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INGHAM-TYPE INEQUALITIES AND RIESZ BASES OF DIVIDED DIFFERENCES

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We study linear combinations of exponentials $e^{i\lambda_n t}, \lambda_n \in \Lambda$ in the case where the distance between some points λ_n tends to zero. We suppose that the sequence Λ is a finite union of uniformly discrete sequences. In (Avdonin and Ivanov, 2001), necessary and sufficient conditions were given for the family of divided differences of exponentials to form a Riesz basis in space $L^2(0,T)$. Here we prove that if the upper uniform density of Λ is less than $T/(2\pi)$, the family of divided differences can be extended to a Riesz basis in $L^2(0,T)$ by adjoining to $\{e^{i\lambda_n t}\}$ a suitable collection of exponentials. Likewise, if the lower uniform density is greater than $T/(2\pi)$, the family of divided differences can be made into a Riesz basis by removing from $\{e^{i\lambda_n t}\}$ a suitable collection of functions $e^{i\lambda_n t}$. Applications of these results to problems of simultaneous control of elastic strings and beams are given.

Keywords: simultaneous controllability, string equation, beam equation, Riesz bases, divided differences

1. Introduction

Families of 'nonharmonic' exponentials $\{e^{i\lambda_n t}\}$ appear in various fields of mathematics and signal processing. One of the central problems arising in all of these applications is the question of the Riesz basis property of an exponential family. For $L^2(0,T)$, this problem was considered for the first time in the classical work of R. Paley and N. Wiener (1934), and since then has motivated a great deal of work by many mathematicians; a number of references are given in (Avdonin and Ivanov, 1995; Khrushchev *et al.*, 1981; Nikol'skiĭ, 1986; Young, 1980). The problem was ultimately given a complete solution (Khrushchev *et al.*, 1981; Minkin, 1991; Pavlov, 1979) on the basis of an approach suggested by B. Pavlov.

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The main result in this direction can be formulated as follows (Pavlov, 1979):

Theorem 1. *Let* $\Lambda := \{\lambda_n \,|\, k \in \mathbb{Z}\}$ *be a countable set in the complex plane. The family* $\{e^{i\lambda_n t}\}$ *forms a Riesz basis in* $L^2(0,T)$ *if and only if the following conditions are satisfied:*

(i) Λ *lies in a strip parallel to the real axis,*

$$\sup_{n\in\mathbb{Z}} |\Im\lambda_n| < \infty,$$

and is uniformly discrete (or separated), i.e.

$$\delta(\Lambda) := \inf_{k\neq n} |\lambda_k - \lambda_n| > 0; \tag{1}$$

(ii) *there exists an entire function* F *of exponential type with indicator diagram of width* T *and zero set* Λ (the generating function of the family $\{e^{i\lambda_n t}\}$ on the interval $(0,T)$) *such that, for some real* h*, the function* $|F(x+ih)|^2$ *satisfies the Helson-Szegö condition: functions* $u, v \in L^\infty(\mathbb{R}), \|v\|_{L^\infty(\mathbb{R})} < \pi/2$ *can be found such that*

$$|F(x+ih)|^2 = \exp\{u(x) + \tilde{v}(x)\}. \tag{2}$$

Here the map $v \mapsto \tilde{v}$ denotes the Hilbert transform for bounded functions:

$$\tilde{v}(x) = \frac{1}{\pi}\ \text{p.v.} \int_{-\infty}^{\infty} v(t)\left\{\frac{1}{x-t} + \frac{t}{t^2+1}\right\} \mathrm{d}t.$$

It is well-known that the Helson-Szegö condition is equivalent to the Muckenhoupt condition (A_2):

$$\sup_{I\in\mathcal{J}} \left\{\frac{1}{|I|}\int_I |F(x+ih)|^2 \,\mathrm{d}x \,\frac{1}{|I|}\int_I |F(x+ih)|^{-2} \,\mathrm{d}x\right\} < \infty,$$

where $\mathcal{J}$ is the set of all intervals of the real axis.

The notion of the generating function mentioned above plays a central role in the modern theory of nonharmonic Fourier series (Avdonin and Ivanov, 1995; Khrushchev *et al.*, 1981). This notion also plays an important role in the theory of exponential bases in Sobolev spaces (Avdonin and Ivanov, 2000; Lyubarskii and Seip, 2000). It is possible to write an explicit expression for this function:

$$F(z) = \lim_{R\to\infty} \prod_{|\lambda_n|\leq R} \left(1 - \frac{z_n}{\lambda_n}\right)$$

(we replace the term $(1-\lambda_n^{-1}z)$ by z if $\lambda_n = 0$).

The theory of nonharmonic Fourier series was successfully applied to control problems for distributed parameter systems and formed the base of the powerful method of moments (Avdonin and Ivanov, 1995; Butkovsky, 1965; Russell, 1978).

This method is based on properties of exponential families (usually in $L^2(0,T)$), the most important of which for control theory are minimality, the Riesz basis property, and also the $\mathcal{L}$*-basis property*. The last notion is introduced to make a Riesz basis meaningful in the closure of its linear span.

Recent investigations into new classes of distributed systems such as hybrid systems and structurally damped systems have raised a number of new difficult problems in the theory of exponential families (see, e.g. Hansen and Zuazua, 1995; Jaffard *et al.*, 1998; Micu and Zuazua, 1997). One of them is connected with the properties of the family $\mathcal{E} = \{e^{i\lambda_n t}\}$ in the case when the set Λ does not satisfy the separation condition (1), and therefore $\mathcal{E}$ does not form a Riesz basis in its span in $L^2(0,T)$ for any $T > 0$. In this case we meet the problem of obtaining a description of Riesz bases of elements which are 'simple and natural' linear combinations of exponentials.

The first result in this direction was obtained by D. Ullrich (1980), who considered sets Λ of the form $\Lambda = \bigcup_{p\in\mathbb{Z}} \Lambda^{(p)}$, where the subsets $\Lambda^{(p)}$ consist of equal numbers (say, N) real points $\lambda_1^{(p)}, \dots, \lambda_N^{(p)}$ close to p, i.e. $|\lambda_j^{(p)} - p| < \varepsilon$ for all j and p. He proved that, for sufficiently small $\varepsilon > 0$ (no estimate of ε was given), the family of particular linear combinations of exponentials $e^{i\lambda_n t}$—the so-called *divided differences* constructed by subsets $\Lambda^{(p)}$ (see Definition 1 in Section 2)—forms a Riesz basis in $L^2(0, 2\pi N)$. Such functions arise in numerical analysis (Isaacson and Keller, 1966; Shilov, 1965), and the divided difference of $e^{i\mu t}$, $e^{i\lambda t}$ of the first order is $(e^{i\mu t} - e^{i\lambda t})/(\mu - \lambda)$. In a sense, Ullrich's result can be considered as a perturbation theorem for the basis family $\{e^{int}, te^{int}, \dots, t^{N-1}e^{int}\}$, $n \in \mathbb{Z}$.

The conditions of this theorem are rather restrictive and, as a result, it cannot be applied to some problems arising in control theory (see, e.g. Baiocchi *et al.*, 1999; Castro and Zuazua, 1998; Jaffard *et al.*, 1998; Lopes and Zuazua, 1998; Micu and Zuazua, 1997). In (Avdonin and Ivanov, 2001), Ullrich's result was generalized in several directions: the set Λ is allowed to be complex and the subsets $\Lambda^{(p)}$ are allowed to contain an arbitrary number of points which are not necessarily 'very' close to each other (or even to some integer). Actually, (Avdonin and Ivanov, 2001) gives a full description of Riesz bases of exponential divided differences and generalized divided differences (the last ones appear in the case of multiple points λ_n).

To be more specific, a sequence Λ which is 'a union' of a finite number of separated sets was decomposed into groups $\Lambda^{(p)}$ of 'close' points. For each group, a family of generalized divided differences (GDDs) was chosen and it was proved that these functions form a Riesz basis in $L^2(0,T)$ if the generating function of the exponential family satisfies the Helson-Szegö condition (2). In the case when Λ is not a finite union of separated sets, a negative result was presented in (Avdonin and Ivanov, 2001): for some ordering of Λ, GDDs do not form a uniformly minimal family.

In the present paper, we continue the study of the GDDs of exponentials and give answers to questions which are very important in applications to control theory. It is known (Avdonin and Ivanov, 1995, Sec. III.3) that the $\mathcal{L}$-basis property of the exponential family is equivalent to the so-called B-controllability of the corresponding dynamical system which, in turn, implies exact controllability. The most efficient way to check that a family forms an $\mathcal{L}$-basis is to prove that it can be extended to a Riesz basis by adjoining a suitable collection of functions.

On the other hand, the strongest negative result about controllability (i.e. the absence of approximate controllability) is equivalent (Avdonin and Ivanov, 1995, Section III.3) to weak linear dependence of the corresponding exponential family. The most efficient way to check that a family is weakly linear dependent is to prove that it can be transformed to a Riesz basis by removing a suitable collection of functions.

The main results of the present paper give simple sufficient conditions for the family of exponential GDDs to be made into a Riesz basis by removing or adjoining a suitable collection of exponentials in terms of the upper and lower uniform densities of the sequence Λ. We also give applications of these results to problems of simultaneous controllability of elastic strings and beams.

2. Main Results

Let $\Lambda = \{\lambda_n\}$ be a sequence in $\mathbb{C}$ ordered in such a way that $\{\Re\lambda_n\}$ forms a non-decreasing sequence. In what follows, we also assume that $\sup|\Im\lambda_n| < \infty$. To each Λ, we associate the exponential family

$$\mathcal{E}(\Lambda) = \{e^{i\lambda_n t}, te^{\lambda_n t}, \ldots, t^{m_n-1}e^{i\lambda_n t}\},$$

where m_n is the multiplicity of $\lambda_n \in \Lambda$.

The sequence Λ is called *uniformly discrete* or *separated* if condition (1) is fulfilled. Note that, in this case, all points λ_n are simple and we do not need to differentiate between a sequence and a set. We say that Λ is *relatively uniformly discrete* if it can be decomposed into a finite number of uniformly discrete subsequences. Sometimes we shall simply say that such a Λ is a finite union of uniformly discrete sets; however, we always consider a point λ_n to be assigned a multiplicity.

We introduce the notations needed to formulate further results. For any $\lambda \in \mathbb{C}$, denote by $D_\lambda(r)$ a disc with centre λ and radius r. Let $G^{(p)}(r)$, $p = 1, 2, \ldots$ be the connected components of the union $\cup_{\lambda\in\Lambda} D_\lambda(r)$, and write $\Lambda^{(p)}(r) = \{\lambda_{j,p}\}$ for the subsequence of Λ lying in $G^{(p)}$, $\Lambda^{(p)}(r) := \Lambda \cap G^{(p)}(r)$.

Denote by $\#\mathcal{A}$ the number of elements in a set or a sequence $\mathcal{A}$. The following two statements are quite obvious (see (Avdonin and Ivanov, 2001) for details):

Lemma 1. *Let Λ be a union of N uniformly discrete sets Λ_j,*

$$\delta(\Lambda_j) := \inf_{\lambda\neq\mu;\ \lambda,\mu\in\Lambda_j} |\lambda - \mu|, \quad \delta := \delta(\Lambda) := \min_j \delta(\Lambda_j).$$

Then, for $r < r_0 := \delta/(2N)$, we have $\mathcal{N}^{(p)}(r) := \#\Lambda^{(p)}(r) \leq N$.

In applications we often meet the case of real Λ's. Then a relatively uniformly discrete set can be characterized using a different parameter than in Lemma 1.

Lemma 2. *A real sequence Λ is a union of N uniformly discrete sets Λ_j if and only if $\inf_n(\lambda_{n+N} - \lambda_n) := \tilde{\delta} > 0$. Along with that, $\min_j \delta(\Lambda_j) \leq \tilde{\delta}$.*

Let μ_k, $k = 1, \dots, m$ be arbitrary complex numbers, not necessarily different.

Definition 1. The generalized divided difference (GDD) of order zero of the function $e^{i\mu t}$ corresponding to the point μ_1 is $[\mu_1](t) := e^{i\mu_1 t}$. The GDD of the order $n-1$, $n \le m$, of $e^{i\mu t}$ corresponding to $\mu_1, \dots, \mu_n$ is

$$[\mu_1, \dots, \mu_n] := \begin{cases} \dfrac{[\mu_1, \dots, \mu_{n-1}] - [\mu_2, \dots, \mu_n]}{\mu_1 - \mu_n}, & \mu_1 \ne \mu_n, \\ \dfrac{\partial}{\partial \mu}[\mu, \mu_2, \dots, \mu_{n-1}]\Big|_{\mu=\mu_1}, & \mu_1 = \mu_n. \end{cases}$$

We use the term *generalized divided differences* when there are multiple points in Λ. If all λ_n have multiplicity one, we use the term *divided differences* (DDs).

If all μ_k are distinct, one can easily derive the explicit formula for the DD:

$$[\mu_1, \dots, \mu_n](t) = \sum_{k=1}^{n} \frac{e^{i\mu_k t}}{\prod_{j \ne k}(\mu_k - \mu_j)}. \tag{3}$$

For any points $\{\mu_k\}$, the following formula is valid (see, e.g. Shilov, 1965, p.228)

$$[\mu_1, \dots, \mu_n](t) = \int_0^1 \mathrm{d}\tau_1 \int_0^{\tau_1} \mathrm{d}\tau_2 \dots \int_0^{\tau_{n-2}} \mathrm{d}\tau_{n-1} (it)^{n-1}$$
$$\exp\Big(it\left[\mu_1 + \tau_1(\mu_2 - \mu_1) + \dots + \tau_{n-1}(\mu_n - \mu_{n-1})\right]\Big).$$

Let $\Lambda^{(p)}(r)$, $p = 1, 2, \dots$ be the subsequences of Λ described above:

$$\Lambda^{(p)}(r) = \{\lambda_{j,p}\}, \quad j = 1, \dots, \mathcal{N}^{(p)}(r).$$

Denote by $\mathcal{E}^{(p)}(\Lambda, r)$ the family of GDDs corresponding to the points $\Lambda^{(p)}(r)$:

$$\mathcal{E}^{(p)}(\Lambda, r) = \{[\lambda_{1,p}], [\lambda_{1,p}, \lambda_{2,p}], \dots, [\lambda_{1,p}, \dots, \lambda_{\mathcal{N}^{(p)},p}]\}.$$

Note that $\mathcal{E}^{(p)}(\Lambda, r)$ depends on the enumeration of $\Lambda^{(p)}$, although every GDD depends symmetrically on its parameters. Write $\{\mathcal{E}^{(p)}(\Lambda, r)\}$ for the family of $\mathcal{E}^{(p)}(\Lambda, r)$ corresponding to all $p = 1, 2, \dots$.

The following theorem describes Riesz bases of GDDs. It is proved in (Avdonin and Ivanov, 2001) using the methods developed in (Avdonin and Ivanov, 1995, Secs. II.2, II.3).

Theorem 2. *Let Λ be a relatively uniformly discrete sequence and $r < r_0$. Then the family $\{\mathcal{E}^{(p)}(\Lambda, r)\}$ forms a Riesz basis in $L^2(0, T)$ if and only if there exists an entire function F of exponential type with indicator diagram of width T and zeros at the points λ_n of multiplicity m_{λ_n}* (the generating function of the family $\mathcal{E}(\Lambda)$ on the interval $(0, T)$) *such that, for some real h, the function $|F(x + ih)|^2$ satisfies the Helson-Szegö condition (2).*

To formulate our next result, we write

$$n^{+}(r) := \sup_{x\in\mathbb{R}} \#\{\Re\Lambda \cap [x, x+r)\}, \quad n^{-}(r) := \inf_{x\in\mathbb{R}} \#\{\Re\Lambda \cap [x, x+r)\},$$

and define in a standard way (see, e.g. Beurling, 1989, p.346) the upper and lower uniform densities of Λ to be respectively

$$\mathcal{D}^{+}(\Lambda) := \lim_{r\to\infty} \frac{n^{+}(r)}{r}, \quad \mathcal{D}^{-}(\Lambda) := \lim_{r\to\infty} \frac{n^{-}(r)}{r}.$$

Both the limits exist due to the subadditivity of $n^{+}(r)$ and superadditivity of $n^{-}(r)$.

The proof of the following theorem is based on the '1/4 in the mean' theorem (Avdonin, 1974; Avdonin and Ivanov, 1995, Section II.4). This theorem generalizes a similar result of K. Seip (1995) concerning the case of uniformly discrete sequences (when instead of GDDs we deal with families of exponentials).

Theorem 3. *Under the conditions of the previous theorem, the following statements are valid:*

(i) *For any $T < 2\pi\mathcal{D}^{-}(\Lambda)$, there exists a subfamily $\mathcal{E}_0$ of $\{\mathcal{E}^{(p)}(\Lambda, r)\}$ which forms a Riesz basis in $L^2(0,T)$; the family $\{\mathcal{E}^{(p)}(\Lambda, r)\} \setminus \mathcal{E}_0$ is infinite.*

(ii) *For any $T > 2\pi\mathcal{D}^{+}(\Lambda)$, the family $\{\mathcal{E}^{(p)}(\Lambda, r)\}$ forms an $\mathcal{L}$-basis in $L^2(0,T)$. Moreover, it can be extended to a family $\mathcal{E}_1$ of GDDs which forms a Riesz basis in this space; the family $\mathcal{E}_1 \setminus \{\mathcal{E}^{(p)}(\Lambda, r)\}$ is infinite.*

Remark 1. The results of this theorem were announced in (Avdonin, 2000). Independently, the $\mathcal{L}$-basis statement of (ii) was announced in (Baiocchi *et al.*, 2000). It should be noted that there, Λ is contained in $\mathbb{R}$.

The fact that exponential divided differences form a Riesz basis or an $\mathcal{L}$-basis in $L^2(0,T)$ can be written as a two-sided inequality which generalizes the classical inequality of Ingham and extends previous results (Baiocchi *et al.*, 1999; Jaffard *et al.*, 1997) in this direction to sets containing an arbitrary number of close points.

Let Λ be a relatively uniformly discrete sequence lying in a strip parallel to the real axis. According to Lemma 1, it can be represented as a union of finite subsequences $\Lambda^{(p)}(r)$ separated from one another, where the number of elements in $\Lambda^{(p)}(r)$ is not greater than some N. Here $\Lambda^{(p)}(r)$ will be denoted by two indices as $\Lambda_n^{(s)}$, where n is the number of elements in this subsequence and s is a numbering of subsequences consisting of n elements. Then we can write Λ in the form

$$\Lambda = \bigcup_{n=1}^{N} \Lambda_n, \quad \text{where} \quad \Lambda_n = \bigcup_{s} \Lambda_n^{(s)},$$

and

$$\operatorname{dist}\left(\Lambda_n^{(s)}, \Lambda \setminus \Lambda_n^{(s)}\right) \geq 2r > 0 \quad \forall n, s.$$

For any finite sequence $(a_\lambda)_{\lambda\in\Lambda}$, we set

$$G_1 = \sum_{\lambda\in\Lambda_1} |a_\lambda|^2, \quad G_2 = \sum_s \sum_{\{\lambda_1,\lambda_2\}\in\Lambda_2^{(s)}} \left(|a_{\lambda_1}+a_{\lambda_2}|^2+|a_{\lambda_2}(\lambda_2-\lambda_1)|^2\right),$$

$$G_3 = \sum_s \sum_{\{\lambda_1,\lambda_2,\lambda_3\}\in\Lambda_3^{(s)}} \Big(|a_{\lambda_1}+a_{\lambda_2}+a_{\lambda_3}|^2+|a_{\lambda_2}(\lambda_2-\lambda_1)+a_{\lambda_3}(\lambda_3-\lambda_1)|^2$$
$$+ |a_{\lambda_3}(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)|^2\Big),$$

$$\vdots$$

$$G_N = \sum_s \sum_{\{\lambda_1,\lambda_2,\ldots,\lambda_N\}\in\Lambda_N^{(s)}} \left(\left|\sum_{k=1}^N a_{\lambda_k}\right|^2\right.$$
$$\left. + \sum_{m=2}^N \left|\sum_{k=m}^N a_{\lambda_k}(\lambda_k-\lambda_1)(\lambda_k-\lambda_2)\ldots(\lambda_k-\lambda_{m-1})\right|^2\right).$$

Formula (3) directly implies the following statement:

Proposition 1. *Let all points λ_n have multiplicity one. If the family $\{\mathcal{E}^{(p)}(\Lambda,r)\}$ of exponential divided differences forms an $\mathcal{L}$-basis in $L^2(0,T)$, then there exist positive constants C_1 and C_2 such that*

$$C_1\sum_{n=1}^N G_n \le \left\|\sum_{\lambda\in\Lambda} a_\lambda e^{i\lambda t}\right\|^2_{L^2(0,T)} \le C_2 \sum_{n=1}^N G_n$$

for any finite sequence $(a_\lambda)_{\lambda\in\Lambda}$.

3. Proof of Theorem 3

We need the following definition from (Avdonin, 1974). Let Λ be, as in Section 2, a sequence of complex numbers in a strip parallel to the real axis, and $\{\alpha_j\}_{j\in\mathbb{Z}}\subset\mathbb{R}$ be an increasing sequence such that $\sup_{j\in\mathbb{Z}} l_j<\infty$, $l_j := \alpha_{j+1}-\alpha_j$. Partition

$$\Lambda = \bigcup_{j\in\mathbb{Z}} \Lambda_j, \quad \Lambda_j := \{\alpha_j \le \Re\lambda_n < \alpha_{j+1}\},$$

is said to be an α-partition of Λ.

The proof of Theorem 3 is based on the following result (Avdonin, 1974).

Proposition 2. *Let $\Lambda=\{\lambda_n\}$ be a zero set of a sine-type function (see the corresponding definition in (Avdonin and Ivanov, 1995, p.61)), $\{\delta_n\}$ a bounded sequence of complex numbers, and F an entire function of the Cartwright class (Avdonin and*

Ivanov, 1995, p.60) with the zero set $\{\lambda_n + \delta_n\}$*. If, for some* α*-partition of* $\{\lambda_n\}$ *and some* $d > 0$,

$$\Big| \sum_{n:\lambda_n \in \Lambda_j} \Re\delta_n \Big| \le d\, l_j,$$

then, for any $d_1 > d$*, functions* $u, v \in L^\infty(\mathbb{R}), \|v\|_{L^\infty(\mathbb{R})} < 2\pi d_1$ *can be found such that*

$$|F(x+ih)|^2 = \exp\{u(x) + \tilde{v}(x)\}$$

for any real h *satisfying that* $|h| > \sup|\Im(\lambda_n + \delta_n)|$.

For the proof of this assertion, one argues using (Avdonin, 1974, Lemma 1), and in much the same way as in the proof of (Avdonin, 1974, Lemma 2), that there exists a sine-type function with zeros μ_n such that, for any $d_1 > d$,

$$d_1 \Re(\mu_{n-1} - \mu_n) \le \Re(\lambda_n + \delta_n - \mu_n) \le d_1 \Re(\mu_{n+1} - \mu_n).$$

Then Proposition 2 follows directly from (Avdonin and Joó, 1988, Lemmas 1 and 2).

The proof of Theorem 3 follows the scheme of the proofs of Theorems 2.3 and 2.4 in (Seip, 1995). Note that the same ideas based on the '1/4 in the mean' theorem (Avdonin, 1974) were also used in (Avdonin *et al.*, 1989; Avdonin and Ivanov, 1995, pp.109, 110).

By a scaling argument, we may assume that, in the conditions of assertion (i) of Theorem 3, $T = 2\pi$ and $\mathcal{D}^-(\Lambda) > 1$. Then an integer L greater than $2\delta(\Lambda)$ can be found such that the number of elements of $\Re\Lambda$ in each interval of length $L - 2\delta(\Lambda)$ is at least $L+1$. Set $M = L(L+1)$ and consider the problem of how to select M elements from the sequence

$$\Lambda(m) := \{\lambda_n \in \Lambda \colon\ \Re\lambda_n \in [mM + 1/2, (m+1)M + 1/2)\}.$$

We choose elements λ_n from the subsequences $\Lambda^{(p)}$ whose real parts are entirely inside the intervals $[mM + 1/2 + kL, mM + 1/2 + (k+1)L]$, $k = 0, 1, \ldots, L$. Such $\Lambda^{(p)}$'s exist in each of the intervals because $\max\{|\lambda - \lambda'| \colon\ \lambda, \lambda' \in \Lambda^{(p)}\} < \delta(\Lambda)$ and L was selected to be large enough. When we choose elements of $\Lambda^{(p)}$, we select the *first* several elements of this sequence. Thus we will be able to obtain a subfamily of $\{\mathcal{E}^{(p)}(\Lambda, r)\}$ which forms a Riesz basis in $L^2(0, 2\pi)$.

We represent the chosen points in the form

$$\lambda_n' = n + \delta_n, \quad n = mM + 1, mM + 2, \ldots, (m+1)M,$$

and consider the following α-partition of the set of integers:

$$\mathbb{Z} = \bigcup_{m\in\mathbb{Z}} \mathbb{Z}_m, \quad \mathbb{Z}_m = \{n \in \mathbb{Z} :\ mM + 1/2 \le n < (m+1)M + 1/2\}.$$

Let $S_{\min}(m)$ and $S_{\max}(m)$ denote respectively the smallest and largest possible values of the sum

$$\sum_{n=mM+1}^{(m+1)M} \Re\delta_n.$$

It is easy to see that

$$S_{\min}(m) < (L+1)\sum_{j=1}^{L}(jL+\frac{1}{2}) - \sum_{j=1}^{M} j = 0$$

and

$$S_{\max}(m) > (L+1)\sum_{j=1}^{L}(M+\frac{1}{2}-jL) - \sum_{j=1}^{M} j = 0.$$

Since $\sup_{n\in\mathbb{Z}}(\Re\lambda_{n+1} - \Re\lambda_n) \le 2L$, it follows that λ'_n can be chosen in such a way that

$$\Big|\sum_{n=mM+1}^{(m+1)M} \Re\delta_n\Big| \le L = \frac{M}{L+1}.$$

Consider the subfamily $\mathcal{E}_0 \subset \{\mathcal{E}^{(p)}(\Lambda, r)\}$, $\mathcal{E}_0 := \mathcal{E}(\Lambda')$, $\Lambda' := \{\lambda'_n\}$, corresponding to such a choice of Λ'. From Proposition 2 it follows that the generating function of the family $\mathcal{E}_0$ satisfies the Helson-Szegö condition if $L > 3$. It is clear from the construction of Λ' that $\mathcal{D}^+(\Lambda') = \mathcal{D}^-(\Lambda') = 1$ and so the width of the indicator diagram of the generating function is equal to 2π. Owing to Theorem 2, the family $\mathcal{E}_0$ forms a Riesz basis in $L^2(0, 2\pi)$; the sequence $\Lambda \setminus \Lambda'$ is infinite by construction. Assertion (i) of Theorem 3 is proved.

Assertion (ii) can be proved in the same way. If $\mathcal{D}^+(\Lambda) < 1$, an integer L greater than $2\delta(\Lambda)$ can be found such that the number of elements of $\Re\Lambda$ in each interval of length L is at most $L-1$. Set $M = L(L+1)$. Consider the problem of how to extend the sequence

$$\Lambda(m) := \{\lambda_n \in \Lambda\colon\ \Re\lambda_n \in [mM+1/2, (m+1)M+1/2)\}$$

consisting at most of $(L-1)(L+1)$ elements, to a sequence consisting of $L(L+1)$ elements. We can take the additional points with real parts from the interval $[mM + 1/2, (m+1)M+1/2)$ in a rather arbitrary way and choose, in particular, all the points adjoined to $\Lambda(m)$ to lie in an interval of length L. It is only important that if we choose additional points from a set $G^{(p)}(r)$, we adjoin them as the *last* elements of the corresponding $\Lambda^{(p)}(r)$, and this allows us to extend the family $\mathcal{E}^{(p)}(\Lambda, r)$ to a Riesz basis.

Let now

$$\lambda'_n = n + \delta_n, \quad n = mM+1, mM+2, \dots, (m+1)M$$

denote an arbitrary extension of $\Lambda(m)$, and let $S_{\min}(m)$ and $S_{\max}(m)$ be respectively the smallest and largest possible values of the sum

$$\sum_{n=mM+1}^{(m+1)M} \Re\delta_n.$$

Then

$$S_{\min}(m) < (L-1)\sum_{j=1}^{L+1}(jL+\frac{1}{2}) + (L+1)(L+\frac{1}{2}) - \sum_{j=1}^{M} j = 0$$

and

$$S_{\max}(m) > (L-1)\sum_{j=0}^{L}(jL+\frac{1}{2}) + (L+1)(M+\frac{1}{2}-L) - \sum_{j=1}^{M} j = 0.$$

The remainder of the proof is the same as that of assertion (i).

4. Applications to Simultaneous Control Problems

Boundary controllability of elastic systems has been intensively studied in recent years (see, e.g. (Avdonin and Ivanov, 2001; Lagnese and Lions, 1988; Lions, 1988) and the references therein). In particular, the following challenging problem was discussed for some distributed parameter systems in (Russell, 1986; Lions, 1988, Ch. 5): For a collection of several exactly controllable systems, find assumptions allowing for a simultaneous control of all systems using the same input function. This property is called *simultaneous controllability.* For two elastic strings it was studied in (Avdonin and Tucsnak, 2001; Baiocchi *et al.*, 1999; Tucsnak and Weiss, 2000), and for two beams in (Baiocchi *et al.*, 1999).

In (Avdonin and Moran, 2001), we investigated the simultaneous controllability of several strings subjected to the Dirichlet boundary conditions. In the present paper, we consider this question for several beams controlled from a common end point and for a string-beam system.

4.1. Several Beams

For $\xi_j > 0,\ j = 1,\dots,N$, we consider the problems

$$\begin{cases} \dfrac{\partial^2 u_j}{\partial t^2}(x,t) + \dfrac{\partial^4 u_j}{\partial x^4}(x,t) = 0 & \forall\, x \in (0,\xi_j), \quad \forall\, t \in (0,\infty), \\ u_j(0,t) = 0, \quad u_j(\xi_j,t) = 0 & \forall\, t \in (0,\infty), \\ \dfrac{\partial^2 u_j}{\partial x^2}(0,t) = f(t), \quad \dfrac{\partial^2 u_j}{\partial x^2}(\xi_j,t) = 0 & \forall\, t \in (0,\infty), \\ u_j(x,0) = 0, \quad \dfrac{\partial u_j}{\partial t}(x,0) = 0 & \forall\, x \in (0,\xi_j). \end{cases} \tag{4}$$

The systems above model the vibrations of several beams controlled at a common end point, $x = 0$.

We introduce now the operators $A_j,\ j = 1,\dots,N$ defined by

$$\mathcal{D}(A_j) = H^2(0,\xi_j) \cap H_0^1(0,\xi_j), \quad A_j\colon\ \mathcal{D}(A_j) \mapsto L^2(0,\xi_j), \quad A_j h = -\frac{\mathrm{d}^2 h}{\mathrm{d}x^2},$$

and the spaces W_j^s and $\mathcal{W}^s$ defined as follows:

$$W_j^s = \mathcal{D}\left(A_j^{s/2}\right) \text{ for } s > 0, \quad W_j^0 = L^2(0,\xi_j), \quad W_j^s = \left(W_j^{-s}\right)' \text{ for } s < 0, \tag{5}$$

$$\mathcal{W}^s = \prod_{j=1}^{N} W_j^{s+1} \times W_j^{s-1}.$$

Each operator A_j possesses a set of eigenfunctions which forms an orthonormal basis in $L^2(0,\xi_j)$:

$$\varphi_n^{(j)}(x) = \sqrt{2/\xi_j}\,\sin(n\pi x/\xi_j), \quad n \in \mathbb{N} \text{ for } \ j = 1,\dots,N.$$

It is well-known (see, e.g. Lions, 1988) that, for $f \in L^2(0,T)$, each of the systems in (4) is well-posed in $H_0^1(0,\xi_j) \times H^{-1}(0,\xi_j)$. This allows us to define the bounded linear operator

$$U^T\colon \quad L^2(0,T) \mapsto \prod_{j=1}^{N} H_0^1(0,\xi_j) \times H^{-1}(0,\xi_j) = \mathcal{W}^0,$$

$$U^T f = \left(u_1(\cdot,T), \dot{u}_1(\cdot,T), u_2(\cdot,T), \dot{u}_2(\cdot,T), \dots, u_N(\cdot,T), \dot{u}_N(\cdot,T)\right),$$

where the upper dot denotes the derivative with respect to time.

The space of the states simultaneously reachable by the systems (4) in the time interval $[0,T]$ is defined as the range $\mathcal{R}^T := U^T(L^2(0,T))$ of the operator U^T. According to the properties of this space, we can define several types of *simultaneous controllability.*

Definition 2.

1. The systems (4) are called *simultaneously approximately controllable* in the time interval $[0,T]$ if $\mathcal{R}^T$ is dense in $\mathcal{W}^0$.

2. The systems (4) are called *simultaneously spectrally controllable* in the time interval $[0,T]$ if, for all $n \geq 1$, the states $(\varphi_n^{(1)}, 0, 0, \dots, 0)$, $(0, \varphi_n^{(1)}, 0, \dots, 0), \dots, (0, 0, \dots, 0, \varphi_n^{(N)}, 0), (0, 0, \dots, 0, \varphi_n^{(N)})$ are reachable at time T, i.e. if they belong to $\mathcal{R}^T$.

3. The systems (4) are called *simultaneously exactly controllable* in the time interval $[0,T]$ with respect to a space $\mathcal{U} \subseteq \mathcal{W}^0$ if $\mathcal{R}^T \supseteq \mathcal{U}$.

The following three theorems describe, respectively, simultaneous approximate controllability, simultaneous spectral controllability and the simultaneously reachable space. We set $\theta_{jk} = \xi_j/\xi_k$, $1 \leq j < k \leq N$.

Theorem 4. *If at least one of* θ_{jk}*'s is rational, the systems* (4) *are not simultaneously approximately controllable for any* $T > 0$.

Theorem 5. *If all θ_{jk}'s are irrational, the systems* (4) *are simultaneously spectrally controllable in the time interval* $[0,T]$ *for any* $T>0$.

To describe further results, we need some concepts from the theory of diophantine approximation. Denote by $\mathcal{S}$ the set of all irrational numbers ρ such that if $[a_0,a_1,\dots,a_n,\dots]$ is the expansion of ρ as a continued fraction, then the sequence of partial quotients (a_n) is bounded. This is the set of 'badly approximable numbers'. Note that $\mathcal{S}$ is obviously uncountable and, by classical results on diophantine approximation (cf. Cassels, 1965, p.120), its Lebesgue measure is equal to zero.

Theorem 6. *For any* $T,\varepsilon>0$ *the following statements are valid:*

(a) *If all θ_{jk}'s belong to $\mathcal{S}$, the systems (4) are simultaneously exactly controllable with respect to* $\prod_{j=1}^N H_0^1(0,\xi_j)\times H^{-1}(0,\xi_j)$.

(b) *For almost all ξ_j's, the systems (4) are simultaneously exactly controllable with respect to* $\prod_{j=1}^N H_0^{1+\varepsilon}(0,\xi_j)\times H^{-1+\varepsilon}(0,\xi_j)$.

To prove Theorems 4–6, we apply the method of moments based on the Fourier method (separation of variables). We represent the solution of initial boundary value problems (4) in a form of series expansions by functions $\varphi_n^{(j)}$:

$$u_j(x,t)=\sum_{n=1}^{\infty}a_n^{(j)}(t)\varphi_n^{(j)}(x).$$

Standard calculations show that the coefficients $a_n^{(j)}$ satisfy the equalities

$$a_n^{(j)}(T)=-\sqrt{2/\xi_j}\,\big(\mu_n^{(j)}\big)^{-1}\int_0^T f(t)\sin\lambda_n^{(j)}(T-t)\,\mathrm{d}t, \tag{6}$$

$$\dot a_n^{(j)}(T)=-\sqrt{2/\xi_j}\mu_n^{(j)}\int_0^T f(t)\cos\lambda_n^{(j)}(T-t)\,\mathrm{d}t, \tag{7}$$

where $\mu_n^{(j)}=n\pi/\xi_j$ and $\lambda_n^{(j)}=(n\pi/\xi_j)^2$. Equalities (6) and (7) can be written in the form

$$c_k^{(j)}(T)=\int_0^T f(t)e^{-i\lambda_k^{(j)}t}\,\mathrm{d}t, \tag{8}$$

where $k\in\mathbb{Z}_*:=\mathbb{Z}\setminus\{0\}$, $\mu_{-n}^{(j)}:=-\mu_n^{(j)},\lambda_{-n}^{(j)}=-\lambda_n^{(j)}$, and

$$c_k^{(j)}(T):=-e^{-i\mu_k^{(j)}T}\sqrt{\xi_j/2}\left[i\mu_k^{(j)}a_k^{(j)}(T)+\dot a_k^{(j)}(T)/\mu_k^{(j)}\right]\quad\text{for}\quad k>0, \tag{9}$$

$$c_k^{(j)}(T):=e^{-i\mu_k^{(j)}T}\sqrt{\xi_j/2}\left[-i\mu_k^{(j)}a_{|k|}^{(j)}(T)+\dot a_{|k|}^{(j)}(T)/\mu_k^{(j)}\right]\quad\text{for}\quad k<0. \tag{10}$$

To proceed further, we introduce the spaces $\ell_s^2, s\in\mathbb{R}$ of sequences $\{c_k\}, k\in\mathbb{Z}_*$ such that

$$\|c_k\|_s^2:=\sum_{k\in\mathbb{Z}_*}|c_k|^2|k|^{2s}<\infty$$

and set $\mathcal{V}_s := \ell_s^2 \times \ell_s^2 \times \cdots \times \ell_s^2$ (N factors). From (9) and (10) it follows that

$$m_j \|\{c_k^{(j)}(T)\}\|_s^2 \le \|u_j(\cdot,T)\|^2_{W_j^{s+1}} + \|\dot{u}_j(\cdot,T)\|^2_{W_j^{s-1}} \le M_j \|\{c_k^{(j)}(T)\}\|_s^2 \quad (11)$$

for some positive constants m_j and M_j.

Therefore (cf. (8) and (11)), the reachable set $\mathcal{R}^T$ is isomorphic to the set of sequences $\{b_k^{(1)}, b_k^{(2)}, \dots, b_k^{(N)}\}$ for which the problem of moments

$$b_k^{(j)} = \left(f, e^{i\lambda_k^{(j)} t}\right)_{L^2(0,T)}, \quad j = 1,\dots,N, \quad k \in \mathbb{Z}_*, \quad (12)$$

has a solution $f \in L^2(0,T)$ in the sense that, for any $s \in \mathbb{R}$, the norm of $U^T f$ in $\mathcal{W}^s$ is equivalent to the norm of $\{b_k^{(1)}, b_k^{(2)}, \dots, b_k^{(N)}\}$ in $\mathcal{V}_s$.

We introduce now the exponential family

$$\mathcal{E} = \bigcup_{j=1}^{N} \mathcal{E}_j, \quad \mathcal{E}_j = \left\{ e^{i\lambda_k^{(j)} t} \right\}_{k \in \mathbb{Z}_*}.$$

In what follows we will extensively use Theorem III.3.10 of (Avdonin and Ivanov, 1995), which connects properties of the reachable set with properties of the family $\mathcal{E}$ in $L^2(0,T)$.

If θ_{jq} is rational for some j, q $(1 \le j < q \le N)$, there are infinitely many $m, n \in \mathbb{N}$ such that $m\pi/\xi_j = n\pi/\xi_q$. Then the family $\mathcal{E}$ is linearly dependent in $L^2(0,T)$ for any T. Theorem 4 follows now from (Avdonin and Ivanov, 1995, Thm. III.3.10(e)). Moreover, the codimension of the reachable set is infinite.

Now we suppose that all θ_{jq}'s are irrational and so all $\lambda_k^{(j)}$'s are distinct.

Proof of Theorem 5. A necessary and sufficient condition of the spectral controllability of systems (4) is the minimality of the family $\mathcal{E}$ (Avdonin and Ivanov, 1995, Thm. III.3.10(d)). By the classical result of Paley and Wiener (see, e.g. Avdonin and Ivanov, 1995, p.99), the family $\mathcal{E}$ is minimal in $L^2(0,T)$ if and only if there exists an entire function F of exponential type not greater than $T/2$ such that

$$F(\lambda_k^{(j)}) = 0 \ \ \forall j,k \ \text{ and } \int_{\mathbb{R}} |F(x)|^2 (1+x^2)^{-1}\, \mathrm{d}x < \infty. \quad (13)$$

Both the upper and lower uniform densities of the sequence $\Lambda := \cup_j \{\lambda_k^{(j)}\}_{k\in\mathbb{Z}_*}$ evidently equal zero. Therefore, exactly as in the proof of the statement (ii) of Theorem 3, for any $T > 0$ one can construct an entire function F of the exponential type $T/2$ which satisfies the first condition of (13) and the Helson-Szegö condition. It is known (Garnett, 1981, Ch. VI) that the Helson-Szegö condition implies the second condition in (13). Theorem 5 is thus proved. ■

Proof of Theorem 6. We set $\delta = \min_{j=1,\dots,N} 3(\pi/\xi_j)^2$ and take $r < \delta/(2N)$. Then, following the algorithm described in Section 2, we decompose the sequence Λ into an infinite number of finite sets $\Lambda^{(p)}(r) = \{\lambda_{qp}\}$, $q = 1,\dots,\mathcal{N}^{(p)} \le N$. Using the new numbering of points of Λ, we also obtain a new numbering $\{b_{qp}\}$ of the sequences on the left-hand side of (12). Denote by $\mathcal{E}^{(p)}(r) = \{\phi_{qp}\}$, $q = 1, \ldots, \mathcal{N}^{(p)}$ the family of the divided differences correponding to $\Lambda^{(p)}(r)$. From (3) it follows that equalities (12) can be rewritten in the form

$$\beta_{qp} = \int_0^T f(t)\phi_{qp}(t)\,\mathrm{d}t, \quad q = 1, \ldots, \mathcal{N}^{(p)}, \quad p = 1, 2, \ldots,$$

where

$$\beta_{qp} = \sum_{i=1}^{q} \frac{b_{ip}}{\prod_{s \neq i}(\lambda_{ip} - \lambda_{sp})}. \tag{14}$$

Using Theorem 2 and the function $F(z)$ constructed in the proof of Theorem 5, we see that the family $\{\mathcal{E}^{(p)}(r)\}$ forms an $\mathcal{L}$-basis for any $T > 0$. Therefore the sequence $\{b_{ip}\}$ corresponds to a solvable problem of moments (12) if and only if the sequence $\{\beta_{qp}\}$ constructed by (14) belongs to ℓ^2.

For some $m, n \in \mathbb{Z}$ and some j, k, we have

$$\lambda_{qp} = \frac{m\pi}{\xi_j}, \quad \lambda_{sp} = \frac{n\pi}{\xi_k}.$$

If all θ_{jk}'s belong to $\mathcal{S}$, then there exists a positive constant C_1 such that

$$\begin{aligned} |\lambda_{qp} - \lambda_{sp}| &= \left|\frac{m\pi}{\xi_j} - \frac{n\pi}{\xi_k}\right| \left|\frac{m\pi}{\xi_j} + \frac{n\pi}{\xi_k}\right| \\ &= \left(\frac{\pi}{\xi_j}\right)^2 |\theta_{jk}n - m|\,|\theta_{jk}n + m| \geq \frac{C_1}{n}n = C_1. \end{aligned}$$

We used the fact that if $\theta \in \mathcal{S}$, then there exists $C_\theta > 0$ such that, for any $n \in \mathbb{N}$,

$$\inf_{m \in \mathbb{N}} |\theta n - m| \geq \frac{C_\theta}{n}.$$

Hence (see (14)),

$$\beta_{qp} \leq C_2 \sum_{i=1}^{q} |b_{ip}|.$$

Since $q \leq N$, it is clear that

$$\{\beta_{qp}\} \in \ell^2 \quad \text{if} \quad \{b_{jp}\} \in \ell^2.$$

Assertion (a) of Theorem 6 is thus proved. Assertion (b) can be proved in exactly the same way using the fact that for almost all θ, the inequality $|\theta n - m| \geq C_\theta n^{-1-\varepsilon}$ is valid (Cassels, 1965, p.120). ∎

4.2. String-Beam Systems

One can study similar problems for several strings and several beams controlled at a common end point. Let M and N be positive integers. For $\xi_j > 0,\ j = 1, \ldots, M$, we consider the initial boundary value problems for the string equations

$$\begin{cases} \dfrac{\partial^2 u_j}{\partial t^2}(x,t) - \dfrac{\partial^2 u_j}{\partial x^2}(x,t) = 0 & \forall\, x \in (0,\xi_j), \quad \forall\, t \in (0,\infty), \\ u_j(0,t) = f(t), \quad u_j(\xi_j,t) = 0 & \forall\, t \in (0,\infty), \\ u_j(x,0) = 0, \quad \dfrac{\partial u_j}{\partial t}(x,0) = 0 & \forall\, x \in (0,\xi_j), \end{cases} \tag{15}$$

and for $\xi_j > 0,\ j = M+1, \ldots, M+N$ we consider the problems as in (4):

$$\begin{cases} \dfrac{\partial^2 u_j}{\partial t^2}(x,t) + \dfrac{\partial^4 u_j}{\partial x^4}(x,t) = 0 & \forall\, x \in (0,\xi_j), \quad \forall\, t \in (0,\infty), \\ u_j(0,t) = 0, \quad u_j(\xi_j,t) = 0 & \forall\, t \in (0,\infty), \\ \dfrac{\partial^2 u_j}{\partial x^2}(0,t) = f(t), \quad \dfrac{\partial^2 u_j}{\partial x^2}(\xi_j,t) = 0 & \forall\, t \in (0,\infty), \\ u_j(x,0) = 0, \quad \dfrac{\partial u_j}{\partial t}(x,0) = 0 & \forall\, x \in (0,\xi_j) \end{cases} \tag{16}$$

Using the spaces W_j^s defined in (5), we introduce the spaces $\mathcal{W}^s$ in the following way:

$$\mathcal{W}^s = \left(\prod_{j=1}^{M} W_j^s \times W_j^{s-1} \right) \times \left(\prod_{j=M+1}^{M+N} W_j^{s+1} \times W_j^{s-1} \right).$$

It is well-known that, for $f \in L^2(0,T)$, each of the systems in (15) is well-posed in $L^2(0,\xi_j) \times H^{-1}(0,\xi_j)$ (see, e.g. Lions, 1988; Avdonin and Ivanov, 1995). This allows us to define the bounded linear operator $U^T: \quad L^2(0,T) \mapsto \mathcal{W}^0$,

$$U^T f = (u_1(\cdot,T), \dot{u}_1(\cdot,T), u_2(\cdot,T), \dot{u}_2(\cdot,T), \ldots, u_{M+N}(\cdot,T), \dot{u}_{M+N}(\cdot,T)).$$

The notions of simultaneous approximate, spectral and exact controllabilities for systems (15), (16) can be introduced as in Definition 2.

We set $T_* = 2\sum_{j=1}^{M} \xi_j$; $\theta_{jk} = \xi_j/\xi_k$ for $1 \le j < k \le M$ and for $M+1 \le j < k \le M+N$, and $\sigma_{jk} = \pi\xi_j/\xi_k^2$ for $1 \le j \le M,\ M+1 \le k \le M+N$. The results concerning, respectively, simultaneous approximate controllability, simultaneous spectral controllability and characterization of the simultaneously reachable space of systems (15), (16) are expressed in the following three theorems:

Theorem 7. *The following statements are valid:*

(a) *If at least one of θ_{jk} or σ_{jk} is rational, the systems (15) and (16) are not simultaneously approximately controllable for any $T > 0$.*

(b) *For any ξ_j, the systems* (15) *and* (16) *are not simultaneously approximately controllable in time $T \le T_*$.*

Theorem 8. *If all θ_{jk} and σ_{jk} are irrational, the systems (15) and (16) are simultaneously spectrally controllable in the time interval $[0,T]$ for any $T > T_*$.*

Theorem 9. *For any $T > T_*, \varepsilon > 0$, the following statements are valid:*

(a) *If all θ_{jk} and σ_{jk} belong to $\mathcal{S}$, the systems (15) and (16) are simultaneously exactly controllable with respect to*

$$\left(\prod_{j=1}^{M} W_j^{M+N-1} \times W_j^{M+N-2}\right) \times \left(\prod_{j=M+1}^{M+N} W_j^{2M+1} \times W_j^{2M-1}\right).$$

(b) *For almost all ξ_j, the systems (15) and (16) are simultaneously exactly controllable with respect to*

$$\left(\prod_{j=1}^{M} W_j^{M+N-1+\varepsilon} \times W_j^{M+N-2+\varepsilon}\right) \times \left(\prod_{j=M+1}^{M+N} W_j^{2M+1+\varepsilon} \times W_j^{2M-1+\varepsilon}\right).$$

The proofs of these theorems can be carried out similarly to those of Theorems 4–6. However, they are more complicated and will be presented in a forthcoming paper.

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