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SHARP REGULARITY OF THE SECOND TIME DERIVATIVE w_{tt} OF SOLUTIONS TO KIRCHHOFF EQUATIONS WITH CLAMPED BOUNDARY CONDITIONS*

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We consider mixed problems for Kirchhoff elastic and thermoelastic systems, subject to boundary control in the clamped Boundary Conditions B.C. ("clamped control"). If w denotes elastic displacement and θ temperature, we establish optimal regularity of $\{w, w_t, w_{tt}\}$ in the elastic case, and of $\{w, w_t, w_{tt}, \theta\}$ in the thermoelastic case. Our results complement those presented in (Lagnese and Lions, 1988), where sharp (optimal) trace regularity results are obtained for the corresponding boundary homogeneous cases. The passage from the boundary homogeneous cases to the corresponding mixed problems involves a duality argument. However, in the present case of clamped B.C., and only in this case, the duality argument in question is both delicate and technical. In this respect, the clamped B.C. are 'exceptional' within the set of canonical B.C. (hinged, clamped, free B.C.). Indeed, it produces new phenomena which are accounted for by introducing new, untraditional factor (quotient) spaces. These are critical in describing both interior regularity and exact controllability of mixed elastic and thermoelastic Kirchhoff problems with clamped controls.

Keywords: Kirchhoff elastic and thermoelastic plate equations, clamped boundary conditions

1. Introduction, Motivation, Statement of Main Results on Regularity of Kirchhoff Systems with Clamped Boundary Controls

The main goal of this note is to provide sharp, in fact optimal, regularity results on the second time derivative w_{tt} of *mixed problems* involving Kirchhoff elastic and thermoelastic systems, with control acting in the clamped Boundary Conditions (B.C.). Some sharp trace regularity results for the corresponding *homogeneous* Kirchhoff elastic and thermoelastic systems are already available in the literature (Lagnese and Lions, 1988, p.123, p.157, p.158). However, the passage—by duality or transposition—from the latter homogeneous problem in (Lagnese and Lions, 1988) to the former mixed problem

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given here is both delicate and technical. In this respect, the clamped B.C. are 'exceptional' within the set of canonical B.C. (hinged, clamped, free B.C.). As we shall see, this passage will require: first, the introduction of untraditional, new function spaces (called $\tilde{L}_2(\Omega)$ and $\tilde{H}_{-1}(\Omega)$ in this paper: see (19) and (57) below); next, the study of their properties (in particular, their key characterizations as appropriate factor, or quotient, spaces, given in Proposition 3 and Proposition 4, respectively, along with the identity in (60)); finally, some untraditional and non-standard dualities, dictated by the intrinsic underlying spaces. Key regularity results of the present paper follow.

1.1. The Elastic and Thermoelastic Mixed Problems

Elastic Kirchhoff equation. Let Ω be an open bounded domain in $\mathbb{R}^n$ with smooth boundary Γ. Consider the following Kirchhoff elastic mixed problem with clamped boundary control in the unknown $w(t,x)$:

$$\begin{cases} w_{tt} - \gamma\Delta w_{tt} + \Delta^2 w = 0 & \text{in } (0,T] \times \Omega \equiv Q, \quad \text{(1a)} \\ w(0,\,\cdot\,) = w_0,\ w_t(0,\,\cdot\,) = w_1 & \text{in } \Omega, \quad \text{(1b)} \\ w|_\Sigma \equiv 0,\ \left.\dfrac{\partial w}{\partial \nu}\right|_\Sigma \equiv u & \text{in } (0,T] \times \Gamma \equiv \Sigma. \quad \text{(1b)} \end{cases}$$

In (1a), γ is a positive constant to be kept fixed throughout this paper: $\gamma > 0$. When $n = 2$, problem (1) describes the evolution of the displacement w of the elastic Kirchhoff plate model, which accounts for rotational inertia. In it, the constant γ is proportional to the square of the thickness of the plate (Lagnese, 1989; Lagnese and Lions, 1988).

Thermoelastic Kirchhoff equations. With Ω, Γ and $\gamma > 0$ as above, consider now the corresponding thermoelastic mixed problem with clamped boundary control in the unknown $\{w(t,x), \theta(t,x)\}$:

$$\begin{cases} w_{tt} - \gamma\Delta w_{tt} + \Delta^2 w + \Delta\theta = 0 & \text{in } (0,T] \times \Omega \equiv Q, \quad \text{(2a)} \\ \theta_t - \Delta\theta - \Delta w_t = 0 & \text{in } Q, \quad \text{(2b)} \\ w(0,\,\cdot\,) = w_0,\ w_t(0,\,\cdot\,) = w_1,\ \theta(0,\,\cdot\,) = \theta_0 & \text{in } \Omega, \quad \text{(2c)} \\ w|_\Sigma \equiv 0;\ \left.\dfrac{\partial w}{\partial \nu}\right|_\Sigma \equiv u;\ \theta|_\Sigma \equiv 0 & \text{in } (0,T] \times \Gamma \equiv \Sigma. \quad \text{(2d)} \end{cases}$$

Again, when $n = 2$, problem (2) describes the evolution of the displacement w and of the temperature θ (with respect to the stress-free temperature) of the thermoelastic Kirchhoff plate model, which accounts for rotational inertia (Lagnese, 1989; Lagnese and Lions, 1988).

1.2. Statement of Main Results: Optimal Interior Regularity

The following results provide optimal regularity properties for the mixed problems (1) and (2). They justify the introduction of the spaces $\tilde{L}_2(\Omega)$ and $\tilde{H}_{-1}(\Omega)$ in Sections 2.2–2.4, and Section 3, respectively.

Theorem 1. *Consider the Kirchhoff elastic problem (1) with $\{w_0, w_1\} = 0$ subject to the hypothesis that*

$$u \in L_2\big(0, T; L_2(\Gamma)\big) \equiv L_2(\Sigma). \tag{3}$$

Then, continuously,

$$\begin{cases} w \in C\big([0,T]; H_0^1(\Omega)\big), & (4) \\ w_t \in C\big([0,T]; \tilde{L}_2(\Omega)\big), & (5) \\ w_{tt} \in L_2\big(0,T; \tilde{H}_{-1}(\Omega)\big). & (6) \end{cases}$$

Conclusions (4)–(5) of Theorem 1 are proved in Section 4.4 of (Lasiecka and Triggiani, 2000b). A complementary subjectivity result is given in Theorem 1.3.1 of (Lasiecka and Triggiani, 2000b). In this note we focus on proving (6) for w_{tt}.

Theorem 2. *Consider the Kirchhoff thermoelastic problem (2) with $\{w_0, w_1, \theta_0\} = 0$, subject to the same hypothesis (3) on u. Then, the map*

$$u \in L_2\big(0,T; L_2(\Gamma)\big) \Rightarrow \begin{cases} \{w, w_t\} \in C\big([0,T]; H_0^1(\Omega) \times \tilde{L}_2(\Omega)\big), & (7) \\ [w_{tt} - \frac{1}{\gamma}\theta] \in L_2\big(0,T; \tilde{H}_{-1}(\Omega)\big), & (8) \\ \theta \in L_p\big(0,T; H^{-1}(\Omega)\big) \cap C\big([0,T]; H^{-1-\epsilon}(\Omega)\big), & \\ \qquad 1 < p < \infty;\ \forall\, \epsilon > 0, & (9) \end{cases}$$

is continuous. However, in addition, we have that

$$\begin{cases} \theta \in C\big([0,T]; L_2(\Omega)\big), \textit{ and } w_{tt} \in L_2\big(0,T; \tilde{H}_{-1}(\Omega)\big), \textit{ but } \text{not} \\ \textit{continuously in } u \in L_2\big(0,T; L_2(\Gamma)\big). \end{cases} \tag{10}$$

Again, conclusions (7)–(8) of Theorem 2 are proved in Section 5 of (Lasiecka and Triggiani, 2000b). In this note, we focus on proving (9) and (10) for w_{tt} and θ.

The present note—as well as (Lasiecka and Triggiani, 2000a; 2000b)—is stimulated by the original dual results given in (Lagnese and Lions, 1988), which deal with the trace regularity of appropriate (dual) problems. However, in the case of *clamped* B.C., the duality argument is delicate and leads to the new (factor) spaces $\tilde{L}_2(\Omega)$, $\tilde{H}_{-1}(\Omega)$, defined in the subsequent sections. Additional information is reported at the end of Section 1 of both (Lasiecka and Triggiani, 2000a; 2000b). A "clamped control" as in (1c) or (2d) was labeled "eminently reasonable" by an expert in theoretical mechanics.

Theorems 1 and 2 are the main results of this note regarding the (optimal) interior regularity of elastic and thermoelastic mixed problems, with clamped boundary controls. To achieve them, we need to introduce and study the properties of two untraditional or new spaces $\tilde{L}_2(\Omega)$ and $\tilde{H}_{-1}(\Omega)$ below, which, by consequence, have a natural invariance property built in with respect to the dynamics. These spaces occur also in describing the regularity of, say, the Kirchhoff elastic problem under irregular right-hand side. This is carried out in Section 4.1 of (Lasiecka and Triggiani, 2000b), which complements results presented in (Triggiani, 1993, Prop. 3.4), which were motivated by point control problems. We finally note that the regularity results of Theorems 1 and 2 are critical in the study of the corresponding exact controllability for elastic Kirchhoff equations, or simultaneous exact/approximate controllability of thermoelastic Kirchhoff equations, under the action of clamped boundary controls, see (Eller *et al.*, 2001a; 2001b; Triggiani, 2000).

2. The space $\tilde{L}_2(\Omega)$ and its properties

We first recall the operators which play a key role in the definition of the space $\tilde{L}_2(\Omega)$. Next, we study their relevant properties (Lasiecka and Triggiani, 2000a; 2000b).

2.1. The Operators A, $\mathcal{A}$, $\mathcal{A}_\gamma$. The Operator $A^{\frac{1}{2}}\mathcal{A}_\gamma^{-1}$

Let Ω be an open bounded domain in $\mathbb{R}^n$ with smooth boundary Γ. We define

$$Af = \Delta^2 f, \quad \mathcal{D}(A) = H^4(\Omega) \cap H_0^2(\Omega), \tag{11}$$

$$\mathcal{A}f = -\Delta f, \quad \mathcal{A}_\gamma = I + \gamma\mathcal{A}, \quad \mathcal{D}(\mathcal{A}_\gamma) = \mathcal{D}(\mathcal{A}) = H^2(\Omega) \cap H_0^1(\Omega), \tag{12}$$

so that, with equivalent norms, we have the following identifications:

$$\mathcal{D}(A^{\frac{1}{2}}) = H_0^2(\Omega), \quad \mathcal{D}(A^{\frac{1}{4}}) = \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) = \mathcal{D}(\mathcal{A}_\gamma^{\frac{1}{2}}) = H_0^1(\Omega). \tag{13}$$

The space $\mathcal{D}(\mathcal{A}_\gamma^{\frac{1}{2}})$ will *always* be endowed with the following inner product, unless specifically noted otherwise:

$$(f_1, f_2)_{\mathcal{D}(\mathcal{A}_\gamma^{\frac{1}{2}})} = (\mathcal{A}_\gamma^{\frac{1}{2}} f_1, \mathcal{A}_\gamma^{\frac{1}{2}} f_2)_{L_2(\Omega)} = (\mathcal{A}_\gamma f_1, f_2)_{L_2(\Omega)}, \quad \forall\ f_1, f_2 \in H_0^1(\Omega), \tag{14}$$

where, at this stage, we denote with the same symbol the $L_2(\Omega)$-inner product and the duality pairing $(\ \cdot\ ,\ \cdot\ )_{V' \times V}$, $V = H_0^1(\Omega)$, $V' = H^{-1}(\Omega)$ with $L_2(\Omega)$ as a pivot space (Aubin, 1972, Thm.1.5, p.51), for the last term in (14).

The following closed subspaces of $L_2(\Omega)$ play a critical role. Consider the null space $\mathcal{N}$ of the operator $(1 - \gamma\Delta) : L_2(\Omega) \to H^{-2}(\Omega) = [\mathcal{D}(A^{\frac{1}{2}})]'$, and so let

$$\mathcal{H} \equiv \big\{h \in L_2(\Omega) : (1 - \gamma\Delta)h = 0 \text{ in } H^{-2}(\Omega)\big\} = \mathcal{N}\big\{(1 - \gamma\Delta)\big\}, \tag{15}$$

be the space of 'generalized harmonic functions' in $L_2(\Omega)$. $\mathcal{H}$ depends on γ, of course. For instance, for $n = 1$, we have $\mathcal{H} = \text{span}\ \{e^{-\sqrt{1/\gamma}\,x}, e^{\sqrt{1/\gamma}\,x}\}$.

Let $\mathcal{H}^{\perp}$ be its orthogonal complement in $L_2(\Omega)$, and $\Pi = \Pi^*$ be the orthogonal projection $L_2(\Omega)$ onto $\mathcal{H}^{\perp}$:

$$\begin{cases} \mathcal{H}^{\perp} = \{f \in L_2(\Omega) : (f,h)_{L_2(\Omega)} = 0, \quad \forall\, h \in \mathcal{H}\}, \\ L_2(\Omega) = \mathcal{H} \oplus \mathcal{H}^{\perp}, \quad \Pi L_2(\Omega) = \mathcal{H}^{\perp}. \end{cases} \tag{16}$$

2.2. Definition of the Space $\tilde{L}_2(\Omega)$. Equivalent Formulations

The definition of the following space arises in duality considerations involving Kirchhoff elastic problems with clamped boundary conditions and their corresponding thermoelastic versions. This was already explained in the PDE duality analysis, beginning with (1.3.5) and leading to (1.3.12) of (Lasiecka and Triggiani, 2000b). This is also explained in Section 4 of (Lasiecka and Triggiani, 2000b): See the critical eqns. (4.1.8) and (4.4.7) of (Lasiecka and Triggiani, 2000b), in a systematic functional analytic approach. See also (Lasiecka, 1989). We consider (see (11)–(14)):

(i) the space $\mathcal{D}(A^{\frac{1}{2}}) \equiv H_0^2(\Omega)$ as a closed subspace of

$$\mathcal{D}(\mathcal{A}_\gamma) \equiv H^2(\Omega) \cap H_0^1(\Omega), \tag{17}$$

(ii) the space $\mathcal{D}(\mathcal{A}_\gamma^{\frac{1}{2}})$ as a pivot space, with norm as in (14),

$$\|f\|^2_{\mathcal{D}(\mathcal{A}_\gamma^{\frac{1}{2}})} = (\mathcal{A}_\gamma^{\frac{1}{2}} f, \mathcal{A}_\gamma^{\frac{1}{2}} f)_{L_2(\Omega)}, \quad \forall\, f \in \mathcal{D}(\mathcal{A}_\gamma^{\frac{1}{2}}) \equiv H_0^1(\Omega). \tag{18}$$

However, the space $\mathcal{D}(A^{\frac{1}{2}})$ is dense in $\mathcal{D}(\mathcal{A}_\gamma^{\frac{1}{2}})$, so the identification result $\mathcal{D}(A^{\frac{1}{2}}) \subset \mathcal{D}(\mathcal{A}_\gamma^{\frac{1}{2}}) \subset [\mathcal{D}(A^{\frac{1}{2}})]'$ in (Aubin, 1972, p.51) applies with duality with respect to $\mathcal{D}(\mathcal{A}_\gamma^{\frac{1}{2}})$ as a pivot space. We then define the (Hilbert) space $\tilde{L}_2(\Omega)$ as follows:

$$\begin{aligned} \tilde{L}_2(\Omega) \;=\; & \text{dual of the space } \mathcal{D}(A^{\frac{1}{2}}) \text{ with respect to the space } \mathcal{D}(\mathcal{A}_\gamma^{\frac{1}{2}}) \\ & \text{as a pivot space, endowed with the norm of (18) or (14).} \end{aligned} \tag{19}$$

This means the following: let $f \in \mathcal{D}(A^{\frac{1}{2}}) \equiv H_0^2(\Omega) \subset \mathcal{D}(\mathcal{A}_\gamma)$, or $\phi = A^{\frac{1}{2}} f \in L_2(\Omega)$. Then:

$$\begin{aligned} g \in \tilde{L}_2(\Omega) \Longleftrightarrow (f,g)_{\mathcal{D}(\mathcal{A}_\gamma^{\frac{1}{2}})} &= (\mathcal{A}_\gamma f, g)_{L_2(\Omega)} = \text{finite}, \quad \forall\, f \in H_0^2(\Omega), \\ &= (f, \mathcal{A}_\gamma g)_{L_2(\Omega)} = (A^{-\frac{1}{2}}\phi, \mathcal{A}_\gamma g)_{L_2(\Omega)} \\ &= (\phi, A^{-\frac{1}{2}}\mathcal{A}_\gamma g)_{L_2(\Omega)} = \text{finite}, \quad \forall\, \phi \in L_2(\Omega), \end{aligned} \tag{20}$$

where we write in the same way inner products and corresponding duality pairings.

Proposition 1. (Lasiecka and Triggiani, 2000b)

(i) *Definition (20) is equivalent to the following restatement:*

$$\begin{aligned} g \in \tilde{L}_2(\Omega) \iff & (\mathcal{A}_\gamma f, g)_{L_2(\Omega)} = (F, g)_{L_2(\Omega)} \\ = & \big((1-\gamma\Delta)f, g\big)_{L_2(\Omega)} = \big(f, (1-\gamma\Delta)g\big)_{L_2(\Omega)} = \textit{finite} \end{aligned} \tag{21}$$

$\forall\ f \in H_0^2(\Omega)$, *or* $\forall\ F \in \mathcal{H}^\perp$, *where* $F = \mathcal{A}_\gamma f = (1-\gamma\Delta)f$.

(ii) *Definition (20) is equivalent to the following restatement:*

$$g \in \tilde{L}_2(\Omega) \iff A^{-\frac{1}{2}}\mathcal{A}_\gamma g \in L_2(\Omega). \tag{22}$$

(iii) *We have the following* set-theoretic *and* algebraic *(but* not *topological, see Proposition 3 below for the topological statement, eqn. (25)) inclusion* $\tilde{L}_2(\Omega) \supset L_2(\Omega)$.

2.3. Further Description of the Space $\tilde{L}_2(\Omega)$

Proposition 2. (Lasiecka and Triggiani, 2000b) (a) *With reference to (21), we have that*

$$g \in \tilde{L}_2(\Omega) \iff \begin{cases} g \textit{ has a component } g_1 \textit{ defined by} \\ g_1 = \Pi g = g|_{\mathcal{H}^\perp} \in \mathcal{H}^\perp \subset L_2(\Omega) \\ \textit{which is the orthogonal projection of } g \\ \textit{onto } \mathcal{H}^\perp, \end{cases} \tag{23a}$$

in which case

(i)

$$(1-\gamma\Delta)g = (1-\gamma\Delta)g_1 \ \textit{in } H^{-2}(\Omega), \tag{23b}$$

(ii)

$$A^{-\frac{1}{2}}\mathcal{A}_\gamma g = A^{-\frac{1}{2}}\mathcal{A}_\gamma g_1 \in L_2(\Omega). \tag{23c}$$

(b) *Let* $g \in \tilde{L}_2(\Omega)$. *Its norm is*

$$\|g\|_{\tilde{L}_2(\Omega)} = \sup_{F\in\mathcal{H}^\perp;\|F\|_{L_2(\Omega)}=1} \{|(F, g_1)_{L_2(\Omega)}|\} = \|g_1\|_{L_2(\Omega)}. \tag{24a}$$

In particular,

$$\|h\|_{\tilde{L}_2(\Omega)} = 0, \quad \forall\ h \in \mathcal{H};\ \|F\|_{\tilde{L}_2(\Omega)} = \|F\|_{L_2(\Omega)}, \quad \forall\ F \in \mathcal{H}^\perp. \tag{24b}$$

2.4. The Space $\tilde{L}_2(\Omega)$ is Isometric to the Factor Space $L_2(\Omega)/\mathcal{H}$

Proposition 2 suggests that $\tilde{L}_2(\Omega)$ is isometric to the factor (or quotient) space $L_2(\Omega)/\mathcal{H}$, hence to $\mathcal{H}^\perp$: in $\tilde{L}_2(\Omega)$, all generalized harmonic functions $h \in \mathcal{H}$ have zero $\tilde{L}_2(\Omega)$-norm. This result is correct and is given below.

Proposition 3. (Lasiecka and Triggiani, 2000b) *The space $\tilde{L}_2(\Omega)$ as defined in (19) is isometrically isomorphic (congruent, in the terminology of [Taylor and Lay, 1980, p.53]) to the factor (or quotient) space $L_2(\Omega)/\mathcal{H}$, where $\mathcal{H}$ is defined by (15). In symbols:*

$$\tilde{L}_2(\Omega) \cong L_2(\Omega)/\mathcal{H} \cong \mathcal{H}^\perp. \tag{25}$$

Thus, if J denotes the isometric isomorphism between $\tilde{L}_2(\Omega)$ and $L_2(\Omega)/\mathcal{H}$, we then have for $g \in \tilde{L}_2(\Omega)$:

$$\|g\|_{\tilde{L}_2(\Omega)} = \big\|[Jg]\big\|_{L_2(\Omega)/\mathcal{H}} = \inf_{h\in\mathcal{H}} \|Jg - h\|_{L_2(\Omega)} = \|g_1\|_{L_2(\Omega)}, \tag{26}$$

for the unique element $g_1 = \Pi g \in \mathcal{H}^\perp$, $g_1 \in [Jg]$ (the latter being the coset or equivalence class of $L_2(\Omega)/\mathcal{H}$ containing the element Jg).

$$(x,y)_{\tilde{L}_2(\Omega)} = \big([Jx],[Jy]\big)_{L_2(\Omega)/\mathcal{H}} = (\xi,\eta)_{L_2(\Omega)} = (x_1,y_1), \quad \forall\, \xi \in [Jx], \eta \in [Jy], \tag{27}$$

where $x_1 = \Pi x$, $y_1 = \Pi y$.

3. The Space $\tilde{H}_{-1}(\Omega) \equiv [H^1(\Omega) \cap \mathcal{H}^\perp]'$ and Its Properties

The consideration of this section is critical to establishing the regularity of the second time derivative w_{tt} of the Kirchhoff elastic or thermoelastic problems with clamped mechanical boundary conditions: see (6) and (9), respectively, to be proved in Theorem 3 and Theorem 4, respectively.

3.1. The Operator $A^{\frac{3}{4}}\mathcal{A}_\gamma^{-1}$

With reference to the operator A in (11), we recall that the space $\mathcal{D}(A^{\frac{3}{4}})$ is given by (Giles, 2000)

$$\mathcal{D}(A^{\frac{3}{4}}) = \left\{ f \in H^3(\Omega) : \; f|_\Gamma = 0, \; \frac{\partial f}{\partial \nu}\bigg|_\Gamma = 0 \right\} \equiv H^3(\Omega) \cap H_0^2(\Omega), \tag{28}$$

with equivalent norms, which complements the identifications in (13).

The counterpart of Lemma 2.1.2 in (Lasiecka and Triggiani, 2000b) for $\tilde{L}_2(\Omega)$ is given next.

Lemma 1. *With reference to (11), (12), (28), (16), we have:*
(a1)

$$\begin{cases} \mathcal{A}_\gamma : \ \textit{continuous } \mathcal{D}(A^{\frac{3}{4}}) \equiv H^3(\Omega) \cap H_0^2(\Omega) \to \big[H^1(\Omega) \cap \mathcal{H}^\perp\big]; \\ \textit{equivalently,} \\ \mathcal{A}_\gamma A^{-\frac{3}{4}} : \ \textit{continuous } L_2(\Omega) \to \big[H^1(\Omega) \cap \mathcal{H}^\perp\big]. \end{cases} \tag{29}$$

(a2) $\mathcal{A}_\gamma A^{-\frac{3}{4}}$ *is injective (one-to-one) on* $L_2(\Omega)$:

$$\mathcal{A}_\gamma A^{-\frac{3}{4}} x = 0, \ x \in L_2(\Omega) \Rightarrow x = 0. \tag{30}$$

(a3) *For* $F \in L_2(\Omega)$, *we have*

$$\mathcal{A}_\gamma^{-1} F \in \mathcal{D}(A^{\frac{3}{4}}) \iff A^{\frac{3}{4}} \mathcal{A}_\gamma^{-1} F \in L_2(\Omega) \iff F \in \big[H^1(\Omega) \cap \mathcal{H}^\perp\big]. \tag{31}$$

Thus, by the closed graph theorem, the operator $A^{\frac{3}{4}} \mathcal{A}_\gamma^{-1}$, *as an operator on* $L_2(\Omega)$, *has the following domain:*

$$\mathcal{D}\big(A^{\frac{3}{4}} \mathcal{A}_\gamma^{-1}\big) \equiv \big[H^1(\Omega) \cap \mathcal{H}^\perp\big]. \tag{32}$$

(a4) *(improving upon (a1))*

$$\begin{cases} \mathcal{A}_\gamma \ \textit{is an isomorphism from } \mathcal{D}(A^{\frac{3}{4}}) \equiv H^3(\Omega) \cap H_0^2(\Omega) \\ \quad \textit{onto } [H^1(\Omega) \cap \mathcal{H}^\perp]; \ \textit{equivalently,} & \text{(33a)} \\ \mathcal{A}_\gamma A^{-\frac{3}{4}} \ \textit{is an isomorphism from } L_2(\Omega) \ \textit{onto } [H^1(\Omega) \cap \mathcal{H}^\perp], & \text{(33b)} \end{cases}$$

with bounded inverse

$$\big(\mathcal{A}_\gamma A^{-\frac{3}{4}}\big)^{-1} = A^{\frac{3}{4}} \mathcal{A}_\gamma^{-1} \ \textit{continuous from } \big[H^1(\Omega) \cap \mathcal{H}^\perp\big] \ \textit{onto } L_2(\Omega). \tag{34}$$

(a5) *The elliptic problem*

$$\begin{cases} (1 - \gamma\Delta)\psi = F & \textit{in } \Omega, \quad \textit{or } \mathcal{A}_\gamma \psi = F, \\ \psi|_\Gamma = 0 & \textit{on } \Gamma, \end{cases} \tag{35}$$

$$\begin{cases} \textit{has a unique solution} \\ \psi \in H^3(\Omega) \cap H_0^2(\Omega) \end{cases} \iff F \in \big[H^1(\Omega) \cap \mathcal{H}^\perp\big]. \tag{36}$$

Proof. (a1) Let $f \in H^3(\Omega) \cap H_0^2(\Omega) \equiv \mathcal{D}(A^{\frac{3}{4}}) \subset \mathcal{D}(\mathcal{A}_\gamma) \equiv H^2(\Omega) \cap H_0^1(\Omega)$, so that $F \equiv \mathcal{A}_\gamma f = (1 - \gamma\Delta) f \in H^1(\Omega)$, as desired. Moreover, if $h \in \mathcal{H}$, see (15), since $f \in H_0^2(\Omega)$ in particular, then Green's identity yields

$$(F, h)_{L_2(\Omega)} = \big((1 - \gamma\Delta) f, h\big)_{L_2(\Omega)} = \big(f, (1 - \gamma\Delta) h\big)_{L_2(\Omega)} = 0, \tag{37}$$

and then $F \in \mathcal{H}^\perp$ as well. Thus, $F \in [H^1(\Omega) \cap \mathcal{H}^\perp]$.

(a2) This is immediate, since $A^{-\frac{3}{4}} x \in \mathcal{D}(\mathcal{A}_\gamma)$ for $x \in L_2(\Omega)$, as noted above in (a1).

(a3) We first show that the right side of (31) implies the left side. Take at first $F \in H^1(\Omega)$ so that $\mathcal{A}_\gamma^{-1}F \in \mathcal{D}(\mathcal{A}_\gamma)$ (conservatively), and

$$\psi \equiv \mathcal{A}_\gamma^{-1}F \Rightarrow \mathcal{A}_\gamma\psi = F \text{ or } \begin{cases} (1-\gamma\Delta)\psi \equiv F & \text{in } \Omega, \\ \psi|_\Gamma = 0 & \text{on } \Gamma. \end{cases} \tag{38}$$

Then elliptic theory (Lions and Magenes, 1972) yields $\psi \in H^3(\Omega) \cap H_0^1(\Omega)$. Next, we recall Lemma 2.1.2 (a3) of (Lasiecka and Triggiani, 2000a; 2000b) stating that for $F \in L_2(\Omega)$,

$$\left.\frac{\partial\psi}{\partial\nu}\right|_\Gamma = \left.\frac{\partial\mathcal{A}_\gamma^{-1}F}{\partial\nu}\right|_\Gamma = 0 \Longleftrightarrow F \in \mathcal{H}^\perp. \tag{39}$$

Thus, using $\Leftarrow$ in (39), we see that $F \in [H^1(\Omega) \cap \mathcal{H}^\perp]$ implies by the argument above that the solution of (38) satisfies $\psi \in H^3(\Omega) \cap H_0^2(\Omega)$. Thus, $\psi \equiv \mathcal{A}_\gamma^{-1}F \in \mathcal{D}(A^{\frac{3}{4}})$ by (28), and then $A^{\frac{3}{4}}\mathcal{A}_\gamma^{-1}F \in L_2(\Omega)$, as desired.

Conversely, we prove that the left side of (31) implies the right side. Let $A^{\frac{3}{4}}\psi \in L_2(\Omega)$ for $\psi \equiv \mathcal{A}_\gamma^{-1}F$, $F \in L_2(\Omega)$. Then, $\psi \in H^3(\Omega) \cap H_0^2(\Omega)$ by (28). Next, the elliptic problem in (38) yields (Lions and Magenes, 1972, p.188) that $F \in H^1(\Omega)$. Moreover, (39) this time from left to right $\Rightarrow$ yields $F \in \mathcal{H}^\perp$. Hence, $F \in [H^1(\Omega) \cap \mathcal{H}^\perp]$, as desired.

(a4) Parts (a1) and (a3) yield part (a4).

(a5) Statement (36) is a PDE reformulation of (31). ■

Remark 1. The above argument in (a3) shows that for $F \in L_2(\Omega)$ we have

$$F \in [H_0^1(\Omega) \cap \mathcal{H}^\perp] \Longleftrightarrow \psi = \mathcal{A}_\gamma^{-1}F \in \mathcal{D}(\mathcal{A}_\gamma^{\frac{3}{2}}),$$

$$\mathcal{D}(\mathcal{A}_\gamma^{\frac{3}{2}}) = \left\{f \in H^3(\Omega) :\ f|_\Gamma = 0,\ \Delta f|_\Gamma = 0\right\}.$$

3.2. The Dual Space $[H^1(\Omega) \cap \mathcal{H}^\perp]'$ is Isometric to the Factor Space $[H^1(\Omega)]'/\mathbb{H}$

We first recall the space $\mathcal{H}$ of 'generalized harmonic functions' defined in (15)

$$\begin{cases} \mathcal{H} = \mathcal{N}\{(1-\gamma\Delta)\} = \{h \in L_2(\Omega) :\ (1-\gamma\Delta)h = 0 \text{ in } H^{-2}(\Omega)\}, \\ L_2(\Omega) = \mathcal{H} + \mathcal{H}^\perp, \text{ where } (1-\gamma\Delta) \text{ is viewed as an operator:} \\ L_2(\Omega) \to H^{-2}(\Omega) \equiv \left[\mathcal{D}(A^{\frac{1}{2}})\right]'. \end{cases} \tag{40}$$

Next, we introduce a new closed space of 'generalized harmonic functions' defined as

$$\begin{cases} \mathbb{H} \;=\; \mathcal{N}\{(1-\gamma\Delta)\} = \Big\{h \in \big[H^1(\Omega)\big]' : (1-\gamma\Delta)h = 0 \text{ in } \big[\mathcal{D}(A^{\frac{3}{4}})\big]'\Big\} \\ \quad = \Big\{h \in \big[H^1(\Omega)\big]' : \big((1-\gamma\Delta)h,\phi\big)_{L_2(\Omega)} \\ \quad = 0 \;\forall\, \phi \in H^3(\Omega)\cap H_0^2(\Omega) \equiv \mathcal{D}(A^{\frac{3}{4}})\Big\}, \\ \text{where here } (1-\gamma\Delta) \text{ is viewed as an operator:} \\ \big[H^1(\Omega)\big]' \to \big[\mathcal{D}(A^{\frac{3}{4}})\big]' = \big[H^3(\Omega)\cap H_0^2(\Omega)\big]'. \\ \text{Moreover, } \big[H^1(\Omega)\big]' = \mathbb{H}\oplus\mathbb{H}^{\perp},\;\; \pi\big[H^1(\Omega)\big]' = \mathbb{H}^{\perp}, \\ \text{where } \pi=\pi^* \text{ is the orthogonal projection of } [H^1(\Omega)]' \text{ onto } \mathbb{H}^{\perp}. \end{cases} \tag{41}$$

We note that $\mathcal{H} \subset \mathbb{H}$. In (40) and (41), $\mathcal{N}$ denotes 'null space,' while $[\;]'$ is always duality with respect to $L_2(\Omega)$ as a pivot space. $\mathcal{H}$ is a closed subspace of $L_2(\Omega)$, while $\mathbb{H}$ is a closed subspace of $[H^1(\Omega)]'$. Next, we observe that

$$\begin{aligned} P \;&\equiv\; \big[H^1(\Omega)\cap\mathcal{H}^{\perp}\big] \\ &\equiv\; \big\{f \in H^1(\Omega) : f \in \mathcal{H}^{\perp}\big\} \text{ is a closed subspace of } H^1(\Omega) \equiv V. \end{aligned} \tag{42}$$

Indeed, let $f_n \in P$ so that $f_n \in H^1(\Omega)$ and $(f_n,h)_{L_2(\Omega)} = 0$, $\forall\, h \in \mathcal{H}$, and let $f_n \to f$ in $H^1(\Omega)$. Then $(f,h)_{L_2(\Omega)} = 0$, $\forall\, h \in \mathcal{H}$ and $f \in P$ as well. We next provide an isometric characterization of the dual space P', which is the counterpart of Proposition 2.

Proposition 4. *The space $[H^1(\Omega)\cap\mathcal{H}^{\perp}]'$, dual of the space in (42) with respect to $L_2(\Omega)$ as a pivot space, is isometrically isomorphic (congruent, in the terminology of (Taylor and Lay, 1980, p.53)) to the factor (or quotient) space $[H^1(\Omega)]'/\mathbb{H}$, where $\mathbb{H}$ is defined by (41). In symbols*

$$\big[H^1(\Omega)\cap\mathcal{H}^{\perp}\big]' \cong \big[H^1(\Omega)\big]'/\mathbb{H} \cong \mathbb{H}^{\perp}. \tag{43}$$

For $g \in [H^1(\Omega)\cap\mathcal{H}^{\perp}]'$,

$$\|g\|_{[H^1(\Omega)\cap\mathcal{H}^{\perp}]'} = \big\|[Jg]\big\|_{[H^1(\Omega)]'/\mathbb{H}} = \inf_{h\in\mathbb{H}} \|Jg-h\|_{[H^1(\Omega)]'} = \|g_1\|_{[H^1(\Omega)]'}, \tag{44}$$

where J denotes the isometric isomorphism from $[H^1(\Omega)\cap\mathcal{H}^{\perp}]'$ onto $[H^1(\Omega)]'/\mathbb{H}$ for the unique element $g_1 = \pi g \in \mathbb{H}^{\perp}$, $g_1 \in [Jg]$ (the latter being the coset or equivalence class of $[H^1(\Omega)]'/\mathbb{H}$ containing the element Jg).

Proof. Step 1. Let P be defined by (42), and $V \equiv H^1(\Omega)$ be defined by (42). By the standard result (Aubin, 1972, Thm.1.6, p.53; Giles, 2000, Thm.6.11, p.118; Taylor and Lay, 1980, Thm.3.3, p.135), we then have that

P' is isometrically isomorphic (congruent) to the factor space $V'/P^{\perp}$, where (45)

$$P^{\perp} \;\equiv\; \big\{f \in V' :\; f(v) = (f,v)_{V'\times V} = 0,\; \forall\, v \in P \subset V\big\}, \tag{46}$$

$$P' \;\equiv\; \text{space of continuous linear functionals on } P, \tag{47}$$

$$V' \;\equiv\; \text{space of continuous linear functionals on } V, \tag{48}$$

and $(\;,\;)_{V'\times V}$ denotes the duality pairing on $V'\times V$.

We now take $L_2(\Omega)$ as a common pivot space. Accordingly, we have the identification

$$V' = [H^1(\Omega)]' = \text{duality of } H^1(\Omega) \text{ with respect to } L_2(\Omega) \text{ as a pivot space.} \quad (49)$$

We next find the corresponnding isometric identification for $P^\perp$ (which is a closed subspace of V'). We note the usual imbedding $V \subset L_2(\Omega) \subset V'$, and we may identify the duality pairing $(\ ,\ )_{V' \times V}$ with the unique extension of the inner product of $L_2(\Omega)$ (Aubin, 1972, Thm.1.5, p.51). Thus, the space $P^\perp$ in V' defined by (46) can be identified with the subspace of $[H^1(\Omega)]'$ defined by

$$P^\perp = \left\{ f \in \left[H^1(\Omega)\right]' : \ (f, v)_{L_2(\Omega)} = 0, \ \forall\, v \in P \equiv \left[H^1(\Omega) \cap \mathcal{H}^1\right] \right\}, \quad (50)$$

and denoted by the same symbol. Since, in (50), we have, in particular, that $(h, v)_{L_2(\Omega)} = 0, \ \forall\, h \in \mathcal{H} \subset L_2(\Omega)$, we see at once that $\mathcal{H} \subset P^\perp$.

Step 2. With reference to $P^\perp$ in (50) and $\mathbb{H}$ in (41), we shall now establish that

$$P^\perp = \mathbb{H}. \quad (51)$$

The proof will be based on Lemma 1(a5): that

$$\begin{cases} \psi \text{ runs over all of } H^3(\Omega) \cap H_0^2(\Omega) \text{ as } F \text{ runs over} \\ \text{all of } \left[H^1(\Omega) \cap \mathcal{H}^\perp\right], \end{cases} \quad (52)$$

where ψ solves the elliptic problem in (35) with right-hand side F.

Next, for any $h \in [H^1(\Omega)]'$ and any $\psi \in H^3(\Omega) \cap H_0^2(\Omega)$, we can write

$$\big((1-\gamma\Delta)h, \psi\big)_{L_2(\Omega)} = \big(h, (1-\gamma\Delta)\psi\big)_{L_2(\Omega)} \quad (53)$$

by Green's identity. We now prove that

$$\mathbb{H} \subset P^\perp. \quad (54)$$

In fact, if $h \in \mathbb{H}$, then, in particular, $h \in [H^1(\Omega)]'$, and by definition (41), we have that the left side of (53) vanishes. Then, the right side of (53) vanishes and hence we have that $(h, F)_{L_2(\Omega)} = 0$ for $(1-\gamma\Delta)\psi = F$, where $F \in P \equiv [H^1(\Omega) \cap \mathcal{H}^\perp]$, see (42). Invoking (52), ultimately Lemma 1(a5), we then see that $h \in P^\perp$ by (50). Thus, (54) is established.

Conversely, we now show that

$$P^\perp \subset \mathbb{H}. \quad (55)$$

Indeed, let $h \in P^\perp$, so that $(h, F)_{L_2(\Omega)} = 0, \ \forall\, F \in P \equiv [H^1(\Omega) \cap \mathcal{H}^\perp]$ by (50). Then, $(h, (1-\gamma\Delta)\psi)_{L_2(\Omega)} = 0$ for all $\psi \in H^3(\Omega) \cap H_0^2(\Omega)$ by (52). As a consequence of this, the left side of (53) vanishes: $((1-\gamma\Delta)h, \psi)_{L_2(\Omega)} = 0, \ \forall\, \psi \in H^3(\Omega) \cap H_0^2(\Omega)$. Then $h \in \mathbb{H}$ by definition (41). Thus, (55) is established.

In conclusion: identity (51) is thus proved.

Returning now to (45), with $P^\perp$ as in (51) and V' as in (49), we conclude that (43) holds true. This finishes the proof. ∎

3.3. Definition of the Space $\tilde{H}_{-1}(\Omega)$. Equivalent Formulations

Paralleling the development of Section 2.2 in (Lasiecka and Triggiani, 2000b), we consider (see (11)–(14), (28))

$$\text{(i)} \qquad \begin{cases} \text{the space } \mathcal{D}(A^{\frac{3}{4}}) \equiv H^3(\Omega) \cap H_0^2(\Omega) \text{ as a closed} \\ \text{subspace of } \mathcal{D}(\mathcal{A}_\gamma) = H^2(\Omega) \cap H_0^1(\Omega); \end{cases} \tag{56}$$

(ii) the space $\mathcal{D}(\mathcal{A}_\gamma^{\frac{1}{2}})$ as a pivot space, with norm as in (14), or (18).

However, the space $\mathcal{D}(A^{\frac{3}{4}})$ is dense in $\mathcal{D}(\mathcal{A}_\gamma^{\frac{1}{2}})$, so (Aubin, 1972, p.51) applies. We then define the (Hilbert) space $\tilde{H}_{-1}(\Omega)$ as follows:

$$\begin{array}{c}\tilde{H}_{-1}(\Omega) \equiv \text{dual of the space } \mathcal{D}(A^{\frac{3}{4}}) \text{ with respect to the space } \mathcal{D}(\mathcal{A}_\gamma^{\frac{1}{2}}) \\ \text{as a pivot space, endowed with the norm of (18).}\end{array} \tag{57}$$

This means the following: let $f \in \mathcal{D}(A^{\frac{3}{4}}) \equiv H^3(\Omega) \cap H_0^2(\Omega) \subset \mathcal{D}(\mathcal{A}_\gamma) \equiv H^2(\Omega) \cap H_0^1(\Omega)$, or $\phi = A^{\frac{3}{4}} f \in L_2(\Omega)$. Then:

$$g \in \tilde{H}_{-1}(\Omega) \Longleftrightarrow (f,g)_{\mathcal{D}(\mathcal{A}_\gamma^{\frac{1}{2}})} = (\mathcal{A}_\gamma f, g)_{L_2(\Omega)} = \text{finite}, \tag{58a}$$

$$\forall\, f \in \mathcal{D}(A^{\frac{3}{4}}) = H^3(\Omega) \cap H_0^2(\Omega),$$

$$= (f, \mathcal{A}_\gamma g)_{L_2(\Omega)} = (A^{-\frac{3}{4}} \phi, \mathcal{A}_\gamma g)_{L_2(\Omega)}$$

$$= (\phi, A^{-\frac{3}{4}} \mathcal{A}_\gamma g)_{L_2(\Omega)} = \text{finite}, \ \forall\, \phi \in L_2(\Omega), \tag{58b}$$

where we write in the same way inner products and corresponding duality pairings.

Proposition 5. (i) *Definition (58) is equivalent to the following restatement:*

$$g \in \tilde{H}_{-1}(\Omega) \iff (\mathcal{A}_\gamma f, g)_{L_2(\Omega)} = (F, g)_{L_2(\Omega)}$$

$$= \big((1-\gamma\Delta) f, g\big)_{L_2(\Omega)} = \big(f, (1-\gamma\Delta) g\big)_{L_2(\Omega)} = \textit{finite}, \tag{59}$$

$\forall\, f \in \mathcal{D}(A^{\frac{3}{4}}) \equiv H^3(\Omega) \cap H_0^2(\Omega)$, *or* $\forall\, F \in H^1(\Omega) \cap \mathcal{H}$, *where* $F = \mathcal{A}_\gamma f = (1-\gamma\Delta) f$;

(ii) *accordingly,*

$$\tilde{H}_{-1}(\Omega) \equiv \big[H^1(\Omega) \cap \mathcal{H}^\perp\big]' \cong \big[H^1(\Omega)\big]'/\mathbb{H} \cong \mathbb{H}^\perp, \tag{60}$$

with duality with respect to $L_2(\Omega)$ *as a pivot space.*

(iii) *Definition (58) is equivalent to the following restatement:*

$$g \in \tilde{H}_{-1}(\Omega) \Longleftrightarrow A^{-\frac{3}{4}} \mathcal{A}_\gamma g \in L_2(\Omega) \tag{61}$$

(iv) *(counterpart of Proposition 1)*

$$g \in \tilde{H}_{-1}(\Omega) \Longleftrightarrow \begin{cases} g \textit{ has a component } g_1 \textit{ defined by } g_1 = \pi g = \\ g|_{\mathbb{H}^\perp} \in \mathbb{H}^\perp \subset \big[H^1(\Omega)\big]', \textit{ which is the orthogo-} \\ \textit{nal projection of } g \textit{ onto } \mathbb{H}^\perp, \end{cases} \tag{62}$$

in which case

$$(1-\gamma\Delta)g = (1-\gamma\Delta)g_1 \ \mathit{in} \ \big[\mathcal{D}(A^{\frac{3}{4}})\big]'. \tag{63}$$

Proof. (i), (ii) Returning to (58), we invoke Lemma 1(a4), eqn. (33a), and obtain $\mathcal{A}_\gamma\mathcal{D}(A^{\frac{3}{4}}) = [H^1(\Omega)\cap\mathcal{H}^\perp]$. Thus, (58) yields (59) and (60), also via Green's identity.

(iii) Part (iii), eqn. (61), follows at once from (59b).

(iv) Counterpart of the proof of Proposition 2.

Since $g \in \tilde{H}_{-1}(\Omega)$ implies *a-fortiori* $g \in [H^1(\Omega)]'$ by (60), therefore (41) implies (62) so that

$$\big((1-\gamma\Delta)f, g\big)_{L_2(\Omega)} = (\mathcal{A}_\gamma f, g)_{L_2(\Omega)} = (\mathcal{A}_\gamma f, g_1)_{L_2(\Omega)} = \big((1-\gamma\Delta)f, g_1\big)_{L_2(\Omega)}, \tag{64}$$

for $f \in H^3(\Omega)\cap H_0^2(\Omega)$. Hence, (64) and Green's identity yield $(f,(1-\gamma\Delta)g)_{L_2(\Omega)} = (f,(1-\gamma\Delta)g_1)_{L_2(\Omega)}$ and (63) is established. ■

4. Implications on Regularity of Kirchhoff Elastic Plate Equations with Clamped B.C.

4.1. PDE Model: (1)

Abstract model. The abstract model of the mixed problem (1) is given by (Lasiecka and Triggiani, 2000b; 2000c; Triggiani, 1993; 2000),

$$(I+\gamma\mathcal{A})w_{tt} = -Aw + AG_2u + F, \tag{65}$$

where A and $\mathcal{A}$ are the operators defined in (11) and (12), respectively. Moreover, G_2 in (65) is the following Green map defined by (Lasiecka and Triggiani, 2000b),

$$v = G_2u \Longleftrightarrow \left\{\Delta^2 v = 0 \text{ in } \Omega;\ v|_\Gamma = 0,\ \frac{\partial v}{\partial\nu}\bigg|_\Gamma = u\right\}, \tag{66}$$

and by elliptic regularity (Grisvard, 1967; Lions and Magenes, 1972), see (Lasiecka and Triggiani, 2000b; Triggiani, 2000),

$$G_2 \ : \ \text{continuous } L_2(\Gamma) \to H^{\frac{3}{2}}(\Omega)\cap H_0^1(\Omega)$$
$$\subset H^{\frac{3}{2}-4\epsilon}(\Omega)\cap H_0^1(\Omega) = \mathcal{D}(A^{\frac{3}{8}-\epsilon}) \tag{67a}$$

$$A^{\frac{3}{8}-\epsilon}G_2 \ : \ \text{continuous } L_2(\Gamma) \to L_2(\Omega). \tag{67b}$$

4.2. The Non-Homogeneous Boundary Case: $u\neq 0$. Proof of Theorem 1, Eqn. (6)

In this section, which is complementary to the previous sections, we consider the mixed problem (1) with

$$w_0 = 0, \quad w_1 = 0, \quad u \in L_2\big(0,T; L_2(\Gamma)\big), \tag{68}$$

whose abstract model is given by (65).

Theorem 3. *Consider problem (1) subject to hypothesis (68). Then, continuously,*

$$\begin{cases} w \in C\big([0,T]; H_0^1(\Omega) \equiv \mathcal{D}(A^{\frac{1}{4}})\big), & (69)\\ w_t \in C\big([0,T]; \tilde{L}_2(\Omega)\big), & (70)\\ w_{tt} \in L_2\big(0,T; \tilde{H}_{-1}(\Omega)\big). & (71)\end{cases}$$

Proof. Conclusions (69), (70) on $\{w, w_t\}$ follow by duality on the sharp trace regularity of the corresponding homogeneous problem, due to (Lagnese and Lions, 1988, Ch.5). Details of the technical duality are given in (Eller *et al.*, 2001a; Lasiecka and Triggiani, 2000b; Triggiani, 2000).

Here we establish (71). We return to the abstract model (65), which we rewrite as

$$A^{-\frac{3}{4}}\mathcal{A}_\gamma w_{tt} = -A^{\frac{1}{4}}w + A^{-\frac{1}{8}+\epsilon}\left(A^{\frac{3}{8}-\epsilon}G_2 u\right) \in L_2\big(0,T; L_2(\Omega)\big), \tag{72}$$

where the regularity noted in (72) follows from $A^{\frac{1}{4}}w \in C([0,T]; L_2(\Omega))$ by (69), as well as from $A^{\frac{3}{8}-\epsilon}G_2 u \in L_2(0,T; L_2(\Omega))$ by (67b), as well as from $A^{\frac{3}{8}-\epsilon}G_2 u \in L_2(0,T; L_2(\Omega))$ by (67b) on G_2 and (68) on u. Thus, as usual via the characterization (61), we see that (72) says that $w_{tt} \in L_2(0,T; \tilde{H}_{-1}(\Omega))$, as claimed in (71). (The above argument shows that, in the present circumstances, the term Aw is the *critical* one, while AG_2u is subordinated to it, in model (65).) ■

5. Implications on Regularity of Mixed Kirchhoff Thermoelastic Plate Equations with Clamped B.C.: Proof of Theorem 2, Eqns. (9) and (10)

In this section we let Ω be an open bounded domain in $\mathbb{R}^n$, for any positive integer n, with smooth boundary Γ. On Ω we consider the following thermoelastic mixed problem in the unknown $\{w(t,x), \theta(t,x)\}$, which is (2) rewritten for convenience:

$$\begin{cases} w_{tt} - \gamma\Delta w_{tt} + \Delta^2 w + \Delta\theta = 0 & \text{in } (0,T]\times\Omega = Q, \quad (73a)\\ \theta_t - \Delta\theta - \Delta w_t = 0 & \text{in } Q, \quad (73b)\\ w(0,\,\cdot\,) = w_0;\ w_t(0,\,\cdot\,) = w_1;\ \theta(0,\,\cdot\,) = \theta_0 & \text{in } \Omega, \quad (73c)\\ w|_\Sigma \equiv 0;\ \left.\dfrac{\partial w}{\partial\nu}\right|_\Sigma = u;\ \theta|_\Sigma \equiv 0 & \text{in } (0,T]\times\Gamma \equiv \Sigma, \quad (73d)\end{cases}$$

where, for our present purposes, it will suffice to take

$$w_0 = 0, \quad w_1 = 0, \quad \theta_0 = 0; \quad u \in L_2\big(0,T; L_2(\Gamma)\big). \tag{74}$$

As in the elastic case of Section 4, in the present thermoelastic case we shall take the constant $\gamma > 0$ throughout. Our goal is to establish the following sharp regularity result.

Theorem 4. *With reference to the mixed problem (73) with $\gamma > 0$ and zero initial conditions as in (74), the following regularity results hold true, where $\mathcal{D}(A^{\frac{1}{4}}) \equiv H_0^1(\Omega)$ (norm equivalence): the map*

$$u \in L_2\big(0,T; L_2(\Gamma)\big) \Rightarrow$$

$$\begin{cases} \{w, w_t\} \in C\big([0,T]; \mathcal{D}(A^{\frac{1}{4}}) \times \tilde{L}_2(\Omega)\big), & (75) \\ \left[w_{tt} - \dfrac{1}{\gamma}\,\theta\right] \in L_2\big(0,T; \tilde{H}_{-1}(\Omega)\big), & (76) \\ \theta \in L_p\big(0,T; H^{-1}(\Omega)\big) \cap C\big([0,T]; H^{-1-\epsilon}(\Omega)\big),\ 1 < p < \infty;\ \forall\ \epsilon > 0, & (77) \end{cases}$$

is continuous. However, in addition, we have

$$\begin{cases} \theta \in C\big([0,T]; L_2(\Omega)\big), \ \textit{and}\ w_{tt} \in L_2\big(0,T; \tilde{H}_{-1}(\Omega)\big), \\ \textit{but}\ \text{not}\ \textit{continuously in}\ u \in L_2\big(0,T; L_2(\Gamma)\big) \end{cases} \tag{78}$$

(that is, the closed graph theorem does not *apply to the maps $u \to \theta$ or $u \to w_{tt}$ in (78)). More precisely, regarding θ, we have*

$$\theta(t) = -w_t(t) + \theta_{1,a}(t) + \theta_{1,b}(t), \tag{79}$$

where w_t satisfies (75), and

$$\begin{cases} \theta_{1,b}(t) = \dfrac{1}{\gamma} \displaystyle\int_0^t e^{-\mathcal{A}(t-\tau)} \theta(\tau)\, \mathrm{d}\tau \in C\big([0,T]; L_2(\Omega)\big) \\ \textit{continuously in}\ u \in L_2\big(0,T; L_2(\Gamma)\big), \end{cases} \tag{80a}$$

while

$$\begin{cases} \theta_{1,a}(t) = \displaystyle\int_0^t e^{-\mathcal{A}(t-\tau)} \left[w_{tt}(\tau) - \dfrac{1}{\gamma}\,\theta(\tau)\right] \mathrm{d}\tau \in C\big([0,T]; L_2(\Omega)\big); \\ \textit{however, not continuously in}\ u \in L_2\big(0,T; L_2(\Gamma)\big). \end{cases} \tag{80b}$$

Proof of Theorem 4. The mechanical regularity (75) for $\{w, w_t\}$ was established in (Triggiani, 2000, Thm.4.1), and it coincides with the mechanical regularity (4.4.2), (4.4.3) in the *elastic* case of Proposition 4.4.1. We shall repeat a sketch of the argument for completeness, following (Triggiani, 2000) or (Eller *et al.*, 2001a).

Step 1. We start with the dual $\{\phi, \eta\}$-thermoelastic problem:

$$\begin{cases} \phi_{tt} - \gamma \Delta \phi_{tt} + \Delta^2 \phi - \Delta \eta = 0 & \text{in } Q = (0,T] \times \Omega, \quad (81a) \\ \eta_t - \Delta \eta + \Delta \phi_t = 0 & \text{in } Q, \quad (81b) \\ \phi(0, \,\cdot\,) = \phi_0, \ \phi_t(0, \,\cdot\,) = \phi_1, \ \eta(0, \,\cdot\,) = \eta_0 & \text{in } \Omega, \quad (81c) \\ \phi|_\Sigma \equiv 0, \ \left.\dfrac{\partial \phi}{\partial \nu}\right|_\Sigma \equiv 0, \ \eta|_\Sigma \equiv 0 & \text{in } \Sigma = (0,T] \times \Gamma, \quad (81d) \end{cases}$$

with initial conditions,

$$\{\phi_0, -\phi_1, \eta_0\} \in Y_\gamma \equiv H_0^2(\Omega) \times H_0^1(\Omega) \times L_2(\Omega). \tag{82}$$

Thus the solution $\{\phi(t), -\phi_t(t), \eta(t)\} = e^{\mathbb{A}_\gamma^* t}[\phi_0, -\phi_1, \eta_0]$ is given by the adjoint semigroup $e^{\mathbb{A}_\gamma^* t}$ on Y_γ to the one $e^{\mathbb{A}_\gamma t}$ claimed (by the Lumer-Phillips theorem) in (Eller *et al.*, 2001a, eqn. (2.10); Triggiani, 2000 below (4.17)), etc. Thus, its *a-priori* regularity is

$$\begin{cases} \{\phi, -\phi_t, \eta\} \in C\big([0,T]; H_0^2(\Omega) \times H_0^1(\Omega) \times L_2(\Omega)\big), & (83a) \\ \eta \in L_2\big(0,T; H_0^1(\Omega)\big), \ \Delta \eta \in L_2\big(0,T; H^{-1}(\Omega)\big), & (83b) \end{cases}$$

using also the usual dissipativity argument for η (Lasiecka and Triggiani, 2000d). Next, we rewrite problem (81) in the following way:

$$\begin{cases} \phi_{tt} - \gamma \Delta \phi_{tt} + \Delta^2 \phi = \Delta \eta \in L_2\big(0,T; H^{-1}(\Omega)\big), & (84a) \\ \phi|_\Sigma \equiv 0, \ \left.\dfrac{\partial \phi}{\partial \nu}\right|_\Sigma \equiv 0. & (84b) \end{cases}$$

Step 2. To problem (84) we apply the same energy method proof as in (Lagnese and Lions, 1988, Ch.5 or Ch.6) by using the multiplier $m \cdot \nabla \phi, m|_\Gamma = \nu$: it yields the following sharp trace regularity:

$$\{\phi_0, \phi_1\} \in H_0^2(\Omega) \times H_0^1(\Omega) \Rightarrow \Delta \phi|_\Gamma \in L_2\big(0,T; L_2(\Gamma)\big), \tag{85}$$

(where, of course, $\eta_0 \in L_2(\Omega)$ as well) since $\int_Q (m \cdot \nabla \phi)(\Delta \eta)\, \mathrm{d}Q$ is finite by (83): $f = m \cdot \nabla \phi \in C([0,T]; H_0^1(\Omega))$ since $f|_\Gamma = m \cdot \nabla \phi|_\Gamma = \frac{\partial \phi}{\partial \nu}|_\Gamma = 0$. Details are given, e.g. in (Eller *et al.*, 2001a, eqns. (3.97)–(3.99), p.129; and also eqns. (C.47)–(C.49), p.206).

Step 3. A duality argument, given in details in (Eller *et al.*, 2001a, Step 1, p.206; Triggiani, 2000, Section 5, Step 3), then shows the following preliminary result: for the mixed problem (73), (74), the map

$$u \in L_2(\Sigma) \rightarrow \begin{cases} \{w, w_t\} \in C\big([0,T]; H_0^1(\Omega) \times \tilde{L}_2(\Omega)\big), \\ \theta \in C\big([0,T]; [\mathcal{D}(\mathcal{A})]'\big) \end{cases} \tag{86}$$

is continuous. This map is *optimal* for $\{w, w_t\}$. The space $\tilde{L}_2(\Omega)$ described in Section 2 arises at this point, in connection with the second component space, as dual of $\mathcal{D}(A^{\frac{1}{2}})$ with respect to $\mathcal{D}(\mathcal{A}_\gamma^{\frac{1}{2}})$ as a pivot space. So far, we have reproduced results of (Eller *et al.*, 2001a; Triggiani, 2000) for the mixed problem (73), (74). Thus, we have established (75) for $\{w, w_t\}$.

Step 4. (Proof of (77)) We next boost the regularity of θ (over (86)) to read: the map

$$u \in L_2\big(0, T; L_2(\Gamma)\big) \to \theta \in L_p\big(0, T; H^{-1}(\Omega)\big) \cap C\big([0, T]; H^{-1-\epsilon}(\Omega)\big),$$
$$1 < p < \infty;\ \forall\ \epsilon > 0 \quad (87)$$

is continuous. To establish (87), we return to eqn. (73b), and integrate by parts to obtain via $w_t|_\Sigma \equiv 0$ and (12):

$$\begin{aligned}\theta(t) &= \int_0^t e^{-\mathcal{A}(t-\tau)} \Delta w_t(\tau)\,\mathrm{d}\tau = -\int_0^t e^{-\mathcal{A}(t-\tau)} \mathcal{A} w_t(\tau)\,\mathrm{d}\tau \\ &= -\mathcal{A}^{\frac{1}{2}} \mathcal{A}^{\frac{1}{2}} w(t) + \mathcal{A}^{\frac{1}{2}+\frac{\epsilon}{2}} \int_0^t \mathcal{A}^{1-\frac{\epsilon}{2}} e^{-\mathcal{A}(t-\tau)} \mathcal{A}^{\frac{1}{2}} w(\tau)\,\mathrm{d}\tau \qquad (88) \\ &\in L_p\big(0, T; H^{-1}(\Omega)\big) \cap C\big([0, T]; H^{-1-\epsilon}(\Omega)\big), \qquad (89)\end{aligned}$$

for $\forall\ 1 < p < \infty$; $\forall\ \epsilon > 0$, since $w_0 = 0$ by (74). The regularity in (89) is obtained by using the regularity (75) for w, along with the following two well-known results for analytic semigroups: the map

$$f \to \int_0^t e^{-\mathcal{A}(t-\tau)} f(\tau)\,\mathrm{d}\tau \qquad (90)$$

is continuous as follows:

$$L_p\big(0, T; L_2(\Omega)\big) \ \to\ L_p\big(0, T; \mathcal{D}(\mathcal{A})\big),\ \forall\ 1 < p < \infty, \qquad (91a)$$

$$L_\infty\big(0, T; L_2(\Omega)\big) \ \to\ C\big([0, T]; \mathcal{D}(\mathcal{A}^{1-\epsilon})\big),\ \forall\ \epsilon > 0, \qquad (91b)$$

see (De Simon, 1964) for (91), (Lasiecka and Triggiani, 2000c, p.4). Thus, (87) is proved via (89).

Remark 2. The weaker result, over (78), that

$$\left\{\begin{array}{l}\theta \in L_p\big(0, T; L_2(\Omega)\big) \cap C\big([0, T]; H^{-\epsilon}(\Omega)\big),\ \forall\ 1 < p < \infty;\ \forall\ \epsilon > 0; \\ \text{however, not continuously in } u \in L_2\big(0, T; L_2(\Gamma)\big),\end{array}\right. \qquad (92)$$

follows at once from (88), via (75) on w_t, (25), (91).

Step 5. (Proof of (76)) The abstract model of the mixed problem (73) is given by (Eller *et al.*, 2001a; Triggiani, 2000):

$$\mathcal{A}_\gamma w_{tt} = -Aw + AG_2 u + \mathcal{A}\theta = -Aw + AG_2 u + \frac{1}{\gamma} \mathcal{A}_\gamma \theta - \frac{1}{\gamma} \theta \qquad (93)$$

(compare with (66)), from which we obtain

$$A^{-\frac{3}{4}}\mathcal{A}_\gamma\left[w_{tt}-\frac{1}{\gamma}\,\theta\right] \;=\; -A^{\frac14}w+A^{-\frac18+\epsilon}(A^{\frac38-\epsilon}G_2u)-\frac{1}{\gamma}A^{-\frac34}\theta \tag{94a}$$

$$\in\; L_2\big(0,T;L_2(\Omega)\big), \tag{94b}$$

continuously in $u\in L_2(0,T;L_2(\Gamma))$. The key regularity noted in (94b) is obtained as follows: first, $A^{\frac14}w\in C([0,T];L_2(\Omega))$ by (75); next, $A^{\frac38-\epsilon}G_2u\in L_2(0,T;L_2(\Omega))$ by (67b) and $u\in L_2(0,T;L_2(\Gamma))$; finally, $A^{-\frac34}\theta\in L_2(0,T;\mathcal{D}(A^{\frac12}))$ by (77) with $p=2$, and $\mathcal{D}(A^{\frac14})=H_0^1(\Omega)$, see (13). Hence, the regularity (94b) is established. Then, as usual, via the characterization (61), we see that (94) says that $[w_{tt}-\frac{1}{\gamma}\theta]\in L_2(0,T;\tilde{H}_{-1}(\Omega))$, continuously in $u\in L_2(0,T;L_2(\Gamma))$, as claimed in (76).

Step 6. (Proof of (78) for θ) We return to eqn. (73b), which we rewrite as

$$(\theta+w_t)_t=\Delta(\theta+w_t)+w_{tt}, \tag{95a}$$

and hence, by (12),

$$(\theta+w)_t=-\mathcal{A}(\theta+w_t)+w_{tt}, \tag{95b}$$

because of the homogeneous B.C. (73d) for w and θ. Solving (95b), we obtain

$$\theta(t)=-w_t(t)+\theta_1(t), \tag{96a}$$

where we shall show that

$$\theta_1(t)=\int_0^t e^{-\mathcal{A}(t-\tau)}w_{tt}(\tau)\,\mathrm{d}\tau\in C\big([0,T];L_2(\Omega)\big); \tag{96b}$$

however, not continuously in $u\in L_2(0,T;L_2(\Gamma))$. We rewrite $\theta_1(t)$ as

$$\theta_1(t)=\theta_{1,a}(t)+\theta_{1,b}(t), \tag{97}$$

$$\left\{\begin{aligned}\theta_{1,a}(t)\;&=\;\int_0^t e^{-\mathcal{A}(t-\tau)}\left[w_{tt}(\tau)-\frac{1}{\gamma}\,\theta(\tau)\right]\mathrm{d}\tau && (98)\\ \theta_{1,b}(t)\;&=\;\frac{1}{\gamma}\int_0^t e^{-\mathcal{A}(t-\tau)}\theta(\tau)\,\mathrm{d}\tau && (99)\end{aligned}\right.$$

where, by (77) with $p=2$, and $H^{-1}(\Omega)=[\mathcal{D}(\mathcal{A}^{\frac12})]'$, i.e. with $\mathcal{A}^{-\frac12}\theta\in L_2(0,T;L_2(\Omega))$, continuously in $u\in L_2(0,T;L_2(\Gamma))$, we have

$$\left\{\begin{array}{l}\gamma\theta_{1,b}(t)=\displaystyle\int_0^t\mathcal{A}^{\frac12}e^{-\mathcal{A}(t-\tau)}\mathcal{A}^{-\frac12}\theta(\tau)\,\mathrm{d}\tau\in C\big([0,T];L_2(\Omega)\big)\\ \text{continuously in } u\in L_2\big(0,T;L_2(\Gamma)\big).\end{array}\right. \tag{100}$$

Moreover, by (76), $[w_{tt}-\frac{1}{\gamma}\theta]\in L_2(0,T;\tilde{H}_{-1}(\Omega))$ continuously in $u\in L_2(0,T;L_2(\Gamma))$. Since $\tilde{H}_{-1}(\Omega)\equiv[H^1(\Omega)]'/\mathbb{H}$ by (60), we also have

$$\left\{\begin{array}{l}\left[w_{tt}-\dfrac{1}{\gamma}\theta\right]\in L_2\big(0,T;[H^1(\Omega)]'\big);\\ \text{however, not continuously in } u\in L_2\big(0,T;L_2(\Gamma)\big).\end{array}\right. \tag{101}$$

Hence, we obtain *a-fortiori* that

$$\begin{cases} \left[w_{tt} - \frac{1}{\gamma}\theta\right] \in L_2\big(0,T;H^{-1}(\Omega)\big), \quad \text{or} \quad \mathcal{A}^{-\frac{1}{2}}\left[w_{tt} - \frac{1}{\gamma}\theta\right] \in L_2\big(0,T;L_2(\Omega)\big); \\ \text{however, not continuously in } u \in L_2\big(0,T;L_2(\Gamma)\big). \end{cases} \tag{102}$$

It then follows from (101) and (98) that

$$\begin{cases} \theta_{1,a}(t) = \displaystyle\int_0^t \mathcal{A}^{\frac{1}{2}} e^{-\mathcal{A}(t-\tau)} \mathcal{A}^{-\frac{1}{2}} \left[w_{tt}(\tau) - \frac{1}{\gamma}\theta(\tau)\right] \mathrm{d}\tau \in C\big([0,T];L_2(\Omega)\big); \\ \text{however, not continuously in } u \in L_2\big(0,T;L_2(\Gamma)\big). \end{cases} \tag{103}$$

Then (100) and (103), used in (97), (96), prove (78) for θ, as desired; in fact, prove precisely (79), (80).

Step 7. (Proof of (78) for w_{tt}) We return to (93) (left form) and obtain

$$\begin{cases} A^{-\frac{3}{4}} \mathcal{A}_\gamma w_{tt} = -A^{\frac{1}{4}} w + A^{-\frac{1}{8}+\epsilon}(A^{-\frac{3}{8}-\epsilon} G_2 u) \\ \qquad\qquad + A^{-\frac{1}{4}}(A^{-\frac{1}{2}}\mathcal{A})\theta \in L_2\big(0,T;L_2(\Omega)\big); \\ \text{however, not continuously in } u \in L_2\big(0,T;L_2(\Gamma)\big). \end{cases} \tag{104}$$

The new term, over the analysis below (94b), is the last one in (104), where now θ satisfies (92) with $p = 2$ (which now suffices without invoking (78)). Since $A^{-\frac{1}{2}}\mathcal{A}$ has a bounded extension on $L_2(\Omega)$ by (12), (13) we conclude that

$$\begin{cases} A^{-\frac{1}{4}}(A^{-\frac{1}{2}}\mathcal{A})\theta \in C\big([0,T];\mathcal{D}(A^{\frac{1}{4}}) = H_0^1(\Omega)\big); \\ \text{however, not continuously in } u \in L_2\big(0,T;L_2(\Gamma)\big). \end{cases} \tag{105}$$

Then the regularity in (104) is established by (105). The usual characterization (61) then yields

$$\begin{cases} w_{tt} \in L_2\big(0,T;\tilde{H}_{-1}(\Omega)\big); \\ \text{however, not continuously in } u \in L_2\big(0,T;L_2(\Gamma)\big). \end{cases} \tag{106}$$

The proof of Theorem 4 is complete. ■

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