Int. J. Appl. Math. Comput. Sci., 2000, Vol.10, No.3, 559–573

LEAST-SQUARES ESTIMATION FOR A LONG-HORIZON PERFORMANCE INDEX

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Estimation of a parametric, discrete-time model for a SISO dynamic plant, derived for minimisation of a performance index determined as a sum of squared prediction errors within some time horizon is considered. A formula for a Long-Horizon Least-Squares (LHLS) off-line solution as well as a theorem for an LHLS recursive on-line scheme are derived. The LHLS scheme reveals some features of Least-Squares (LS) estimation and Instrumental-Variable (IV) estimation. An algorithm for the on-line LHLS scheme is presented and compared with LS and IV estimation schemes for a linear, second-order system. The fast convergence of the derived LHLS on-line scheme is demonstrated in the case of detecting changes in parameters of a non-stationary system.

Keywords: identification, least-squares estimation, prediction, recursive scheme

1. Introduction

Recent developments in LSI chips have broadened applications of advanced methods in industrial control systems. Intelligent measurement systems are equipped with various procedures for self-testing and automatic diagnosis. PID controllers possess many additional options including self-tuning and adaptation schemes. The control methods of 'upper shelf', called now the advanced control, are usually based on predictive schemes, where parametric models of plants or disturbances are used. The main idea of the predictive approach is based on the evaluation of control effects within some interval in the future on which these effects are expected. The discrete-time models used for these problems may be of different nature. Linear, discrete-time difference equations constitute local approximations to the investigated process dynamics. Models parameterised by fuzzy sets of different impacts offer better approximations. Artificial neural nets are supposed to be most general representations of system dynamics. All these representations share one common feature: the corresponding determination rules are derived for prediction of the model output $\hat{y}(\cdot)$ for a single step (one or a multiple of a sampling interval Δ) ahead with respect to the actual time t,

$$Q = \|\hat{y}(t + p\Delta)\|, \quad p \geq 1. \tag{1}$$

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The index of the model quality Q is defined by some measure $||\cdot||$, e.g. an estimated value of the squared prediction error $||\hat{y}(t+p\Delta)|| = [\hat{y}(t+p\Delta) - y(t+p\Delta)]^2$ for the measured output $y(\cdot)$. In predictive control, weighted effects of control over a prediction horizon H usually estimate the system quality

$$Q_C = \sum_{i=1}^{H} \alpha_i \, ||y_d(t+i\Delta)||_C \quad \alpha_i > 0, \quad i = 1, \dots, H. \tag{2}$$

In much the same way as in (1), the measure $||\cdot||_C$ of the controller quality can be defined as the difference between a desired output $y_d(\cdot)$ and $y(\cdot)$ in the form $||y_d(t+p\Delta)|| = [y_d(t+p\Delta) - y(t+p\Delta)]^2$. The weighting coefficients α_i are equal for $i = 1, \dots, H$ or they can be decreasing. A discrete-time model (1) does not fully match the predictive control problem (2) which corresponds to some interval $[t+\Delta,\, t+H\Delta]$ in the future. We wish to investigate the problem whether it is possible to derive a model for an index more general than (1), e.g. based on some time interval in the future and corresponding to the structure defined by (2). This index will be defined as

$$Q_I = \sum_{p=1}^{H} \alpha_p \Big[y(t+p\Delta) - \hat{y}(t+p\Delta)\Big]^2, \quad \alpha_p \geq 0. \tag{3}$$

The positive coefficients α_p weigh the prediction errors for different p's. All the predicted values of the output $\hat{y}(t+p\Delta)$ are calculated with the same model M. This index of the model estimation quality will better fit to the application in the sense of the predictive controller index Q_C. However, in spite of the fact that Q_C and Q_I have similar forms, they are completely different. The controller index Q_C usually contains optimally-tuned output values of the controller $u(\cdot)$ and is referred to as the desired trajectory $y_d(\cdot)$. The estimation index Q_I is defined by the differences between the measured and predicted process output values. Both the indices produce averaging effects on the prediction errors over some time interval $[t+\Delta,\, t+H\Delta]$.

An immediate question is whether the discrete-time models determined with evaluation (1) of the one-step prediction error (Box and Jenkins, 1970; Draper and Smith, 1981; Eykhoff, 1974; Ljung, 1987; Söderstrom and Stoica, 1994) do not cover this problem and what differences are expected. The answer is not easy. In the case of well-conditioned identification tasks, stationary linear plants with well-prepared experiments (sufficient plant excitations, not excessive disturbances with proper statistic distribution) are usually investigated. A proper estimation scheme is used and a proper model structure is known, etc. There is some justified hope (Ljung, 1987; Söderstrom and Stoica, 1994) for determining a model that will correspond to a real representation of the system under investigation. This model will be the best and there will be no real need for another. In practical applications, the above-mentioned conditions are not common, and therefore the problem of estimating a model fitted to the index other than (3) should rather be investigated.

Some practical observations can support this way of model estimation. The predictive control schemes usually represent different behaviours rather than other strategies of advanced control, e.g. state-space controllers with observers. The state-space control schemes are tuned to the performance index (2), but with prediction horizon $H = 1$. They have superb dynamics and very fast transients in step responses, but require powerful controller outputs with large instantaneous variations. In the predictive approach, the controller looks forward, anticipates future reactions of the plant for a horizon $H > 1$ and does not need violent actions as is the case in state-space control. The reactions of the predictive-control system are smooth and demand less controller effort than in state-space control. A usual way of introducing more refined control in industrial applications is the predictive control approach. Rare applications of state-space control are limited to the cases when the plant is well-known and there is a demand for a very fast reaction of the system. This observation has suggested the idea of the proposed approach to fit the estimated model of the plant to the task of predicting plant reactions for some interval in the future, not only for one time instant.

The derivation of a least-squares estimation scheme for a discrete-time model with the LH criterion (3) is presented in the next section. The recursive scheme for a least-squares-like estimation algorithm is presented in Section 3. Section 4 contains simulation examples and comparisons with the results for simple LS and instrumental-variable (IV) schemes.

2. Long-Horizon Least-Squares (LHLS) Estimation

Let us consider a linear difference equation of order n representing a dynamic SISO plant with a sampling interval Δ:

$$y(t) = \sum_{i=1}^{n} \alpha_i y(t - i\Delta) + \sum_{i=0}^{n} \beta_i u(t - i\Delta - d\Delta) + \eta(t). \tag{4}$$

The delay of the output signal $y(t)$ in reacting to changes in the input $u(t)$ is equal to $d\Delta$. The impact of non-observable disturbances is represented by an additive zero-mean random signal $\eta(t)$. In the sequel, the discrete-time argument $t = k\Delta$ is replaced by k and any time shift $p\Delta$ by p. The following relation constitutes the one-step prediction model based on the estimates $\hat{\alpha}_i$ and $\hat{\beta}_i$ which form the vector of model coefficients θ:

$$\hat{y}_1(k) = \sum_{i=1}^{n} \hat{\alpha}_i y(k - i) + \sum_{i=0}^{n} \hat{\beta}_i u(k - i - d) = w_1(k)\theta,$$

where

$$w_1(k) = \Big[y(k-1), y(k-2), \dots, y(k-n), u(k-d), u(k-1-d), \dots, u(k-n-d)\Big], \tag{5}$$

$$\theta = [\hat{\alpha}_1, \hat{\alpha}_2, \dots, \hat{\alpha}_n, \hat{\beta}_0, \hat{\beta}_1, \dots, \hat{\beta}_n]^T.$$

The vector of model inputs $w_1(k)$ is determined for one-step prediction. The next relations define the predictions $\hat{y}_p(k)$ of the output signal for p steps ahead, composed of the values of the measured output $y(k - i\Delta)$, predicted output $\hat{y}_{p-i}(k)$, $i = 1, \dots, p-1$, and input signal $u(\cdot)$:

$$\hat{y}_p(k) = \sum_{i=1}^{p-1} \hat{\alpha}_i \hat{y}_{p-i}(k-i) + \sum_{i=p}^{n} \hat{\alpha}_i y(k-i) + \sum_{i=0}^{n} \hat{\beta}_i u(k-i-d) = w_p(k)\theta,$$

where

$$\begin{aligned} w_p(k) &= \Big[\hat{y}_{p-1}(k-1), \dots, \hat{y}_1(k-p+1), y(k-p), y(k-p-1), \dots, y(k-n), \\ &\qquad u(k-d), u(k-1-d), \dots, u(k-n-d)\Big], \quad p = 1, \dots, H, \qquad (6) \\ \theta &= [\hat{\alpha}_1, \hat{\alpha}_2, \dots, \hat{\alpha}_n, \hat{\beta}_0, \hat{\beta}_1, \dots, \hat{\beta}_n]^T. \end{aligned}$$

The components of the input vectors $w_p(k)$, corresponding to the future values of the output signal (with respect to the time instant $k - p$) are unknown and replaced by their estimates. The values of the input signal $u(k + i)$ are known. The components of the input vectors (5) and (6) are of a different nature (measurements or estimates), but the model coefficients are the same.

Now let us introduce the following vector notation for the input-output data:

$$W_p(M) = \begin{bmatrix} w_p(1) \\ \vdots \\ w_p(M) \end{bmatrix}, \quad Y(M) = \begin{bmatrix} y(1) \\ \vdots \\ y(M) \end{bmatrix}. \qquad (7)$$

The vector of the output predictions for p steps and the corresponding vector of the model error values determined by (6) are respectively given by

$$\hat{Y}_p(M) = W_p(M)\theta, \quad \varepsilon_p(M) = Y(M) - \hat{Y}_p(M). \qquad (8)$$

The LS estimation of the coefficients in (4) consists in minimisation of the performance index (1) where the measure $\|\cdot\|$ is the sum of the squared errors in $\varepsilon_1(M)$, i.e.

$$Q_{LS} = \varepsilon_1^T(M)\varepsilon_1(M). \qquad (9)$$

In the case of estimation for multi-step prediction, as in (3), θ is has to minimise the index

$$Q_{LH} = \sum_{p=1}^{H} \alpha_p \varepsilon_p^T(M)\varepsilon_p(M), \quad \alpha_p > 0. \qquad (10)$$

The problem is similar to the minimisation of (9), since θ has to minimise the quadratic form (10). With the notation (7)–(8), the performance index Q_{LH} can be rewritten in the form

$$\begin{aligned} Q_{LH} &= \sum_{p=1}^{H} \alpha_p \varepsilon_p^T(M)\varepsilon_p(M) \\ &= \sum_{p=1}^{H} \alpha_p \Big(Y(M) - W_p(M)\theta\Big)^T \Big(Y(M) - W_p(M)\theta\Big). \qquad (11) \end{aligned}$$

The minimum of Q_{LH} entails the condition $\partial Q_{LH}/\partial\theta = 0$, which amounts to

$$\sum_{p=1}^{H} \alpha_p W_p(M)^T Y(M) = \left[\sum_{p=1}^{H} \alpha_p W_p(M)^T W_p(M)\right] \theta. \tag{12}$$

If $\sum_{p=1}^{H} \alpha_p W_p(M)^T W_p(M)$, is invertible, the above relation yields the LHLS estimate

$$\theta_{LHLS} \triangleq \left[\sum_{p=1}^{H} \alpha_p W_p(M)^T W_p(M)\right]^{-1} \sum_{p=1}^{H} \alpha_p W_p(M)^T Y(M). \tag{13}$$

Theorem 1. *The estimate (13) of the discrete-time model (4) minimises the long-horizon performance index (3).*

Proof. The relation $\partial Q_{LH}/\partial\theta = 0$ is a necessary and sufficient condition for a global minimum of the convex form (11). The definition (7) of $W_p(M)$ and weight coefficients α_p in (10) induce the non-singularity of the matrix $\sum_{p=1}^{H} \alpha_p W_p(M)^T W_p(M)$, for a sufficiently exciting input signal $u(\cdot)$. ■

For the one-step prediction model and $\alpha_1 = 1$, (13) reduces to the simple LS estimate (Ljung, 1987; Söderstrom and Stoica, 1994; Strejc, 1981)

$$\theta_{LS} \triangleq \left[W_1(M)^T W_1(M)\right]^{-1} W_1(M)^T Y(M). \tag{14}$$

The pseudo-inversion formula for a single matrix $W_1(M)^T W_1(M)$ in (14) is known (Isermann, 1982) and an on-line LS estimation algorithm can be introduced. The LHLS estimate (13) consists of many terms corresponding to different values of p and, consequently, the matrix inverted in (13) is more complex than for LS estimation (14). This difference can produce some difficulties while deriving the on-line LHLS estimation scheme.

Theorem 2. *Consider a symmetric matrix Q of the form*

$$Q = \alpha_1 Q_1 + \alpha_2 Q_2 + \cdots + \alpha_n Q_n, \quad \alpha_i > 0 \quad i = 1, \ldots, n, \tag{15}$$

where the matrices $Q_1, Q_2, \ldots, Q_n \in \mathbb{R}^{m\times m}$ are symmetric and perturbed by $\Delta Q_i = q_i^T q_i$, $q_i \in \mathbb{R}^m$, $i = 1, \ldots, n$, respectively. If the inverse of Q is known, then the inverse of the modified matrix $Q' = Q + \alpha_1 \Delta Q_1 + \cdots + \alpha_n \Delta Q_n$ can be calculated by successive application of the matrix inversion rule

$$[A + BCD]^{-1} = A^{-1} - A^{-1}B\left[DA^{-1}B + C^{-1}\right]^{-1} DA^{-1}, \tag{16}$$

where $A \in \mathbb{R}^{m\times m}$, $B \in \mathbb{R}^{m\times p}$, $C \in \mathbb{R}^{p\times p}$, $D \in \mathbb{R}^{p\times m}$.

Proof. Define $P \in \mathbb{R}^{m\times m}$ as the inverse of the positive definite, square matrix Q and consider the change in Q_1 by the product $(\alpha_1^{1/2} q_1)(\alpha_1^{1/2} q_1^T)$ for the column vector $\alpha_1^{1/2} q_1, q_1 \in \mathbb{R}^m$. The formula (16) can be used for inversion of the matrix

$$Q^{(1)} = (\alpha_1 Q_1 + \alpha_1 q_1^T q_1) + \cdots + \alpha_n Q_n = Q + \alpha_1 q_1^T q_1$$

after setting $A = P^{-1}$, $C = 1 \in \mathbb{R}$, $B = \alpha_1^{1/2} q_1 \in \mathbb{R}^m$, $D = \alpha_1^{1/2} q_1^T \in \mathbb{R}^m$. We get

$$\begin{aligned}\left[Q^{(1)}\right]^{-1} &= \left[Q + \alpha_1 q_1^T q_1\right]^{-1} = P - P\alpha_1^{1/2} q_1 \left[\alpha_1^{1/2} q_1^T P \alpha_1^{1/2} q_1 + 1\right]^{-1} \alpha_1^{1/2} q_1^T P \\ &= P - \frac{\alpha_1 P q_1 (P q_1)^T}{\alpha_1 q_1^T P q_1 + 1} = P - \gamma V_1 V_1^T = P^{(1)}, \end{aligned} \tag{17}$$

where

$$\gamma = \frac{\alpha_1}{\alpha_1 q_1^T P q_1 + 1}, \quad V_1 = P q_1.$$

The next modification of $Q^{(1)}$, introduced by the term $(\alpha_2^{1/2} q_2)(\alpha_2^{1/2} q_2^T)$ corresponding to the variation in Q_2, results in a transformation similar to (17), where $P^{(1)}$ is used in place of P, α_2 substituted for α_1, and q_1 replaced by q_2. The other steps follow by simple induction. ■

The above theorem shows that for n increments of matrices Q_i, $i = 1, \dots, n$ the calculation corresponding to (17) should be repeated n times. It is not possible to shorten this approach by one-step modification of the matrix Q in form $\Delta Q = \alpha_1 \Delta Q_1 + \cdots + \alpha_n \Delta Q_n$. The modification ΔQ cannot be represented as a product qq^T for some vector $q \in \mathbb{R}^m$ in a general case.

When a non-stationary behaviour of the identified plant is expected, the factor 1 in the denominator of the definition of γ can be replaced by a forgetting factor ρ lying within the interval $(0.9, 1.0]$.

3. Recursive LHLS-Estimation Algorithm

Theorem 2 provides a basis for the recursive LHLS scheme. For that purpose, consider a model of the form (4). It is represented by a difference equation of order n with discrete time delay d. In the case of a prediction horizon $H > 1$ the calculations have to be shifted H steps back with respect to the current discrete time instant k. The input values up to time moment k are considered as known, but the values of the output are considered as measured up to time $k - H$ and all the consecutive values of the output are replaced by the corresponding estimates.

The steps of the estimation in the LHLS sense are as follows:

1. At a discrete-time instant k determine the values

$$\begin{aligned} e_1(k - H + 1) &= y(k - H + 1) - \hat{y}(k - H + 1) = y(k - H + 1) \\ &\quad - w_1(k - H + 1)\theta_k, \end{aligned}$$

$$
\begin{aligned}
w_1(k-H+1) &= \Big[y(k-H),\dots,y(k-H-n),u(k-H-d+1),\dots,\\
&\qquad u(k-H-d-n+1)\Big],\\
e_2(k-H+2) &= y(k-H+2)-\hat{y}(k-H+2)\\
&= y(k-H+2)-w_2(k-H+2)\theta_k,\\
w_2(k-H+2) &= \Big[\hat{y}(k-H+1),y(k-H)\dots,y(k-H-n+1),\dots,\\
&\qquad u(k-H-d-n+2)\Big],\\
\hat{y}(k-H+1) &= w_1(k)\theta_k \qquad (18)\\
&\ \vdots\\
e_H(k) &= y(k)-\hat{y}(k) = y(k)-w_H(k)\theta_k,\\
w_H(k) &= \Big[\hat{y}(k-1),\hat{y}(k-2)\dots,\hat{y}(k-n),u(k-d),\dots,\\
&\qquad u(k-d-n)\Big],\quad k>H,\\
\hat{y}(k-1) &= w_{H-1}(k-1)\theta_k,
\end{aligned}
$$

where the estimated values of the output $\hat{y}(k-H+1),\hat{y}(k-H+2),\dots$ are sequentially substituted for the measured values of the output $y(k-H+1),y(k-H+2),\dots$ in the relations for the model inputs $w_2(k-H+2),w_3(k-H+3),\dots$.

2. Determine the LHLS estimates of the model by successive recalculation of the coefficients of θ^i_{k+1}:

$$
\begin{aligned}
&P^0(k+1) = P^H(k),\\
&V_i = P^{i-1}(k+1)w_i(k)^T,\quad i=1,\dots,H,\\
&\gamma_i = \frac{\alpha_i}{\rho+\alpha_i w_i(k)V_i}, \qquad (19)\\
&P^i(k+1) = P^{i-1}(k+1)-\gamma_i V_i V_i^T,\\
&\mu_i = \gamma_i P^i(k+1)w_i(k)^T,\\
&\theta^i_{k+1} = \theta^{i-1}_{k+1}+\mu_i e_i(k),
\end{aligned}
$$

where $P^H(k)$ is the last modification of the matrix P determined for the discrete-time instant $k-1$. The forgetting factor ρ can be defined as in Section 2.

The initial conditions for this scheme are classical (Ljung, 1987; Söderstrom and Stoica, 1994):

$$P^1(0) = \operatorname{diag}(\lambda) \in \mathbb{R}^{m\times m}, \quad \lambda \in [10^5, 10^7], \quad \theta_0^1 = 0, \quad \theta_0 \in \mathbb{R}^m. \tag{20}$$

Investigation of the convergence properties of (18)–(20) is rather complex. As regards the model input vectors $w_i(k)$, $i = 1,\ldots,n-1$ in (18), one can observe the components $y(k-i)$ which contribute to the output disturbances. These terms may introduce a bias in estimates as is observed in the usual LS estimation. On the other hand, these terms are subsequently replaced by estimated terms with index i increasing within $i = 2,\ldots,H$. In the case of the prediction index $i > n$, these vectors consist only of the estimated output values, as is the case in the instrumental-variable estimation. Hence the algorithm (19) can reveal some features inherent to LS estimation (robust convergence) and appropriate to IV estimates (very fast, unbiased estimates). The final evaluation of the quality of the introduced algorithm and conclusions can be formulated after some tests of a statistic character. Some results of these tests will be presented in the next section.

The above scheme has also been used for the SISO model of the form (4), but it can be used for other representations such as AR, FIR or MISO models.

4. Tests of LHLS-Estimation

Tests of LHLS estimation were performed for identification of a second-order, SISO linear stationary system described by the difference equation

$$y(k) = a_1 y(k-1) + a_2 y(k-2) + b_1 u(k-1) + b_2 u(k-2) + \lambda\varepsilon(k), \tag{21}$$

where $a_1 = 1.5$, $a_2 = -0.7$, $b_1 = 1.0$, $b_2 = -0.5$, $k = 1,\ldots,500$, $y(l) = u(l) = \varepsilon(l) = 0$, $l \leq 2$. The signals $y(k)$ and $u(k)$ are the measured output and input of the system, respectively, and $\varepsilon(k)$ represents a non-observable disturbance. The exciting signals $u(k)$ and $\varepsilon(k)$ were simulated by random generators. The influence of the disturbance $\varepsilon(k)$, expressed by the factor λ, was varying in the interval $[0.05, 1.0]$.

Initial trials with the proposed scheme revealed relatively fast convergence of the estimates for the model coefficients. This property can be very interesting in the case of tracking model changes and therefore, in this study this aspect was the main focus of interest.

The estimation quality could be compared using statistically determined (averaged values estimated in L simulation runs) final values of the estimated model coefficients with appropriate standard coefficient deviations. However, this information would have not represented the quality of the estimation convergence. For a better expression of the convergence speed and accuracy of estimation, an overall index of the absolute errors for each coefficient was introduced:

$$\rho\theta_i = \frac{1}{L}\sum_{p=1}^{L}\frac{1}{450}\sum_{k=50}^{500}\left|\hat{\theta}_{ip}(k) - \theta_i^0\right|, \tag{22}$$

where the upper value of discrete time $k = 500$ means that each simulation run contained 500 steps, but the quality of estimation was evaluated from step $k = 50$, for avoiding excessive errors that sometimes appeared in initial estimation steps.

The results of recursive LHLS estimation (19) of the coefficients a_1, a_2, b_1, b_2 in (21) were compared with the results determined by a simple recursive LS scheme with the same algorithm (19), but for $H = 1$, and the results obtained from the instrumental variable (IV) algorithm. In the case of the IV estimation scheme, the first 50 steps were calculated with the LS algorithm and then, in the vector of instruments, the output values were replaced by the corresponding estimates.

The LHLS estimate (18)–(19) was calculated for $H = 2, 4, 6$ with the weighting coefficients

$$\alpha_i = \frac{1}{1+0.1i}, \quad i = 1, \dots, H. \tag{23}$$

The coefficients α_i were determined as slightly decreasing within the prediction interval. In the case of equal weights, failures in the convergence were sometimes observed. This small decrease introduced by (23) was sufficient to yield very good convergence features.

The noise impact was determined by factor $\lambda = 0.05$, 0.1, 0.2, 0.5 and 1.0. For a more precise evaluation of noise, the following index was calculated:

$$N/S = \frac{1}{L}\sum_{p=1}^{L} \frac{\sum\limits_{k=1}^{500} y_\varepsilon(k)^2 - \sum\limits_{k=1}^{500} y(k)^2}{\sum\limits_{k=1}^{500} y(k)^2}. \tag{24}$$

In the above expression, $y(k)$ denotes the value of the simulated output for the simulation performed with the noise amplification factor $\lambda = 0$ and $y_\varepsilon(k)$ denotes the value for the same random sequence of the excitation signals u and ε, but with $\lambda \neq 0$.

The mean results based on $L = 20$ simulation runs for a variable impact λ of the white noise disturbance ε are presented in Table 1. In the case of the LHLS algorithm the results calculated for a short prediction horizon $H = 2$ are presented. The results of this estimation procedure determined for larger numbers of steps $H = 4$ and $H = 6$ yielded errors $\rho\theta$ in (22) approximately 15 and 30% greater than for $H = 2$, respectively.

The direct inspection of the results shown in Table 1 confirms the obvious conclusion that an increase in the noise impact yields deterioration of the estimates. However, there is no straight relation between the magnitudes of noise and errors. The increment in the noise by the factor of 10 (in magnitude) is not followed by the corresponding magnitude of the errors. In the case of the LS estimates, the errors increased by only 40–50%. This influence is more visible in LHLS estimates. The errors were increased by 5 to 6 times. The results for IV estimation and a low magnitude of disturbances were reasonable, but for $\lambda \geq 0.5$ this method failed. Poor estimations of the IV algorithm resulted from the bad quality of the initial values

Table 1. Estimates of model coefficients with white noise disturbances.

Noise impact λ and N/S ratio	Method	$\hat{a}_1$ ($a_1^0 = 1.5$)	$\hat{a}_2$ ($a_2^0 = -0.7$)	$\hat{b}_1$ ($b_1^0 = 1.0$)	$\hat{b}_2$ ($b_2^0 = -0.5$)
$\lambda = 0.1$ $N/S = 0.11$	LS	1.468	−0.681	0.980	−0.482
		(0.085)	(0.052)	(0.055)	(0.054)
	IV	1.536	−0.733	0.983	−0.550
		(0.077)	(0.057)	(0.054)	(0.070)
	LHLS	1.500	−0.700	1.002	−0.499
		(0.010)	(0.008)	(0.005)	(0.012)
$\lambda = 0.2$ $N/S = 0.28$	LS	1.467	−0.681	0.982	−0.481
		(0.087)	(0.052)	(0.055)	(0.052)
	IV	1.525	−0.725	0.983	−0.543
		(0.070)	().049)	(0.054)	(0.059)
	LHLS	1.498	−0.700	1.003	−0.496
		(0.019)	(0.016)	(0.010)	(0.023)
$\lambda = 0.5$ $N/S = 0.79$	LS	1.452	−0.675	0.993	−0.471
		(0.106)	(0.061)	(0.053)	(0.065)
	IV	3.472	−1.224	1.431	−2.868
		0.114	0.064	0.060	0.081
	LHLS	1.484	−0.695	1.014	−0.472
		(0.050)	(0.042)	(0.036)	(0.062)
$\lambda = 1.0$ $N/S = 1.62$	LS	1.449	−0.672	1.008	−0.473
		(0.109)	(0.069)	(0.063)	(0.080)
	IV	—	—	—	—
		—	—	—	—
	LHLS	1.487	−0.693	0.995	−0.491
		(0.058)	(0.050)	(0.074)	(0.097)

for the instruments, sufficiently precise after only 50 steps of the LS algorithm. In the case of repetitive calculation, performed in all off-line schemes, this method is more efficient. It can be observed that the LHLS estimation algorithm for low and intermediate noise levels $\lambda = 0.1 \div 0.5$ is visible better than for other schemes. For $\lambda = 0.1$ the errors $\rho\theta$ are less than 1%. This means that after the initial 50 steps of the estimation the mean error in each coefficient was about 1% of the nominal value.

The other methods produce visibly worse results. In the case of $\lambda = 1.0$ the results of LHLS are comparable with the LS algorithm.

The white-noise distortions usually constitute an effect of measurement disturbance. In the case of modern equipment this influence is rather low with respect to the magnitude of the measured signal. Hence the observed good efficiency of the LHLS estimation at a low level of white noise distortions is more important than in the case of large noise magnitudes.

Table 2 shows the results determined for estimation tests organised as in the case of white noise, but with the noise simulated in the form of the correlated signal

$$\varepsilon(k) = 0.9\varepsilon(k-1) + 0.1\eta(k). \tag{25}$$

The signal $\eta(k)$ was a random sequence. The comparison of the estimation efficiency for all methods is presented in Table 2.

In the presence of coloured noise, the LHLS estimation confirmed its superiority, mainly for low magnitudes of disturbances. In this case a longer horizon $H = 6$ was more suitable. The LS and IV for all the magnitudes of disturbances provided very similar results. The errors in the estimated coefficients a_1, a_2, b_1 were less for the IV than for the LS-method, but the IV yielded significant errors in the estimation of the b_2 coefficient. It can be noted that the IV method failed at a proper evaluation of the sign of the gain for the input u when $\lambda = 0.1$. The same effect can be observed for the LS estimates but for $\lambda = 1.0$. This problem did not appear for LHLS estimates. The better estimates for the LHLS algorithm can be explained by the fact that in each step of the recursive LHLS algorithm the calculation of the estimates is repeated many times (six for $H = 6$) and hence there is a possibility of a significant improvement in the convergence rate and accuracy.

After the inspection of the estimates one can observe the influence of the number H of predictions instants on the convergence speed and accuracy. In the case of a white noise distortion a lower number of prediction instants $H = 2$ was more suitable, because the accuracy of the estimates for $H = 4, 6$ was not sufficient to compensate for the errors in the model output created by white noise distortions. For the correlated noise the output errors can be modelled to some extent by a proper estimation of the model coefficients and thus the increased number $H = 6$ was justified. The influence of the number H on the convergence speed is shown in Fig. 1. The estimates of the coefficient a_1 determined from simulated data with correlated noise disturbance for factor $\lambda = 0.2$ are reported. Figure 1 presents the transients of the estimates of a_1 calculated for $H = 2, 4, \ldots, 14$ and the LS estimation. The LHLS-6 or LHLS-8 estimation procedures started with delays of 4 and 6 time instants in comparison with the LS estimation, respectively, but the desired level was reached at 12–14 instants before the LS estimation.

The advantage of the very fast convergence of the estimates can be very useful in identification of non-stationary systems or detection of abrupt changes in model parameters. This effect was investigated by estimation of the coefficient b_1 simulated in a non-stationary way:

$$b_1(k) = 1 - \sin(0.015k), \quad k = 1, \ldots, 500. \tag{26}$$

Table 2. Estimates of the model coefficients for coloured noise disturbances.

Noise impact λ and N/S ratio	Method	$\hat{a}_1$ ($a_1^0 = 1.5$)	$\hat{a}_2$ ($a_2^0 = -0.7$)	$\hat{b}_1$ ($b_1^0 = 1.0$)	$\hat{b}_2$ ($b_2^0 = -0.5$)
$\lambda = 0.05$ $N/S = 0.001$	LS	1.412	−0.576	0.958	−0.738
		(0.302)	(0.293)	(0.122)	(0.282)
	IV	1.720	−0.741	0.935	−1.057
		(0.279)	(0.229)	(0.136)	(0.496)
	LHLS	1.494	−0.694	1.003	−0.496
		(0.017)	(0.012)	(0.018)	(0.032)
$\lambda = 0.1$ $N/S = 0.034$	LS	1.436	−0.588	0.960	−0.769
		(0.284)	(0.279)	(0.123)	(0.296)
	IV	1.722	−0.745	0.937	−1.058
		(0.277)	(0.228)	(0.135)	(0.488)
	LHLS	1.498	−0.690	1.011	−0.516
		(0.035)	(0.024)	(0.034)	(0.067)
$\lambda = 0.2$ $N/S = 0.192$	LS	1.489	−0.616	0.964	−0.837
		(0.265)	(0.250)	(0.124)	(0.330)
	IV	1.725	−0.751	0.941	−1.060
		(0.271)	(0.222)	(0.135)	(0.478)
	LHLS	1.529	−0.690	1.022	−0.597
		(0.070)	(0.042)	(0.064)	(0.152)
$\lambda = 0.5$ $N/S = 0.614$	LS	1.592	−0.683	0.979	−0.948
		(0.254)	(0.208)	(0.117)	(0.440)
	IV	1.711	−0.745	0.953	−1.059
		(0.261)	(0.202)	(0.124)	(0.514)
	LHLS	1.598	−0.700	1.027	−0.738
		(0.124)	(0.105)	(0.060)	(0.358)
$\lambda = 1.0$ $N/S = 1.28$	LS	1.656	−0.737	1.002	−1.005
		(0.234)	(0.178)	(0.133)	(0.529)
	IV	1.726	−0.765	0.980	−1.070
		(0.238)	(0.182)	(0.141)	(0.569)
	LHLS	1.760	−0.824	1.031	−0.886
		(0.258)	(0.128)	(0.104)	(0.414)

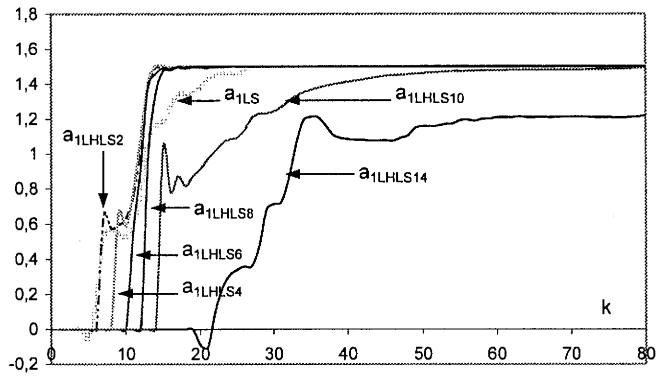


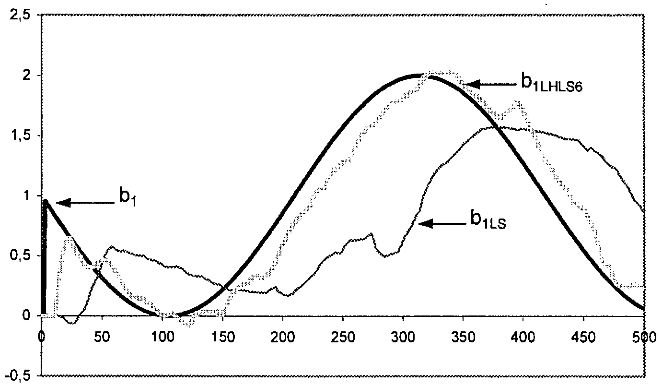
Fig. 1. Convergence rate for the LHLS estimates with different prediction horizons H.

The simulation of the system (21) was performed with the white noise disturbance for $\lambda = 0.2$. The forgetting factor ρ (19) should exert an influence on the adaptation rate of the model (Ljung, 1987; Spriet and Vansteenkiste, 1982), but a more efficient method was a periodic increase in the main diagonal of the pseudo-inverse matrix P in the algorithm (19) (Ljung, 1987). This method was used for both the recursive LS and LHLS-6 algorithms. The corresponding estimates of b_1 are presented in Fig. 2(a). A mutual correlation is usually observed at transients of the model estimates. This correlation is very strong between the coefficients corresponding to the same input or between the auto-regressive parts of the model. Then it can be reasonable to investigate the transients of the other model coefficient related to the input signal $-b_2$. In the case of a very prompt identification the estimate of this coefficient has to be constant. In Fig. 2(b) some transients of the b_2 estimates are presented. The LHLS estimates are very similar to the proper value of -0.5 when the LS estimate is visibly drifting towards this value.

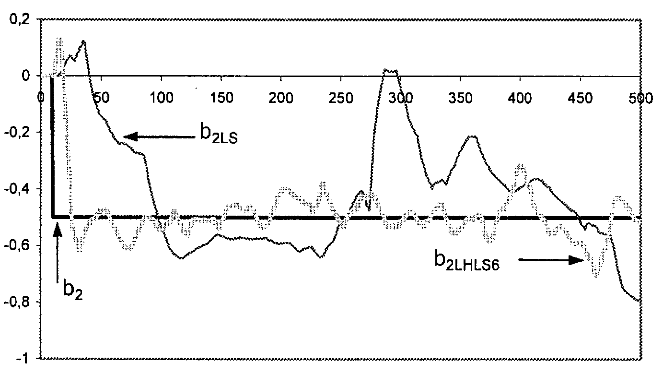
5. Conclusions

The presented idea of the LHLS estimation of the model coefficients, based on evaluation of the prediction quality for more than one sampling interval ahead, leads to the derivation of a simple and numerically effective estimation algorithm. The advantage of the LHLS approach is that the robustness and stability of the estimates are combined with very fast convergence and reduced sensitivity resulting in a bias in the estimates that are characteristic for the LS or IV algorithms. The fitting of the length of the prediction horizon H and the weighting coefficients α_i need more attention in the application of the method, but they raise a possibility for a better adaptation of the estimation algorithm to the investigated problem.

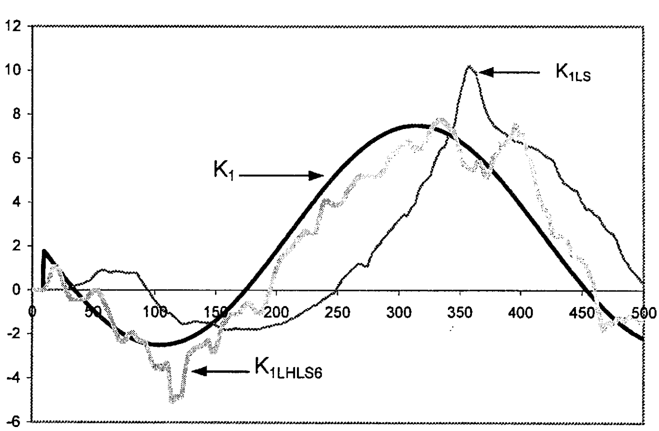
The most important advantage of the derived algorithm is the observed very fast rate of convergence that suggests positive results in the estimation of non-stationary processes. However, the examples presented here do not establish the superiority of



(a)



(b)



(c)

Fig. 2. Estimation of the b_{11} coefficient (a), the b_{12} coefficient (b), and the gain K_1 (c).

our approach in comparison with the other algorithms, but the preliminary tests are promising. This first presentation of the algorithm has to be followed by further tests made on real plants and application to industrial control and diagnostic systems.

Acknowledgements

This work was partly supported by a CATID research program in the year 2000.

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Received: 14 March 2000
Revised: 27 July 2000