos k\varphi), \quad k = 0, 1, 2, \dots, \end{aligned} \tag{9}$$

and the real initial state $\operatorname{Im} x^0$, is an output-zeroing input which produces the solution to the state equation of (1) of the form

$$\begin{aligned} x(k, \operatorname{Im} x^0) &= -j\frac{1}{2} x^0 \lambda^k + j\frac{1}{2} \bar{x}^0 \bar{\lambda}^k \\ &= |\lambda|^k (\operatorname{Re} x^0 \sin k\varphi + \operatorname{Im} x^0 \cos k\varphi), \quad k = 0, 1, 2, \dots, \end{aligned} \tag{10}$$

and the system response

$$y(k) = Cx(k, \operatorname{Im} x^0) + Du(k, \operatorname{Im} x^0) \equiv 0 \quad \text{for} \quad k = 0, 1, 2, \dots .$$

The following result (together with Examples 1 and 2) shows that the adopted definition of invariant zeros (Definition 1(i)) constitutes an extension of the notion of Smith invariant zeros. Recall that a number $\lambda \in \mathbb{C}$ is a Smith invariant zero of (1) if and only if $\operatorname{rank} P(\lambda) <$ normal $\operatorname{rank} P(z)$.

Lemma 2. (Tokarzewski *et al.*, 1999) *If* $\lambda \in \mathbb{C}$ *is a Smith invariant zero of (1), then* λ *is also an invariant zero of (1) according to Definition 1(i).*

Proof. For the proof it suffices to show that if $\operatorname{rank} P(\lambda) <$ normal $\operatorname{rank} P(z)$, then the equation

$$P(\lambda) \begin{bmatrix} x \\ u \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

possesses a solution $\begin{bmatrix} x^0 \\ g \end{bmatrix}$ in which $x^0 \neq 0$. Suppose that a number $\lambda \in \mathbb{C}$ is a Smith invariant zero of (1), i.e.

$$\operatorname{rank} P(\lambda) < \text{normal rank}\, P(z) \leq n + \min\{m, r\}.$$

This means that the system $P(\lambda)\begin{bmatrix} x \\ u \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ of $n+r$ linear equations with $n+m$ unknowns has at least one non-zero solution. We shall discuss separately two cases.

In the first case, let the matrix $\begin{bmatrix} -B \\ D \end{bmatrix}$ be monic (i.e. of full column rank) and let a vector $\begin{bmatrix} 0 \\ g \end{bmatrix}$, $g \neq 0$, be a non-zero solution to the considered system of linear equations. Then we have

$$\begin{bmatrix} -B \\ D \end{bmatrix} g = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

and consequently, $g = 0$. This contradiction proves that in any non-zero solution $\begin{bmatrix} x^0 \\ g \end{bmatrix}$ of the considered system of equations we must have $x^0 \neq 0$, i.e. λ satisfies Definition 1(i).

In the other case, assume that $\begin{bmatrix} -B \\ D \end{bmatrix}$ is not monic and its rank is equal to $m' < m$. Without loss of generality, we can assume that the first m' columns of that matrix are linearly independent and the matrix composed of these columns is denoted by $\begin{bmatrix} -B' \\ D' \end{bmatrix}$. It is clear that

$$P'(z) = \begin{bmatrix} zI - A & -B' \\ C & D' \end{bmatrix}$$

is of the same normal rank as $P(z)$, and for any fixed $z \in \mathbb{C}$ we have $\operatorname{rank} P'(z) = \operatorname{rank} P(z)$ (where rank is taken in the field of complex numbers). This enables us to write

$$\operatorname{rank} P'(\lambda) = \operatorname{rank} P(\lambda) < \text{normal rank}\, P(z) = \text{normal rank}\, P'(z) \leq n + \min\{m', r).$$

This means, by virtue of the first part of the proof, that the system

$$P'(\lambda)\begin{bmatrix} x \\ u' \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

of $n+r$ linear equations with $n+m'$ unknowns has a solution $\begin{bmatrix} x^0 \\ g' \end{bmatrix}$ such that $x^0 \neq 0$, $g' \in \mathbb{C}^{m'}$, i.e. we have

$$P'(\lambda)\begin{bmatrix} x^0 \\ g' \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Now, adding $m - m'$ zero rows to the vector $g' \in \mathbb{C}^{m'}$, i.e. taking $g = \begin{bmatrix} g' \\ 0 \end{bmatrix}$, where $0 \in \mathbb{C}^{m-m'}$, we obtain

$$P(\lambda)\begin{bmatrix} x^0 \\ g \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

i.e. λ satisfies Definition 1(i). ∎

Thus, although the Smith invariant zeros are defined merely as complex numbers, Lemma 2 enables us to assign to each such a zero a non-zero state-zero direction (x^0) and an input-zero direction (g), and consequently, to assign a physical (geometric) interpretation described in Remark 1.

Corollary 1. *Let the system matrix $P(z)$ in (1) have full column normal rank. Then a complex number λ is an invariant zero of (1) if and only if λ is a Smith invariant zero of the system.*

Proof. In view of Lemma 2 it is sufficient to show that each invariant zero of (1) is also a Smith invariant zero of the system. However, from Definition 1(i) it follows that if λ satisfies (2), then columns of $P(\lambda)$ are linearly dependent over $\mathbb{C}$. Thus we can write

$$\operatorname{rank} P(\lambda) < \text{normal rank} P(z) = n + m,$$

which means that λ is a Smith invariant zero of system (1). ■

When the system (1) is minimal (reachable and observable), we can formulate the following characterization of invariant zeros (the complete proof of Lemma 3 can be found in (Tokarzewski *et al.*, 1999, p.1102)).

Lemma 3. (Tokarzewski *et al.*, 1999) *A number $\lambda \in \mathbb{C}$ is an invariant zero of a minimal system (1) if and only if there exist $0 \neq x^0 \in \mathbb{C}^n$ and $0 \neq g \in \mathbb{C}^m$ such that the input sequence of the form (5) applied to (1) at the initial condition $x(0) = x^0$ yields the system response $\tilde{y}(k) \equiv 0$, $k = 0, 1, 2, \ldots$.*

Proof. If $\lambda \in \mathbb{C}$ is an invariant zero, i.e. a triple λ, $x^0 \neq 0$, g satisfies (2), then, as we know from Lemma 1, the input sequence of the form (5) applied to the system at $x(0) = x^0$ yields the identically zero output sequence. It remains to show that $g \neq 0$. However, supposing $g = 0$ in the considered triple we would obtain $(\lambda I - A)x^0 = 0$ and $Cx^0 = 0$ which would contradict the observability assumption. Thus we must have $g \neq 0$.

In order to prove the converse implication, we should show that if a triple λ, $x^0 \neq 0$, $g \neq 0$ is such that the input (5) applied to (1) at the initial condition x^0 yields $\tilde{y}(k) \equiv 0$, $k = 0, 1, 2, \ldots$, then this triple satisfies (2). The idea of the proof is as follows. We first show that for each $k = 0, 1, 2, \ldots$ the equality $CA^k((\lambda I - A)x^0 - Bg) = 0$ holds. Then, by virtue of the observability assumption, we will have $(\lambda I - A)x^0 - Bg = 0$. The second equality in (2) will follow immediately from the assumption $\tilde{y}(0) = 0$. ■

Definition 2. (Tokarzewski *et al.*, 1999) A number $\lambda \in \mathbb{C}$ is a zero of a proper transfer matrix $G(z)$ iff λ is an invariant zero of any given minimal realization of $G(z)$.

Making use of Lemma 2 it is not difficult to relate Definition 2 to other commonly known definitions of zeros of a transfer matrix. To this end, we take into account the Desoer-Schulman definition and the definition of Smith zeros of $G(z)$ employed by

(Emami-Naeini and Van Dooren, 1982) (see also Misra *et al.*, 1994, p.1923). Recall that the latter is based on a minimal state-space realization of $G(z)$ and defines the Smith zeros of $G(z)$ as the points where the rank of the system matrix drops below its normal rank. On the other hand, the definition of the zeros attributed to Desoer and Schulman concerns merely transfer matrices with full normal rank and exploits the matrix coprime fraction description $G(z) = D_l^{-1}(z)N_l(z)$. Then a number $\lambda \in \mathbb{C}$ is said to be a Desoer-Schulman zero of an $r \times m$ $G(z)$ if and only if $\operatorname{rank} N_l(z)$ at $z = \lambda$ falls below its normal rank, i.e. $\operatorname{rank} N_l(\lambda) < \min\{m, r\}$. The Desoer-Schulman zero of $G(z)$ can also be defined by using dynamic equations. Namely (Chen, 1984, Theorem H-6), if $S(A, B, C, D)$ denotes an irreducible n-dimensional realization of $G(z)$, then $\lambda \in \mathbb{C}$ is a Desoer-Schulman zero of $G(z)$ if and only if

$$\operatorname{rank} P(\lambda) = \operatorname{rank} \begin{bmatrix} \lambda I - A & -B \\ C & D \end{bmatrix} < \text{normal rank}\, P(z) = n + \min\{m, r\}.$$

Now we can formulate the desired result.

Corollary 2. (Tokarzewski *et al.*, 1999) *Consider an $r \times m$ transfer matrix $G(z)$ and its n-dimensional state-space irreducible realization $S(A, B, C, D)$. Then:*

(a) If $\lambda \in \mathbb{C}$ is a Smith zero of $G(z)$, then λ is also a zero of $G(z)$ in the sense of Definition 2 (i.e. there exist $0 \neq x^0 \in \mathbb{C}^n$ and $0 \neq g \in \mathbb{C}^m$ such that the triple λ, x^0, g satisfies (2), where $P(z)$ is determined by the matrices of $S(A, B, C, D)$).

(b) If $G(z)$ has full normal rank and $\lambda \in \mathbb{C}$ is its Desoer-Schulman zero, then λ is also a zero of $G(z)$ in the sense of Definition 2 (i.e. λ is an invariant zero of $S(A, B, C, D)$ in the sense of Definition 1(i)).

Proof. (a) The proof follows immediately from Lemma 2 when applied to a minimal system $S(A, B, C, D)$.

(b) In this case we have the relation

$$\operatorname{rank} P(\lambda) < \text{normal rank}\, P(z) = n + \min\{m, r\}$$

and the remaining part of the proof follows from Lemma 2. ■

Now, consider an $r \times m$ transfer matrix $G(z)$ with its n-dimensional state-space irreducible realization $S(A, B, C, D)$ and assume additionally that $G(z)$ has full column normal rank. Then, as we shall see below, each zero of $G(z)$ in the sense of Definition 2 is also a Desoer-Schulman zero of $G(z)$. In the proof we make use of the following relation (Chen, 1984, Appendix H, p.630):

$$\text{normal rank}\, P(z) = \text{normal rank} \begin{bmatrix} zI - A & -B \\ C & D \end{bmatrix} = n + \text{normal rank}\, G(z),$$

which is valid without any assumption concerning the rank of $G(z)$. Now, if $G(z)$ has full column normal rank, then $m \leq r$ and normal rank $P(z) = n + m$, i.e. $P(z)$ has full column normal rank.

From the above discussion and Corollary 1 we obtain immediately the following result.

Corollary 3. (Tokarzewski *et al.*, 1999) *If $G(z)$ has full column normal rank, then λ is a zero of $G(z)$ in the sense of Definition 2 if and only if λ is a Desoer-Schulman zero of $G(z)$.*

As we will show in Section 4 (Examples 1 and 2) the above result fails when $G(z)$ is not of full column normal rank. For square transfer matrices of full normal rank we can establish the following characterization of their zeros.

Corollary 4. (Tokarzewski *et al.*, 1999) *If $G(z)$ is square and of full normal rank, then the following statements are equivalent:*

- $\lambda \in \mathbb{C}$ *is a zero of $G(z)$ in the sense of Definition 2,*
- $\lambda \in \mathbb{C}$ *is a Desoer-Schulman zero of $G(z)$,*
- $\det P(\lambda) = 0$.

From the above discussion it follows that each zero of a transfer matrix defined as the point where the system matrix loses its normal rank admits a physical interpretation described in Remark 1.

3. Output-Zeroing Problem in Decouplable Systems

Consider a square, m-input m-output system

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k), \\ y(k) &= Cx(k), \end{aligned} \tag{11}$$

$k = 0, 1, 2, \ldots$, in which

$$C = \begin{bmatrix} c_1 \\ \vdots \\ c_m \end{bmatrix}$$

and c_s, $s = 1, \ldots, m$ stand for the consecutive rows of C. Denote by $S_s(A, B, c_s)$, $s = 1, \ldots, m$ the subsystems of (11) with m inputs and one output. The first nonzero Markov parameters of $S_s(A, B, c_s)$ are denoted by $c_s A^{\nu_s} B$, i.e. we have

$$c_s B = \cdots = c_s A^{\nu_s - 1} B = 0, \quad c_s A^{\nu_s} B \neq 0 \tag{12}$$

for some integer $0 \leq \nu_s \leq n-1$. Define a square $m \times m$ matrix

$$L := \begin{bmatrix} c_1 A^{\nu_1} B \\ \vdots \\ c_m A^{\nu_m} B \end{bmatrix} \tag{13}$$

whose rows are formed from the first non-zero Markov parameters of subsequent subsystems $S_s(A, B, c_s)$, $s = 1, \ldots, m$. We say that system (11) has vector relative degree $\nu + 1 = \mathrm{col}\,[\nu_1 + 1, \nu_2 + 1, \ldots, \nu_m + 1]$ if and only if L is non-singular. In the remainder of this section it is assumed that (11) has vector relative degree $\nu + 1$.

Remark 2. As is well-known (Chen, 1984, p.372), non-singularity of L is a necessary and sufficient condition for (11) to be decoupled by a constant state fedback and a non-singular precompensator. Recall that a multivariable system is said to be decoupled if its transfer matrix is diagonal and non-singular.

Set

$$M := \begin{bmatrix} c_1 A^{\nu_1} \\ \vdots \\ c_m A^{\nu_m} \end{bmatrix}. \tag{14}$$

Then

$$L = MB. \tag{15}$$

Define an $n \times n$ matrix

$$K := I - BL^{-1}M. \tag{16}$$

Lemma 4. *The matrix K has the following properties:*

(i) $K^2 = K$,

(ii) $$\begin{cases} \Sigma := \{x : Kx = x\} = \mathrm{Ker}\, M, & \dim \Sigma = n - m, \\ \Omega := \{x : Kx = 0\} = \mathrm{Im}\, B, & \dim \Omega = m, \\ \mathbb{R}^n\,(\mathbb{C}^n) = \Sigma \oplus \Omega, \end{cases}$$

(iii) $KB = 0, \quad MK = 0,$

(iv) $$c_s(KA)^l = \begin{cases} c_s A^l & \text{for} \quad 0 \le l \le \nu_s \\ 0 & \text{for} \quad l \ge \nu_s + 1 \end{cases}, \quad s = 1, \ldots, m.$$

Proof. The proof of (i) follows from (16) and (15) by direct calculation. Similarly, relations (iii) follow directly from (15) and (16). In (ii) it is sufficient to show that $\Sigma = \mathrm{Ker}\, M$, $\Omega = \mathrm{Im}\, B$. Since L is non-singular, from (15) it follows that M is epic and B is monic. From (16) and the fact that B is monic we obtain

$$Kx = x \Leftrightarrow BL^{-1}Mx = 0 \Leftrightarrow Mx = 0 \Leftrightarrow x \in \mathrm{Ker}\, M$$

which proves that $\Sigma = \mathrm{Ker}\, M$. Because M is epic, we have $\dim \mathrm{Ker}\, M = n - m$. Now, let $x \in \mathrm{Im}\, B$. Since $KB = 0$, we have $Kx = 0$, i.e. $x \in \Omega$. Conversely, if

$x \in \Omega$, then $Kx = 0 \Rightarrow B(L^{-1}Mx) = x \Rightarrow x \in \operatorname{Im} B$. Consequently, $\Omega = \operatorname{Im} B$. The relation $\dim \operatorname{Im} B = m$ appears.

For the proof of (iv), we verify first that for each $s = 1, \dots, m$ and for any real $m \times n$ matrix F the conditions

$$c_s B = \cdots = c_s A^{\nu_s - 1} B = 0$$

imply

$$c_s (A + BF)^l = c_s A^l \quad \text{for} \quad l = 0, 1, \dots, \nu_s$$

and

$$c_s (A + BF)^l = c_s A^{\nu_s} (A + BF)^{l - \nu_s} \quad \text{for} \quad l \geq \nu_s + 1.$$

Consequently,

$$c_s (A + BF)^l B = c_s A^l B = 0 \quad \text{for} \quad l = 0, 1, \dots, \nu_s - 1$$

and

$$c_s (A + BF)^{\nu_s} B = c_s A^{\nu_s} B \neq 0.$$

In particular, for $F = -L^{-1}MA$, we get

$$c_s (A + BF)^{\nu_s + 1} = c_s A^{\nu_s} (A + BF) = c_s A^{\nu_s} KA = 0,$$

where $c_s A^{\nu_s} KA = 0$ follows from $MK = 0$ (iii) (since $c_s A^{\nu_s}$ is the s-th row of M, we have $c_s A^{\nu_s} K = 0$). Thus we have obtained $c_s (KA)^{\nu_s + 1} = 0$ and consequently, $c_s (KA)^l = 0$ for each $l \geq \nu_s + 1$. Finally, for $F = -L^{-1}MA$ the equalities $c_s (A + BF)^l = c_s A^l$, $l = 0, 1, \dots, \nu_s$ take the form $c_s (KA)^l = c_s A^l$. ■

The following lemma characterizes the invariant zeros of (11). We denote by $\rho(\cdot)$ the spectrum of a matrix.

Lemma 5. *A number $\lambda \in \mathbb{C}$ is an invariant zero of (11) if and only if $\lambda \in \rho(KA)$ and there exists an associated eigenvector x_0 such that $x_0 \in \operatorname{Ker} C$.*

In the proof of Lemma 5 we will employ the following result.

Lemma 6. *If $\lambda \in \mathbb{C}$ is an invariant zero of (11), i.e. a triple λ, $x_0 \neq 0$, g satisfies (2), then*

$$x_0 \in \bigcap_{s=1}^{m} \left(\bigcap_{l=0}^{\nu_s} \operatorname{Ker} c_s A^l \right) \subset \operatorname{Ker} M = \Sigma, \tag{17}$$

$$g = -L^{-1} M A x_0. \tag{18}$$

Proof. For any subsystem $S_s(A, B, c_s)$, $s = 1, \dots, m$ we have, by virtue of (2),

(i) $$\begin{cases} \lambda x_0 - A x_0 = Bg, \\ c_s x_0 = 0. \end{cases}$$

Premultiplying successively the first equality of (i) by $c_s, \ldots, c_s A^{\nu_s - 1}$, we obtain, in view of (12),

$$\text{(ii)} \qquad \begin{cases} c_s x_0 = 0, \\ \vdots \\ c_s A^{\nu_s} x_0 = 0, \end{cases}$$

i.e. $x_0 \in \bigcap_{l=0}^{\nu_s} \operatorname{Ker} c_s A^l$.

Premultiplying the first equality of (i) by $c_s A^{\nu_s}$, we get

$$\text{(iii)} \qquad -c_s A^{\nu_s + 1} x_0 = c_s A^{\nu_s} B g.$$

Because (ii) and (iii) are valid for each $s = 1, \ldots, m$, by virtue of (14) and Lemma 4(ii), we infer that (17) holds and (iii) can be written in the following compact form:

$$\text{(iv)} \qquad -MAx_0 = Lg.$$

Finally, from (iv) one gets (18). ■

Proof of Lemma 5. Suppose first that $\lambda, x_0 \neq 0, g$ satisfy (2). Because, as we know from (17), $x_0 \in \operatorname{Ker} M = \Sigma$ (which means, via Lemma 4(ii), that $Kx_0 = x_0$), premultiplying the first equality of (2) by K and taking into account that $KB = 0$ (see Lemma 4(iii)), we can write (2) in the form

$$\text{(v)} \qquad \begin{cases} \lambda x_0 - KAx_0 = 0, \\ Cx_0 = 0. \end{cases}$$

Thus $\lambda \in \rho(KA)$ and x_0 is an associated eigenvector which belongs to $\operatorname{Ker} C$. Moreover, via Lemma 6, $g = -L^{-1} MAx_0$. Conversely, if (v) is fulfilled for a pair $\lambda \in \mathbb{C}$, $x_0 \neq 0$, then setting $g = -L^{-1} MAx_0$ and taking into account (16) we can write the first equality of (v) in the form $\lambda x_0 - Ax_0 = Bg$. This means that the triple λ, $x_0 \neq 0$, $g = -L^{-1} MAx_0$ satisfies (2), i.e. λ is an invariant zero of (11). ■

Remark 3. As follows from the above proof, Lemma 5 can be formulated in a somewhat more detailed form. Namely, a triple λ, $x_0 \neq 0$, g satisfies (2) if and only if λ is an eigenvalue of KA, x_0 is an associated eigenvector which lies in $\operatorname{Ker} C$ and $g = -L^{-1} MAx_0$.

Lemma 7. *Let $P(z)$ denote the system matrix for (11). Then we have*

$$\det(zI - KA) = \det(L^{-1}) z^{m + (\nu_1 + \cdots + \nu_m)} \det P(z). \tag{19}$$

Proof. For the proof we take into account the closed-loop system $S(KA, BL^{-1}, C)$ obtained from (11) by introducing the state-feedback matrix $-L^{-1} MA$ and the precompensator L^{-1}. The transfer matrix of the closed-loop system is

$$G_{cl}(z) = C(zI - KA)^{-1} BL^{-1}. \tag{20}$$

Using Lemma 4 (iv), one can check that $G_{cl}(z)$ is diagonal and has the form

$$G_{cl}(z) = \begin{bmatrix} \frac{1}{z^{\nu_1+1}} & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & \frac{1}{z^{\nu_m+1}} \end{bmatrix}. \tag{21}$$

To this end, it is sufficient to expand $G_{cl}(z)$ in (20) in power series around $z = \infty$ and to note that, by virtue of (12) and Lemma 4(iv), for the s-th row of (20) we can write

$$c_s \sum_{l=0}^{\infty} \frac{(KA)^l}{z^{l+1}} BL^{-1} = \sum_{l=0}^{\nu_s} \frac{c_s A^l}{z^{l+1}} BL^{-1} = \frac{c_s A^{\nu_s} B}{z^{\nu_s+1}} L^{-1}. \tag{22}$$

Now, employing (22) and (13), we get (21). The system matrix of the closed-loop system can be expressed as

$$P_{cl}(z) = \begin{bmatrix} zI - KA & -BL^{-1} \\ C & 0 \end{bmatrix} = \begin{bmatrix} zI - A & -B \\ C & 0 \end{bmatrix} \begin{bmatrix} I & 0 \\ -L^{-1}MA & L^{-1} \end{bmatrix}. \tag{23}$$

The left-hand side of (23) and (21) enable us to write

$$\det P_{cl}(z) = \det(zI - KA) \det G_{cl}(z) = z^{-m} z^{-(\nu_1+\cdots+\nu_m)} \det(zI - KA), \tag{24}$$

whereas the right-hand side of (23) yields

$$\det P_{cl}(z) = \det(L^{-1}) \det P(z). \tag{25}$$

Now (19) follows easily from (24) and (25). ∎

From (19) it follows that $\det P(z) \neq 0$, i.e. $\det P(z)$ is not identically zero. This implies, by virtue of Corollary 1, that a complex number is an invariant zero (in the sense of Definition 1(i)) of (11) if and only if it is a Smith invariant zero of (11) (i.e. if and only if it is a root of $\det P(z)$). This fact and Lemma 7 mean that the number of invariant zeros of (11) is equal to $n - m - (\nu_1 + \cdots + \nu_m)$. In particular, the system has no invariant zeros if and only if $n = m + (\nu_1 + \cdots + \nu_m)$ and this is possible if and only if $\det P(z) = \text{const} \neq 0$.

In what follows we will need the following characterization of the invariant zeros of (11).

Lemma 8. *A number $\lambda \in \mathbb{C}$ is an invariant zero of (11) if and only if λ is an o.d. zero of $S(KA, BL^{-1}, C)$.*

Proof. The claim follows immediately from Definition 1(iii) and Lemma 5. ∎

Lemma 9. *In the system (11), let*

$$u_0(k) = -L^{-1}MA(KA)^k x^0, \quad x^0 \in \mathbb{R}^n, \quad k = 0, 1, \ldots \tag{26}$$

denote an input vector sequence. Then:

(a) the corresponding solution to the state equation of (11) which passes at $k = 0$ *through an arbitrary point* $x(0) \in \mathbb{R}^n$ *has the form*

$$x(k, x(0)) = A^k(x(0) - x^0) + (KA)^k x^0, \quad k = 0, 1, 2, \dots \tag{27}$$

and the system response is equal to

$$y(k) = CA^k(x(0) - x^0) + C(KA)^k x^0, \tag{28}$$

where the s-th row, $s = 1, \dots, m$ *in the term* $C(KA)^k x^0$ *has the form*

$$\begin{cases} c_s A^k x^0 & \text{for} \quad 0 \le k \le \nu_s, \\ 0 & \text{for} \quad k \ge \nu_s + 1. \end{cases} \tag{29}$$

(b) Furthermore, if $x^0 \in S$, *where*

$$S := \bigcap_{s=1}^{m} \left(\bigcap_{l=0}^{\nu_s} \operatorname{Ker} c_s A^l \right) \subset \mathbb{R}^n$$

and $x(0) = x^0$, *then for each* $k = 0, 1, 2, \dots$ *we have* $x(k, x(0)) = (KA)^k x^0 \in S$ *and* $y(k) = 0$.

Proof. The proof of (27) follows by direct verification that $x(k, x(0))$ of (27) and $u_0(k)$ of (26) satisfy $x(k+1) = Ax(k) + Bu(k)$ at the initial condition $x(0)$. For the proof of (29) it is sufficient to use relation (iv) of Lemma 4. If $x^0 \in S$, the sequence (29) becomes zero identically. The point (iv) of Lemma 4 enables us to show that S is a KA-invariant subspace, i.e. $KA(S) \subset S$. This proves (in (b)) that $x(k, x(0)) \in S$ for each $k = 0, 1, 2, \dots$. ∎

Remark 4. Note that the first component on the right-hand side of (27) constitutes a solution to the homogeneous equation $x(k+1) = Ax(k)$ at the initial condition $x(0) - x^0$. The second component, i.e. the sequence $(KA)^k x^0$, is a particular solution to the non-homogeneous equation $x(k+1) = Ax(k) + Bu_0(k)$ which passes through the point x^0 at $k = 0$.

Remark 5. From (28) and (29) it follows that if $x(0) = x^0$, then for all k satisfying $k \ge \max_{s=1,\dots,m}\{\nu_s\} + 1$ we have $y(k) = 0$, i.e. the system response becomes equal to zero after a finite number of steps. Due to this property, the input sequence (26) can be called 'almost output-zeroing'. If (11) is asymptotically stable, then $y(k) \to 0$ as $k \to \infty$ at any fixed points $x(0)$ and x^0 of the state space.

Remark 6. The point (b) in Lemma 9 tells us that any pair $(x^0, u_0(k))$, where $x^0 \in S$ and $u_0(k)$ is of the form (26), is an output-zeroing input for (11). We can also prove the converse implication i.e. that in any output-zeroing input $(x^0, u_0(k))$ we must have $x^0 \in S$ and $u_0(k)$ of the form (26).

Lemma 10. *Let a pair $(x^0, u_0(k))$ be an output-zeroing input for (11). Then $x^0 \in S$ and $u_0(k)$ is of the form (26). Moreover, the corresponding solution $x(k, x^0) = (KA)^k x^0$ to the state equation is entirely contained in S.*

Proof. Let $y_s(k)$ denote the s-th component of the output sequence $y(k)$ (which, by assumption, equals zero identically). Moreover, let $(x^0, u_0(k))$ be an output-zeroing input and let $x(k, x^0)$ denote the corresponding solution to the state equation of (11). Thus we can write the following equalities:

$$x(k+1, x^0) = Ax(k, x^0) + Bu_0(k), \quad k = 0, 1, 2, \dots, \tag{30}$$

$$\begin{aligned} y_s(k+1) &= c_s x(k+1, x^0) \\ &= c_s A^{k+1} x^0 + \sum_{l=0}^{k} c_s A^{k-l} B u_0(l) = 0, \quad k = 0, 1, 2, \dots, \end{aligned} \tag{31}$$

$$y_s(0) = c_s x(0, x^0) = c_s x^0 = 0. \tag{32}$$

From (31), taking into account successively $k = 0, 1, \dots, \nu_s - 1$, as well as employing (12) and (32), we obtain $c_s x^0 = \dots = c_s A^{\nu_s} x^0 = 0$. Since these relations are valid for every $s = 1, \dots, m$, we have $x^0 \in S$. Premultiplying the equalities of (30) successively by $c_s, c_s A, \dots, c_s A^{\nu_s - 1}$ and taking into account that $y_s(k) = c_s x(k, x^0) = 0$, we get

$$x(k, x^0) \in \bigcap_{l=0}^{\nu_s} \operatorname{Ker} c_s A^l.$$

This, in turn, implies $x(k, x^0) \in S$, i.e. that the solution under consideration is entirely contained in the subspace S. Premultiplying (30) by $c_s A^{\nu_s}$, we obtain

$$c_s A^{\nu_s + 1} x(k, x^0) + c_s A^{\nu_s} B u_0(k) = 0, \quad s = 1, 2, \dots, m.$$

This yields

$$u_0(k) = -L^{-1} M A x(k, x^0). \tag{33}$$

Substituting (33) into (30) and using (16), we write (30) in the form

$$x(k+1, x^0) = KAx(k, x^0). \tag{34}$$

Substituting the solution to the initial-value problem (34), $x(0, x^0) = x^0$, into (33), one gets the desired form of $u_0(k)$. ∎

Combining Lemmas 9 and 10, we obtain the following characterization of the output-zeroing problem for decouplable systems.

Theorem 1. *A pair $(x^0, u_0(k))$ is an output-zeroing input for system (11) if and only if $x^0 \in S$ and $u_0(k) = -L^{-1} M A (KA)^k x^0$. Moreover, the solution to the state equation corresponding to the output-zeroing input $(x^0, u_0(k))$ is of the form $x(k, x^0) = (KA)^k x^0$ and is entirely contained in S.*

Suppose now that the system (11) is minimal. Then $S(KA, BL^{-1}, C)$ is reachable, but not necessarily observable. Let $\bar{x} = Hx$ denote a change of coordinates which leads to the decomposition of $S(KA, BL^{-1}, C)$ into an unobservable and an observable part:

$$\begin{bmatrix} (\overline{KA})_{11} & (\overline{KA})_{12} \\ 0 & (\overline{KA})_{22} \end{bmatrix}, \quad \begin{bmatrix} \bar{B}'_1 \\ \bar{B}'_2 \end{bmatrix}, \quad \begin{bmatrix} 0 & \bar{C}_2 \end{bmatrix}, \tag{35}$$

where

$$\bar{x} = \begin{bmatrix} \bar{x}_{11} \\ \bar{x}_{22} \end{bmatrix}$$

and the subsystem $((\overline{KA})_{11}, \bar{B}'_1, 0)$ is unobservable and reachable, while the subsystem $((\overline{KA})_{22}, \bar{B}'_2, \bar{C}_2)$ is a minimal realization of $G_{cl}(z)$ in (21). The matrix $(\overline{KA})_{22}$ is of order $m + (k_1 + \cdots + k_m)$ and its spectrum consists of only zero eigenvalues. These eigenvalues represent the zeros at infinity of (11) (Weller, 1999). The matrix H can be constructed in the well-known manner (Chen, 1984, p.203) from the observability matrix for $S(KA, BL^{-1}, C)$. According to Lemma 8, $\rho((\overline{KA})_{11})$ represents all the invariant zeros of (11). Moreover, using Lemma 4(iv), one can observe that the subspace S is equal to the kernel of the observability matrix for the closed-loop system, i.e.

$$\operatorname{Ker} \begin{bmatrix} C \\ C(KA) \\ \vdots \\ C(KA)^{n-1} \end{bmatrix} = S = \bigcap_{s=1}^{m} \left(\bigcap_{l=0}^{k_s} \operatorname{Ker} c_s A^l \right), \tag{36}$$

and $\dim S = n - m - (k_1 + \cdots + k_m)$.

In the new coordinates the output-zeroing problem for the system $S(\bar{A}, \bar{B}, \bar{C})$, where $\bar{A} = HAH^{-1}$, $\bar{B} = HB$, $\bar{C} = CH^{-1}$, becomes significantly simpler. At first, the image of the subspace S in (36) under H (i.e. $\bar{S} = H(S)$) takes the form

$$\bar{S} = \left\{ \bar{x} = \begin{bmatrix} \bar{x}_{11} \\ \bar{x}_{22} \end{bmatrix} : \bar{x}_{22} = 0 \right\},$$

where each vector $\bar{x}_{11} \in \mathbb{R}^{n-m-(k_1+\cdots+k_m)}$ is spanned by the eigenvectors and pseudoeigenvectors (or their real and imaginary parts) corresponding to the eigenvalues of $(\overline{KA})_{11}$ (i.e. to invariant zeros of (11)). Moreover, the general form of the output-zeroing inputs for $S(\bar{A}, \bar{B}, \bar{C})$ is

$$\left(\begin{bmatrix} \bar{x}_{11} \\ 0 \end{bmatrix}, L^{-1}MAH^{-1} \begin{bmatrix} (\overline{KA})_{11}^k \bar{x}_{11} \\ 0 \end{bmatrix} \right),$$

whereas the so-called zero dynamics (Isidori, 1995, p.164) for this system are described as

$$\bar{x}_{11}(k+1) = (\overline{KA})_{11} \bar{x}_{11}(k).$$

4. Examples

Example 1. (Latawiec, 1998; Tokarzewski *et al.*, 1999) Consider the transfer function of full row rank

$$G(z) = \begin{bmatrix} \dfrac{z-2}{z^2} & \dfrac{2z+1}{z^2} \end{bmatrix}$$

and its irreducible realization $S(A, B, C)$ with the matrices

$$A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} -2 & 1 \\ 1 & 2 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 \end{bmatrix}.$$

It is easy to check directly via Definition 1(i) that $S(A, B, C)$ is degenerate. This means, by virtue of Definition 2, that $G(z)$ possesses an infinite number of zeros. On the other hand, the rank of the system matrix does not fall below its normal rank at any point of the complex plane. This means that $G(z)$ has no Desoer-Schulman zeros.

Example 2. (Latawiec, 1998; Tokarzewski *et al.*, 1999) Consider the transfer matrix

$$G(z) = \begin{bmatrix} \dfrac{z-2}{(2z-1)(3z+1)} & \dfrac{1}{(2z-1)(3z+1)} \end{bmatrix}$$

and its minimal state-space realization with the matrices

$$A = \begin{bmatrix} -\dfrac{11}{6} & 1 \\ -\dfrac{21}{6} & 2 \end{bmatrix}, \quad B = \begin{bmatrix} \dfrac{1}{6} & 0 \\ 0 & \dfrac{1}{6} \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

In the sense of Definitions 1(i) and 2, the system is degenerate, i.e. each complex number is a zero of $G(z)$. In particular, the triples

$$\lambda = 2 + j1, \quad x^0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad g = \begin{bmatrix} -6 \\ j6 \end{bmatrix}$$

and

$$\lambda = 2 - j1, \quad x^0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad g = \begin{bmatrix} -6 \\ -j6 \end{bmatrix}$$

satisfy (2). For these triples we can find, according to Remark 1, the following output-zeroing input sequences:

$$u(k, \operatorname{Re} x^0) = (\sqrt{5})^k \begin{bmatrix} -6\cos k\varphi \\ -6\sin k\varphi \end{bmatrix}$$

and

$$u(k, \operatorname{Im} x^0) = (\sqrt{5})^k \begin{bmatrix} -6\sin k\varphi \\ 6\cos k\varphi \end{bmatrix},$$

where $\varphi = \arctan \frac{1}{2}$, $k = 0, 1, 2, \ldots$.

The input sequence $u(k, \mathrm{Re}\, x^0)$ yields the solution to the state equation of the form

$$x(k, \mathrm{Re}\, x^0) = (\sqrt{5})^k \begin{bmatrix} 0 \\ \cos k\varphi \end{bmatrix}$$

provided that $u(k, \mathrm{Re}\, x^0)$ is applied to the system exactly at the moment when the initial state of the system is equal to

$$\mathrm{Re}\, x^0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Similarly, the input sequence $u(k, \mathrm{Im}\, x^0)$ produces the solution

$$x(k, \mathrm{Im}\, x^0) = (\sqrt{5})^k \begin{bmatrix} 0 \\ \sin k\varphi \end{bmatrix},$$

provided that this input is applied at the initial state

$$\mathrm{Im}\, x^0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

5. Concluding Remarks

The definition of invariant zeros adopted in this paper (Definition 1(i)), based in a natural way on the notions of state-zero and input-zero directions introduced in (MacFarlane and Karcanias, 1976), is strictly linked with the output-zeroing problem and for this reason these zeros have a clear dynamical interpretation. In view of Lemma 2, each Smith invariant zero can be easily interpreted in the context of the output-zeroing problem, although, as indicated in Examples 1 and 2, the Smith invariant zeros do not characterize completely the problem.

Of course, degeneracy is not restricted only to MIMO systems. It may also appear in SISO systems. However, as is possible to show, a SISO system is degenerate if and only if its transfer function equals zero identically (i.e. $G(z) \equiv 0$). As an example of a degenerate system, one can consider the system (1) with the matrices

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & \frac{1}{2} & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$$

which, according to Definition 1(i), is degenerate and, on the other hand, it has exactly one Smith invariant zero $\lambda = 1/2$.

Finally, some remark concerning the Davison-Wang zeros should be made. As is known (Hewer and Martin, 1984; Schrader and Sain, 1989), the Davison-Wang definition determines the zeros of (1) as those points of the complex plane for which the rank of the system matrix drops below $n+\min\{m,r\}$. This definition also admits an infinite number of zeros (Hewer and Martin, 1984). However, as the following examples show, Definition 1(i) and the Davison-Wang definition are in general not comparable. The system of Example 1 is degenerate in the sense of Definition 1(i) and has no Davison-Wang zeros. On the other hand, a minimal system (1) with the matrices

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 3 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

has no zeros according to Definition 1(i), while each complex number is its Davison-Wang zero.

Further research should be focused on questions concerning algebraic criteria of degeneracy (non-degeneracy), as well as on zeros in sampled data systems (cf. Weller, 1999).

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