SPATIAL COMPENSATION OF BOUNDARY DISTURBANCES BY BOUNDARY ACTUATORS

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In this paper we show how to find convenient boundary actuators, termed boundary efficient actuators, ensuring finite-time space compensation of any boundary disturbance. This is the so-called remediability problem. Then we study the relationship between this remediability notion and controllability by boundary actuators, and hence the relationship between boundary strategic and boundary efficient actuators. We also determine the set of boundary remediable disturbances, and for a boundary disturbance, we give the optimal control ensuring its compensation.

Keywords: distributed-parameter systems, remediability, controllability, actuators, sensors

1. Problem Statement

This paper deals with the notion of remediability and efficient actuators introduced in (Afifi *et al.*, 1998; 1999; 2000) for a class of linear distributed systems. The remediability problem consists in studying the existence of a convenient input operator (efficient actuators), ensuring the compensation of any disturbance acting on the considered system. The previous works on the problem of remediability are focused on the compensation of internal disturbances, and this paper constitutes an extension to the boundary case. The proposed approach and the problem itself are different from those considered in previous works on disturbance problems, the so-called disturbance rejection or decoupling problems, particularly studied for finite-dimensional systems (Malabre and Rabah, 1993; Otsuka, 1991; Pandolfi, 1986; Rabah and Malabre, 1997; Senamel *et al.*, 1995).

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In this paper without loss of generality, we focus our attention on a class of disturbed linear systems described by the following state equation:

$$(S_P)\begin{cases} \dfrac{\partial z}{\partial t}(x,t) &= \Delta z(x,t) \qquad \text{in} \quad \Omega\times]0,T[, \\ \dfrac{\partial z}{\partial \nu}(\cdot,t) &= f(\cdot,t)+B(\cdot)u(t) \quad \text{on} \quad \Gamma\times]0,T[, \\ z(x,0) &= 0 \qquad \text{in} \quad \Omega, \end{cases}$$

where Ω is an open and bounded subset of $\mathbb{R}^n$ with a sufficiently regular boundary $\Gamma=\partial\Omega$, $B\in\mathcal{L}(\mathcal{U};L^2(\Gamma))$, $u\in L^2(0,T;\,\mathcal{U})$, $\mathcal{U}$ is a Hilbert space (control space), and $\partial/\partial\nu$ is the partial derivative with respect to the outward unit normal of $\partial\Omega$. The disturbance $f\in L^2(0,T;L^2(\Gamma))$ is in general unknown. Let A be the operator defined by

$$\mathcal{D}(A)=\left\{z\in H^2(\Omega)\;\Big|\;\frac{\partial z}{\partial \nu}_{/\Gamma}=0\right\} \quad \text{and} \quad Az=\Delta z \quad \text{for} \quad z\in\mathcal{D}(A).$$

A generates a strongly continuous semigroup (s.c.s.g.) $(S(t))_{t\geq 0}$ which is self-adjoint and analytic.

The system (S_P) is augmented by the output equation

$$(E)\quad y(t)=Cz(t),$$

where $C\in\mathcal{L}(L^2(\Omega),Y)$, Y is a Hilbert space (observation space) and $z(t)$ is identified with $z(\cdot,t)$.

In this paper we show how to find convenient (efficient) boundary actuators ensuring the compensation of any known or unknown boundary disturbances. This is the basic concept of remediability, which turns out to be a weaker notion than controllability. We also determine the set of boundary disturbances which are exactly remediable, and we construct, using an extension of the Hilbert Uniqueness Method (H.U.M., Lions, 1988; Lions and Magenes, 1968), an optimal control which ensures the exact compensation of a boundary disturbance acting on the system.

Let G be the Green operator (Necas, 1967) defined by

$$G:\begin{array}{ccc} L^2(\Gamma) & \longrightarrow & L^2(\Omega), \\ g & \longmapsto & Gg=h, \end{array} \tag{1}$$

with

$$\begin{cases} h-\Delta h = 0 \ \text{ in } \ \Omega \\ \dfrac{\partial h}{\partial \nu} = g \ \text{ on } \ \Gamma. \end{cases}$$

The solution to (S_P), denoted by $z_{u,f}$, has the from

$$\begin{aligned} z_{u,f}(t) \;=\; & -\int_0^t AS(t-s)GBu(s)\,\mathrm{d}s + \int_0^t S(t-s)GBu(s)\,\mathrm{d}s \\ & -\int_0^t AS(t-s)Gf(s)\,\mathrm{d}s + \int_0^t S(t-s)Gf(s)\,\mathrm{d}s. \end{aligned}$$

We have

$$z_{u,f}(T) = Hu + \tilde{H}f, \tag{2}$$

where H and $\tilde{H}$ are the linear operators defined as follows:

$$H : \begin{array}{rcl} L^2(0,T;\mathcal{U}) & \longrightarrow & L^2(\Omega), \\ u & \longmapsto & Hu = -\displaystyle\int_0^T AS(T-s)GBu(s)\,\mathrm{d}s \\ & & \quad + \displaystyle\int_0^T S(T-s)GBu(s)\,\mathrm{d}s, \end{array} \tag{3}$$

and

$$\tilde{H} : \begin{array}{rcl} L^2\big(0,T;L^2(\Gamma)\big) & \longrightarrow & L^2(\Omega) \\ f & \longmapsto & \tilde{H}f = -\displaystyle\int_0^T AS(T-s)Gf(s)\,\mathrm{d}s \\ & & \quad + \displaystyle\int_0^T S(T-s)Gf(s)\,\mathrm{d}s. \end{array} \tag{4}$$

Hence

$$y_{u,f}(T) = Cz_{u,f}(T) = CHu + C\tilde{H}f. \tag{5}$$

Let R be the linear operator defined by

$$R : \begin{array}{rcl} L^2\big(0,T;L^2(\Gamma)\big) & \longrightarrow & Y, \\ f & \longmapsto & Rf = C\tilde{H}f. \end{array} \tag{6}$$

We then have

$$y_{u,f}(T) = CHu + Rf. \tag{7}$$

If $f = 0$ and $u = 0$, the observation is given by $y_{0,0}(t) = 0$, but if $f \neq 0$ and $u = 0$, then

$$y_{0,f}(t) = -\int_0^t CAS(t-s)Gf(s)\,\mathrm{d}s + \int_0^t CS(t-s)Gf(s)\,\mathrm{d}s.$$

The problem consists in studying the existence of an input operator B (actuators), with respect to a given output operator C (sensors), ensuring finite-time compensation of any boundary disturbance; in other words, we wish to show that for any

$f \in L^2(0,T;L^2(\Gamma))$ there exists $u \in L^2(0,T;\mathcal{U})$ such that $y_{u,f}(T) = 0$, or for a given $\epsilon > 0$ and any $f \in L^2(0,T;L^2(\Gamma))$ there exists $u \in L^2(0,T;\mathcal{U})$ such that $\|y_{u,f}(T)\| < \epsilon$.

This work is organized as follows. In Section 2 we define and characterize the notions of exact and weak remediabilities as well as efficient boundary actuators. In Section 3 we study the problem of exact remediability with minimum energy using an extention of the H.U.M. Then we characterize the set of boundary disturbances which are exactly remediable, and we construct on optimal control which compensates exactly an arbitrary disturbance acting on the boundary of the considered system. In Section 4 we recall the notions of controllability and strategic actuators in the boundary case and study the relationship between controllability and remediability, and hence between strategic and efficient actuators. As applications we consider the cases where the geometrical domain is a rectangle or a disc.

2. Remediability

2.1. Definition and Characterization

The definitions of exact and weak remediabilities are analoguous to the case of internal disturbances, but the characterization results are different, as they reflect the boundary aspect of the considered problem.

Definition 1.

(i) We say that the system (S_P) augmented by the output equation (E) (or $(S_P)+(E)$) is *exactly remediable* on $[0,T]$ if for every $f \in L^2(0,T;L^2(\Gamma))$ there exists $u \in L^2(0,T;\mathcal{U})$ such that

$$CHu + Rf = 0. \tag{8}$$

(ii) We say that $(S_P) + (E)$ is *weakly remediable* on $[0,T]$ if for every $f \in L^2(0,T;L^2(\Gamma))$ and every $\epsilon > 0$ there exists $u \in L^2(0,T;\mathcal{U})$ such that

$$\|CHu + Rf\| < \epsilon. \tag{9}$$

The exact remediability characterization is given by the result below.

Proposition 1. *The following conditions are equivalent:*

(i) $(S_P) + (E)$ *is exactly remediable on* $[0,T]$,

(ii) *operators* R *and* CH *satisfy*

$$\mathrm{Im}(R) \subset \mathrm{Im}(CH), \tag{10}$$

(iii) $\exists\, \gamma > 0$ *such that* $\forall\, \theta \in Y^\star$

$$\begin{aligned}\big\|\big(- G^\star S^\star(T-\cdot)A^\star + G^\star S^\star(T-\cdot)\big)C^\star\theta\big\|_{L^2(0,T;L^2(\Gamma))} \\ \le \gamma\big\|\big(- B^\star G^\star S^\star(T-\cdot)A^\star + B^\star G^\star S^\star(T-\cdot)\big)C^\star\theta\big\|_{L^2(0,T;\,\mathcal{U}')},\end{aligned} \tag{11}$$

where in the general case $P^\star$ *is the adjoint operator of* P, *and* Z' *stands for the dual space of* Z.

Proof. The equivalence between (i) and (ii) follows easily from Definition 1. The equivalence between (i) and (ii) results from

$$R^\star = \tilde{H}^\star C^\star = \big(- G^\star S^\star(T-\cdot)A^\star + G^\star S^\star(T-\cdot)\big)C^\star, \tag{12}$$

$$H^\star C^\star = \big(- B^\star G^\star S^\star(T-\cdot)A^\star + B^\star G^\star S^\star(T-\cdot)\big)C^\star \tag{13}$$

and Lemma 1 below. ■

Lemma 1. (Curtain and Pritchard, 1978; El Jai and Pritchard, 1988) *Let* X, Y *and* Z *be reflexive Banach spaces and* $P \in \mathcal{L}(X, Z)$, $Q \in \mathcal{L}(Y, Z)$. *Then the following properties are equivalent:*

(i) $\mathrm{Im}(P) \subset \mathrm{Im}(Q)$,

(ii) $\exists \gamma > 0$ *such that* $\|P^\star z^\star\|_{X'} \le \gamma \|Q^\star z^\star\|_{Y'}, \quad \forall\, z^\star \in Z'$.

Let us remark that if $B = I$, i.e. in the case of an action distributed over all the boundary, the system $(S_P) + (E)$ is exactly remediable for every output operator C (this follows from (11) in Proposition 1).

Proposition 2. *The following conditions are equivalent:*

(i) $(S_P) + (E)$ *is weakly remediable on* $[0, T]$,

(ii) *operators* R *and* CH *satisfy*

$$\mathrm{Im}(R) \subset \overline{\mathrm{Im}(CH)}, \tag{14}$$

(iii) *for the adjoint operators we have*

$$\ker(B^\star R^\star) = \ker(R^\star). \tag{15}$$

Proof. The equivalence between (i) and (ii) follows easily from Definition 1. The equivalence between (ii) and (iii) results from the inclusion

$$\mathrm{Im}(CH) \subset \mathrm{Im}(R), \tag{16}$$

because for $f = -Bu$ we have $Rf = -CHu$. We also see that

$$H^\star C^\star = B^\star R^\star, \tag{17}$$

which follows from (12) and (13), and from the use of orthogonal subspaces in (14). ■

Remark 1. If the observation space has finite dimension, or the observation is given by a finite number of sensors, the weak and exact remediabilities are equivalent.

In the case of p boundary actuators $(\Gamma_i, g_i)_{i=1:p}$, a characterization of the exact remediability is given by the following result:

Proposition 3. *$(S_P)+(E)$ is exactly remediable on $[0,T]$ if and only if $\exists \gamma > 0$ such that $\forall\, \theta \in Y'$*

$$\int_0^T \big\| \big(-G^\star S^\star(T-s)A^\star + G^\star S^\star(T-s)\big) C^\star \theta \big\|^2_{L^2(\Gamma)}\, \mathrm{d}s$$
$$\leq \gamma \int_0^T \sum_{i=1}^p \big\langle g_i, \big(-G^\star S^\star(T-s)A^\star + G^\star S^\star(T-s)\big) C^\star \theta \big\rangle^2_\Gamma \,\mathrm{d}s, \quad (18)$$

where $\langle\cdot,\cdot\rangle_\Gamma$ is the inner product in $L^2(\Gamma)$.

Proof. The result follows directly from Proposition 1. ■

It is well-known that $L^2(\Omega)$ has a complete orthonormal system of eigenfunctions $(\psi_{nj})_{\substack{n\geq 1\\ j=1:r_n}}$ of A, associated with the eigenvalues $(\lambda_n)_{n\geq 1}$, r_n being the multiplicity of λ_n, and the semi group $(S(t))_{t\geq 0}$ generated by A is given by

$$S(t)z = \sum_{n\geq 1} e^{\lambda_n t} \sum_{j=1}^{r_n} \langle z, \psi_{nj}\rangle_\Omega \psi_{nj}. \quad (19)$$

Corollary 1. *$(S_P)+(E)$ is exactly remediable on $[0,T]$ if and only if there exists $\gamma > 0$ such that $\forall\, \theta \in Y'$*

$$\int_0^T \Big\| \sum_{n\geq 1} e^{\lambda_n (T-s)} \sum_{j=1}^{r_n} \langle C^\star\theta, \psi_{nj}\rangle_\Omega L_{nj} \Big\|^2_{L^2(\Gamma)} \mathrm{d}s$$
$$\leq \gamma \int_0^T \sum_{i=1}^p \Big[\sum_{n\geq 1} e^{\lambda_n (T-s)} \sum_{j=1}^{r_n} \langle C^\star\theta, \psi_{nj}\rangle_\Omega \langle g_i, \psi_{nj}\rangle_\Gamma \Big]^2 \mathrm{d}s, \quad (20)$$

with

$$L_{nj}(z) = \langle \psi_{nj}, z\rangle_\Gamma, \quad \forall\, z \in L^2(\Gamma). \quad (21)$$

Proof. The result follows from the fact that, for $z \in L^2(\Gamma)$, the Green formula yields

$$-\langle A\psi_{nj}, Gz\rangle_\Omega + \langle \psi_{nj}, Gz\rangle_\Omega = \langle \psi_{nj}, z\rangle_\Gamma.$$

■

If the system output is given by q sensors $(D_i, h_i)_{i=1:q}$, where $h_i \in L^2(D_i)$ with $D_i = \operatorname{supp}(h_i) \subset \Omega$ for $i = 1, \dots, q$ and $D_i \cap D_j = \emptyset$ for $i \neq j$, we have the following result:

Corollary 2. *$(S_P) + (E)$ is exactly remediable on $[0, T]$ if and only if there exists $\gamma > 0$ such that $\forall\, \theta = (\theta_1, \dots, \theta_q)^{\mathrm{tr}} \in \mathbb{R}^q$*

$$\int_0^T \Big\| \sum_{n\geq 1} e^{\lambda_n (T-s)} \sum_{j=1}^{r_n} \sum_{l=1}^{q} \theta_l \langle h_l, \psi_{nj} \rangle_\Omega L_{nj} \Big\|^2_{L^2(\Gamma)} \mathrm{d}s$$

$$\leq \gamma \int_0^T \sum_{i=1}^{p} \Big[\sum_{n\geq 1} e^{\lambda_n (T-s)} \sum_{j=1}^{r_n} \langle g_i, \psi_{nj} \rangle_\Gamma \sum_{l=1}^{q} \theta_l \langle h_l, \psi_{nj} \rangle_\Omega \Big]^2 \mathrm{d}s. \quad (22)$$

2.2. Efficient Actuators

In this part we define the notion of boundary efficient actuators, and we give some characterizations of these actuators with respect to sensors.

Definition 2. Actuators $(\Gamma_i, g_i)_{i=1:p}$ ensuring the weak remediability of $(S_P) + (E)$ are said to be boundary efficient actuators.

In the multi-actuator case and an s.c.s.g. given by (19), we have the following characterization.

Proposition 4. *Boundary actuators $(\Gamma_i, g_i)_{i=1:p}$ are efficient if and only if*

$$\bigcap_{n\geq 1} \ker(L_n^{\mathrm{tr}} f_n) = \bigcap_{n\geq 1} \ker(M_n f_n), \quad (23)$$

where, for $n \geq 1$,

$$M_n = \big(\langle g_i, \psi_{nj} \rangle_\Gamma \big)_{\substack{i=1:p \\ j=1:r_n}}, \quad (24)$$

$$f_n : \begin{array}{ccl} Y' & \longrightarrow & \mathbb{R}^{r_n} \\ \theta & \longmapsto & f_n(\theta) = \big(\langle C^\star \theta, \psi_{n1} \rangle_\Omega, \dots, \langle C^\star \theta, \psi_{nr_n} \rangle_\Omega \big)^{\mathrm{tr}} \end{array} \quad (25)$$

and

$$L_n = (L_{n1}, \dots, L_{nr_n})^{\mathrm{tr}}, \quad (26)$$

L_{nj} being given by (21), and in the general case N^{tr} is the transpose of N.

Proof. For $\theta \in Y'$, we have

$$R^\star \theta = \sum_{n\geq 1} e^{\lambda_n (T-\cdot)} \sum_{j=1}^{r_n} \langle C^\star \theta, \psi_{nj} \rangle_\Omega (G^\star - G^\star A^\star) \psi_{nj}.$$

By analyticity, we obtain then

$$R^\star\theta = 0 \Longleftrightarrow \sum_{j=1}^{r_n} \langle C^\star\theta, \psi_{nj}\rangle_\Omega (G^\star - G^\star A^\star)\psi_{nj} = 0, \quad \forall\, n \geq 1.$$

Since $A^\star = A$, from the Green formula we have $(G^\star - G^\star A^\star)\psi_{nj} = L_{nj}$, and hence

$$R^\star\theta = 0 \Longleftrightarrow \sum_{j=1}^{r_n} \langle C^\star\theta, \psi_{nj}\rangle_\Omega L_{nj} = 0, \quad \forall\, n \geq 1.$$

Consequently,

$$\ker(R^\star) = \bigcap_{n\geq 1} \ker(L_n^{\mathrm{tr}} f_n). \tag{27}$$

On the other hand,

$$B^\star R^\star\theta = \Big(\sum_{n\geq 1} e^{\lambda_n(T-\cdot)} \sum_{j=1}^{r_n} \langle (G^\star - G^\star A^\star)\psi_{nj}, g_l\rangle_\Gamma \langle C^\star\theta, \psi_{nj}\rangle_\Omega\Big)^{\mathrm{tr}}_{l=1:p}.$$

Similarly, we obtain

$$B^\star R^\star\theta = 0 \Longleftrightarrow \sum_{j=1}^{r_n} \langle g_l, \psi_{nj}\rangle_\Gamma \langle C^\star\theta, \psi_{nj}\rangle_\Omega = 0, \quad \forall\, n \geq 1,\ \ \forall\, l = 1:p.$$

Using (24) and (25), we have

$$B^\star R^\star\theta = 0 \Longleftrightarrow M_n f_n(\theta) = 0, \quad \forall\, n \geq 1,$$

and hence

$$\ker(B^\star R^\star) = \bigcap_{n\geq 1} \ker(M_n f_n). \tag{28}$$

Consequently, (15) in Proposition 2 becomes

$$\bigcap_{n\geq 1} \ker(L_n^{\mathrm{tr}} f_n) = \bigcap_{n\geq 1} \ker(M_n f_n).$$

■

Corollary 3. *If, for every $n \geq 1$, the vectors $(L_{nj})_{j=1:r_n}$ are linearly independent, then the boundary actuators $(\Gamma_i, g_i)_{i=1:p}$ are efficient if and only if*

$$\bigcap_{n\geq 1} \ker(M_n f_n) = \ker(C^\star). \tag{29}$$

Proof. The result follows from the fact that $\ker(R^\star) = \ker(C^\star)$ and from the equalities (28) and (15). ■

Now, if the output is given by q sensors $(D_i, h_i)_{i=1:q}$, the characterization of boundary efficient actuators is given in the following proposition:

Proposition 5. *Boundary actuators* $(\Gamma_i, g_i)_{i=1:p}$ *are efficient if and only if*

$$\bigcap_{n\geq 1} \ker\left(M_n G_n^{\mathrm{tr}}\right) = \bigcap_{n\geq 1} \ker(G_n L_n)^{\mathrm{tr}}, \tag{30}$$

where

$$G_n = \left(\langle h_i, \psi_{nj}\rangle_\Omega\right)_{\substack{i=1:q\\ j=1:r_n}}.$$

Proof. For $\theta = (\theta_1, \dots, \theta_q)^{\mathrm{tr}} \in \mathbb{R}^q$, we have

$$R^\star\theta = 0 \iff \sum_{j=1}^{r_n} \langle C^\star\theta, \psi_{nj}\rangle_\Omega L_{nj} = 0, \quad \forall\, n \geq 1.$$

Since $C^\star\theta = \sum_{i=1}^{q} \theta_i h_i$, it follows that

$$\begin{aligned} R^\star\theta = 0 &\iff \sum_{j=1}^{r_n}\sum_{i=1}^{q} \theta_i \langle h_i, \psi_{nj}\rangle_\Omega L_{nj} = 0, \quad \forall\, n \geq 1 \\ &\iff \sum_{i=1}^{q} \theta_i \sum_{j=1}^{r_n} \langle h_i, \psi_{nj}\rangle_\Omega L_{nj} = 0, \quad \forall\, n \geq 1 \\ &\iff (G_n L_n)^{\mathrm{tr}}\theta = 0, \quad \forall\, n \geq 1. \end{aligned}$$

Then

$$\ker(R^\star) = \bigcap_{n\geq 1} \ker(G_n L_n)^{\mathrm{tr}}. \tag{31}$$

On the other hand,

$$B^\star R^\star\theta = \Big(\sum_{n\geq 1} e^{\lambda_n(T-\cdot)} \sum_{j=1}^{r_n} \langle (G^\star - G^\star A^\star)\psi_{nj}, g_l\rangle_\Gamma \sum_{i=1}^{q} \theta_i \langle h_i, \psi_{nj}\rangle_\Omega\Big)^{\mathrm{tr}}_{l=1:p}.$$

By analyticity and the Green formula, we obtain

$$\begin{aligned} B^\star R^\star\theta = 0 &\iff \sum_{j=1}^{r_n} \langle g_l, \psi_{nj}\rangle_\Gamma \sum_{i=1}^{q} \theta_i \langle h_i, \psi_{nj}\rangle_\Omega = 0, \quad \forall\, l = 1:p,\ \forall\, n \geq 1 \\ &\iff M_n G_n^{\mathrm{tr}}\theta = 0, \quad \forall\, n \geq 1. \end{aligned}$$

Hence

$$\ker(B^\star R^\star) = \bigcap_{n\geq 1} \ker(M_n G_n^{\mathrm{tr}}), \tag{32}$$

and the result follows from (15). ■

Corollary 4. *If for every $n \geq 1$ the vectors $(L_{nj})_{j=1:r_n}$ are linearly independent, then the boundary actuators $(\Gamma_i, g_i)_{i=1:p}$ are efficient if and only if*

$$\bigcap_{n\geq 1} \ker[M_n G_n^{\mathrm{tr}}] = \{0\}. \tag{33}$$

Proof. The result follows from the fact that $\ker(R^\star) = \{0\}$, and from equalities (32), (36) and (15). ■

Corollary 5. *If for every $n \geq 1$ the vectors $(L_{nj})_{j=1:r_n}$ are linearly independent, and there exists $n_0 \geq 1$ such that*

$$\mathrm{rank}(M_{n_0} G_{n_0}^{\mathrm{tr}}) = q, \tag{34}$$

then the boundary actuators $(\Gamma_i, g_i)_{i=1:p}$ are efficient.

The results are analoguous in the case of boundary pointwise actuators or pointwise sensors, with some technical precautions.

3. Exact Remediability with Minimal Energy

In this section we consider the following exact remediability problem: For $f \in L^2(0,T;L^2(\Gamma))$, does there exist an optimal control $u \in L^2(0,T;\ \mathcal{U})$ such that $y_{u,f}(T) = CS(T)z_0$, i.e. minimizing the function $J(v) = \|v\|^2$ on the set $\{v \in L^2(0,T;\ \mathcal{U}) \mid y_{v,f}(T) = CS(T)z_0\}$?

This problem will be solved using an extension of the H.U.M. approach. For $\theta \in Y' \equiv Y$, let

$$\|\theta\|_{\mathcal{F}} = \left[\int_0^T \big\|\big(- B^\star G^\star S^\star(T-s)A^\star + B^\star G^\star S^\star(T-s)\big)C^\star\theta\big\|_{\mathcal{U}'}^2 \,\mathrm{d}s\right]^{\frac{1}{2}}, \tag{35}$$

where $\mathcal{F}$ is a space which will be precised later. Note that $\|\cdot\|_{\mathcal{F}}$ is a semi-norm, but not necessarily a norm.

Lemma 2. *If for every $n \geq 1$ the vectors $(L_{nj})_{j=1:r_n}$ are linearly independent and* $\ker(C^\star) = \{0\}$*, then the following conditions are equivalent:*

(i) $(S_P) + (E)$ *is weakly remediable on* $[0,T]$,

(ii) $\ker(H^\star C^\star) = \{0\}$,

(iii) $\|\cdot\|_{\mathcal{F}}$ *is a norm on* Y.

Proof.
(i) $\Longleftrightarrow$ (ii) It follows from (17), Proposition 2, and the fact that

$$\ker(R^\star) = \ker(C^\star) = \{0\}.$$

(ii) $\Longrightarrow$ (iii) Let $\theta \in Y$ such that $\|\theta\|_{\mathcal{F}} = 0$. This is equivalent to

$$\big\|\big(-B^\star G^\star S^\star(T-s)A^\star + B^\star G^\star S^\star(T-s)\big)C^\star\theta\big\|^2_{\mathcal{U}'} = 0, \quad \forall s \in [0,T].$$

Then

$$\big(-B^\star G^\star S^\star(T-\cdot)A^\star + B^\star G^\star S^\star(T-s)\big)C^\star\theta = 0,$$

which means $H^\star C^\star\theta = 0$. Since $\ker(H^\star C^\star) = \{0\}$, we have $\theta = 0$.

(iii) $\Longrightarrow$ (ii) It follows from (13). ■

Let us consider the operator $\Lambda = CHH^\star C^\star$. For $\theta \in Y' \equiv Y$, we have

$$\Lambda\theta = \int_0^T C\big(-AS(T-s)GB + S(T-s)GB\big)\big(-B^\star G^\star S^\star(T-s)A^\star + B^\star G^\star S^\star(T-s)\big)C^\star\theta\,\mathrm{d}s \in Y \tag{36}$$

From Lemma 2, it is easy to deduce the following result.

Lemma 3. *If, for every $n \geq 1$, the vectors $(L_{nj})_{j=1:r_n}$ are linearly independent and* $\ker(C^\star) = \{0\}$*, then the following conditions are equivalent:*

(i) $(S_P) + (E)$ *is weakly remediable on* $[0,T]$*,*

(ii) *the operator* $\Lambda = CHH^\star C^\star$ *is positive definite.*

Remark 2. If the system output is given by sensors, then $\ker(C^\star) = \{0\}$.

Suppose that $\|\cdot\|_{\mathcal{F}}$ is a norm and let $\mathcal{F}$ be the completion of the space Y with respect to the norm $\|\cdot\|_{\mathcal{F}}$, i.e.

$$\mathcal{F} = \overline{Y}^{\|\cdot\|_{\mathcal{F}}}. \tag{37}$$

$\mathcal{F}$ is a Hilbert space, with the inner product defined by

$$\langle\theta,\sigma\rangle_{\mathcal{F}} = \int_0^T \big\langle\big(-B^\star G^\star S^\star(T-s)A^\star + B^\star G^\star S^\star(T-s)\big)C^\star\theta, \big(-B^\star G^\star S^\star(T-s)A^\star + B^\star G^\star S^\star(T-s)\big)C^\star\sigma\big\rangle_{\mathcal{U}'}\,\mathrm{d}s \tag{38}$$

$\forall\,\theta,\ \sigma \in \mathcal{F}$.

Proposition 6.

(i) Y *is contained in* $\mathcal{F}$ *with continuous injection.*

(ii) *We have*

$$\langle\Lambda\theta,\sigma\rangle_Y = \langle\theta,\sigma\rangle_{\mathcal{F}}, \quad \forall\,\theta,\sigma \in Y. \tag{39}$$

(iii) Λ *has a unique extension as an isomorphism from* $\mathcal{F}$ *to* $\mathcal{F}'$ *such that*

$$\langle \Lambda\theta, \sigma\rangle_Y = \langle \theta, \sigma\rangle_{\mathcal{F}}, \quad \forall\, \theta, \sigma \in \mathcal{F} \tag{40}$$

and

$$\|\Lambda\theta\|_{\mathcal{F}'} = \|\theta\|_{\mathcal{F}}, \quad \forall\, \theta \in \mathcal{F}. \tag{41}$$

Proof. (i) It follows from the fact that, for $\theta \in Y$, we have

$$\begin{aligned} \|\theta\|^2_{\mathcal{F}} &= \int_0^T \big\|\big(- B^\star G^\star S^\star(T-s)A^\star + B^\star G^\star S^\star(T-s)\big)C^\star\theta\big\|^2\,\mathrm{d}s \\ &\le \int_0^T \|\big(- B^\star G^\star S^\star(T-s)A^\star + B^\star G^\star S^\star(T-s)\big)C^\star\|^2\,\mathrm{d}s\, \|\theta\|^2_Y \le \gamma\|\theta\|^2_Y. \end{aligned}$$

(ii) Let $\theta, \sigma \in Y$. We have

$$\begin{aligned} \langle \Lambda\theta, \sigma\rangle_Y &= \Big\langle \int_0^T C(-AS(T-s)GB + S(T-s)GB)\big(- B^\star G^\star S^\star(T-s)A^\star \\ &\qquad + B^\star G^\star S^\star(T-s)\big)C^\star\theta\,\mathrm{d}s, \sigma\Big\rangle_Y \\ &= \int_0^T \Big\langle\big(- B^\star G^\star S^\star(T-s)A^\star + B^\star G^\star S^\star(T-s)\big)C^\star\theta, \\ &\qquad \big(- B^\star G^\star S^\star(T-s)A^\star + B^\star G^\star S^\star(T-s)\big)C^\star\sigma\Big\rangle\,\mathrm{d}s = \langle\theta,\sigma\rangle_{\mathcal{F}}. \end{aligned}$$

(iii) For $\theta \in \mathcal{F}$, we consider the linear mapping $\Lambda\theta : \sigma \in Y \longmapsto \langle \Lambda\theta, \sigma\rangle_Y \in \mathbb{R}$. We have

$$|(\Lambda\theta)(\sigma)| = |\langle\Lambda\theta,\sigma\rangle_Y| = |\langle\theta,\sigma\rangle_{\mathcal{F}}| \le \|\theta\|_{\mathcal{F}}\|\sigma\|_{\mathcal{F}}.$$

$\Lambda\theta$ is then continuous on Y for the topology of $\mathcal{F}$, so it can be continuously extended in a unique way to $\mathcal{F}$. Hence $\Lambda\theta \in \mathcal{F}'$ and $\langle\Lambda\theta,\sigma\rangle_Y = \langle\theta,\sigma\rangle_{\mathcal{F}}$, $\forall\sigma \in Y$, and then $\|\Lambda\theta\|_{\mathcal{F}'} = \|\theta\|_{\mathcal{F}}$.

The operator $\Lambda : \mathcal{F} \longmapsto \mathcal{F}'$ is linear and injective. Indeed, for $\theta \in \mathcal{F}$ such that $\Lambda\theta = 0$, we have $\langle\Lambda\theta,\theta\rangle = 0$. This means that $\|\theta\|^2_{\mathcal{F}} = 0$ and then $\theta = 0$. Λ is also surjective, using the Riesz theorem. Λ is then an isomorphism from $\mathcal{F}$ to $\mathcal{F}'$. ■

As regards the problem of exact remediability with minimal energy, we have the following result.

Proposition 7. *If the observation* $y_f = Rf \in \mathcal{F}'$, *then there exists a unique element* $\theta_f \in \mathcal{F}$ *such that* $\Lambda\theta_f = -y_f$, *and the control*

$$u_{\theta_f}(t) = \big(- B^\star G^\star S^\star(T-t)A^\star + B^\star G^\star S^\star(T-t)\big)C^\star\theta_f \tag{42}$$

satisfies

$$CHu_{\theta_f} + y_f = 0. \tag{43}$$

Moreover, u_{θ_f} *is optimal, with*

$$\|u_{\theta_f}\|_{L^2(0,T;\,\mathcal{U})} = \|\theta_f\|_{\mathcal{F}}. \tag{44}$$

Proof. We have

$$\begin{aligned}\Lambda\theta_f &= \int_0^T C\big(-AS(T-s)GB + S(T-s)GB\big)\big(-B^\star G^\star S^\star(T-s)A^\star \\ &\qquad + B^\star G^\star S^\star(T-s)\big)C^\star\theta\,\mathrm{d}s \\ &= \int_0^T C\big(-AS(T-s)GB + S(T-s)GB\big)u_{\theta_f}\,\mathrm{d}s = CHu_{\theta_f} = -y_f.\end{aligned}$$

On the other hand, consider the set

$$\mathcal{C} = \big\{u \in L^2(0,T;\,\mathcal{U}) \text{ such that } y_{u,f}(T) = 0\big\}.$$

$\mathcal{C}$ is convex, closed and non-empty, because $u_{\theta_f} \in \mathcal{C}$. Consider the function

$$J(u) = \|CHu + y_f\|^2 + \|u\|^2.$$

For $u \in \mathcal{C}$ we have $J(u) = \|u\|^2$. J is strictly convex on $\mathcal{C}$, hence it admits a unique minimum in $u^\star \in \mathcal{C}$ with $u^\star$ characterized by

$$\langle u^\star, v - u^\star\rangle \geq 0, \quad \forall v \in \mathcal{C}.$$

For $v \in \mathcal{C}$, we have

$$\begin{aligned}\langle u_{\theta_f}, v - u_{\theta_f}\rangle_{L^2(0,T;\,\mathcal{U})} &= \int_0^T \langle u_{\theta_f}(t), v(t) - u_{\theta_f}(t)\rangle\,\mathrm{d}t \\ &= \int_0^T \big\langle\big(-B^\star G^\star S^\star(T-t)A^\star + B^\star G^\star S^\star(T-t)\big)C^\star\theta_f, v(t) - u_{\theta_f}(t)\big\rangle\,\mathrm{d}t \\ &= \Big\langle\theta_f, \int_0^T C(-AS(T-t)GB + S(T-t)GB)v(t)\,\mathrm{d}t \\ &\qquad - \int_0^T C(-AS(T-t)GB + S(T-t)GB)u_{\theta_f}(t)\,\mathrm{d}t\Big\rangle \\ &= \langle\theta_f, CHv - CHu_{\theta_f}\rangle = \langle\theta_f, -y_f + y_f\rangle = 0.\end{aligned}$$

Since $u^\star$ is unique, we have $u^\star = u_{\theta_f}$, and u_{θ_f} is optimal with

$$\begin{aligned}\|u_{\theta_f}\|^2_{L^2(0,T;\,\mathcal{U})} &= \int_0^T \|u_{\theta_f}(t)\|^2\,\mathrm{d}t \\ &= \int_0^T \big\|\big(-B^\star G^\star S^\star(T-s)A^\star + B^\star G^\star S^\star(T-s)\big)C^\star\theta_f\big\|^2\mathrm{d}t = \|\theta_f\|^2_{\mathcal{F}}.\end{aligned}$$

∎

Consider now the set

$$\mathcal{E} = \big\{ f \in L^2\big(0,T;L^2(\Gamma)\big) \mid \exists u \in L^2(0,T;\,\mathcal{U}) \text{ which satisfies } CHu + Rf = 0 \big\}. \quad (45)$$

Proposition 8. *$\mathcal{E}$ is the inverse image of $\mathcal{F}'$ by R, i.e.*

$$R\mathcal{E} = \mathcal{F}' \quad (46)$$

Proof. Let $y \in \mathcal{F}'$. There exists a unique $\theta \in \mathcal{F}$ such that $\Lambda\theta = y$. Let

$$\int_0^T C(AS(T-s)GB + S(T-s)GB)\big(- B^\star G^\star S^\star(T-s)A^\star + B^\star G^\star S^\star(T-s)\big)C^\star\theta \,\mathrm{d}s = y.$$

If u is the control defined by

$$u(\cdot) = \big(- B^\star G^\star S^\star(T-\cdot)A^\star + B^\star G^\star S^\star(T-\cdot)\big)C^\star\theta \ \in \mathcal{C},$$

we have

$$\Lambda\theta = \int_0^T C\big(AS(T-s)GB + S(T-s)GB\big)u(s)\,\mathrm{d}s = y,$$

i.e. $CHu = y$, and for $f = -Bu \in L^2(0,T;L^2(\Gamma))$ we have $CHu = -Rf = y$. Then $y \in R\mathcal{E}$.

Conversely, let $y \in R\mathcal{E}$. There exists $f \in L^2(0,T;L^2(\Gamma))$ such that $y = Rf$ and $CHu + Rf = 0$ with $u \in \mathcal{C}$. If we identify CHu and the linear mapping $L : \theta \in Y \longmapsto \langle CHu, \theta\rangle$, we have

$$\begin{aligned} L(\theta) &= \langle CHu, \theta\rangle \\ &= \Big\langle C\int_0^T (AS(T-s)GB + S(T-s)GB)u(s)\,\mathrm{d}s, \theta\Big\rangle \\ &= \int_0^T \big\langle u(s), \big(- B^\star G^\star S^\star(T-s)A^\star + B^\star G^\star S^\star(T-s)\big)\theta\big\rangle\,\mathrm{d}s. \end{aligned}$$

Using (35), we get

$$|L(\theta)| \le \ \|u\|_{L^2(0,T;\,\mathcal{U})}\ \|\theta\|_{\mathcal{F}}.$$

L is then a continuous linear mapping on Y for the topology of $\mathcal{F}$, and hence it has a unique continuous extension to $\mathcal{F}$. Hence $L \in \mathcal{F}'$, $CHu = -Rf = -y \in \mathcal{F}'$, and therefore $y \in \mathcal{F}'$. ■

In the case of q sensors, we have $Y = \mathbb{R}^q = \mathcal{F} \equiv \mathcal{F}'$. Then the set $\mathcal{E}$ defined in (45) is $L^2(0,T;L^2(\Gamma))$.

4. Remediability and Controllability

In this section we study the relationship between the notions of controllability by boundary actions and remediability, and hence the relationship between strategic and efficient boundary actuators. Let us recall first the notion of controllability and strategic boundary actuators.

4.1. Controllability

We consider the system described by the following state equation:

$$(S)\begin{cases}\dfrac{\partial z}{\partial t}(x,t) = \Delta z(x,t) & \text{in} \quad \Omega\times]0,T[,\\ \dfrac{\partial z}{\partial \nu}(\cdot,t) = B(\cdot)u(t) & \text{on} \quad \Gamma\times]0,T[,\\ z(x,0) = 0 & \text{in} \quad \Omega.\end{cases}$$

System (S) has a unique weak solution given by

$$z_u(t) = -\int_0^t AS(t-s)GBu(s)\,\mathrm{d}s + \int_0^t S(t-s)GBu(s)\,\mathrm{d}s. \tag{47}$$

Definition 3. The system (S) is said to be
(i) *exactly controllable* on $[0,T]$ if $\mathrm{Im}(H) = L^2(\Omega)$.
(ii) *weakly controllable* on $[0,T]$ if $\overline{\mathrm{Im}(H)} = L^2(\Omega)$.

Proposition 9. *System* (S) *is*
(i) *exactly controllable on* $[0,T] \Longleftrightarrow \exists\gamma > 0$ *such that*

$$\|z^\star\|_{L^2(\Omega)} \le \gamma\big\|\big(-B^\star G^\star S^\star(T-\cdot)A^\star + B^\star G^\star S^\star(T-\cdot)\big)z^\star\big\|_{L^2(0,T;\,\mathcal{U}')}, \quad \forall z^\star \in L^2(\Omega) \tag{48}$$

$$\Longleftrightarrow \textit{the operator } M = HH^\star : L^2(\Omega) \longrightarrow L^2(\Omega) \textit{ is coercive}; \tag{49}$$

(ii) *weakly controllable on* $[0,T]$

$$\Longleftrightarrow \ker(H^\star) = \{0\} \tag{50}$$

$$\Longleftrightarrow \textit{the operator } M = HH^\star \textit{ is positive definite.} \tag{51}$$

Remark 3. Exact controllability implies weak one and the converse is not true (Berrahmoune, 1984; El Jai and Pritchard, 1988).

4.1.1. Controllability and Actuators

In the case of p zone boundary actuators $(\Gamma_i, g_i)_{i=1:p}$, we have $\mathcal{U} = \mathbb{R}^p$ and (El Jai, 1991; El Jai and Pritchard, 1988)

$$B: \begin{array}{ccc} \mathbb{R}^p & \longrightarrow & L^2(\Gamma), \\ u(t) & \longmapsto & Bu(t) = \sum_{i=1}^{p} g_i u_i(t), \end{array}$$

where $u = (u_1, \dots, u_p)^{\mathrm{tr}} \in L^2(0, T; \mathbb{R}^p)$ and $g_i \in L^2(\Gamma_i)$ with $\Gamma_i = \operatorname{supp}(g_i) \subset \Gamma$ for $i = 1 : p$ and $\Gamma_i \cap \Gamma_j = \emptyset$. For $i \neq j$, we have

$$B^\star z = \big(\langle g_1, z\rangle_{\Gamma_1}, \dots, \langle g_p, z\rangle_{\Gamma_p}\big)^{\mathrm{tr}} \quad \text{for } z \in L^2(\Gamma).$$

Definition 4. We say that actuators are *strategic* if the corresponding system (S) is weakly controllable.

Proposition 10. (Berrahmoune, 1984; El Jai and Pritchard, 1988) *The actuators* $(\Gamma_i, g_i)_{i=1:p}$ *are strategic if and only if*

$$\begin{cases} p \geq r_n, \quad \forall\, n \geq 1, \\ \operatorname{rank}(M_n) = r_n, \quad \forall\, n \geq 1, \end{cases} \tag{52}$$

where M_n *is defined in (24).*

Let us remark that the condition $p \geq \sup_n r_n$ is necessary for boundary actuators $(\Gamma_i, g_i)_{i=1:p}$ to be strategic, but it is not necessary for them to be efficient.

Remark 4. In the case of pointwise boundary actuators $(b_i, \delta_{b_i})_{i=1:p}$, we have $z(\cdot) \in L^2(0, T; V)$ where $V' \subset L^2(\Omega) \subset V$, with continuous injections, and the characterization of strategic pointwise actuators is similar to (52) for zone actuators, with $M_n = (\psi_{nj}(b_i))_{\substack{i=1:p \\ j=1:r_n}}$.

The following results show that remediability is a weaker notion than controllability.

Proposition 11. *If* (S) *is exactly controllable on* $[0, T]$*, then* $(S_P) + (E)$ *is exactly remediable on* $[0, T]$*.*

Proof. For $\theta \in Y'$, we have

$$\begin{aligned} &\big\|\big(-G^\star S^\star(T-\cdot)A^\star + G^\star S^\star(T-\cdot)\big)C^\star\theta\big\|^2_{L^2(0,T;L^2(\Gamma))} \\ &\quad = \int_0^T \big\|\big(-G^\star S^\star(T-s)A^\star + G^\star S^\star(T-s)\big)C^\star\theta\big\|^2_{L^2(\Gamma)}\,\mathrm{d}s \\ &\quad \leq \int_0^T \|-G^\star S^\star(T-s)A^\star + G^\star S^\star(T-s)\|^2\,\mathrm{d}s\,\|C^\star\theta\|^2_{L^2(\Omega)} \leq M\|C^\star\theta\|^2_{L^2(\Omega)} \end{aligned}$$

with $M > 0$. On the other hand, using the exact controllability hypothesis, there exists $\gamma_1 > 0$ such that

$$\|C^\star\theta\|_{L^2(\Omega)} \le \gamma_1 \big\| \big(-B^\star G^\star S^\star(T-\cdot)A^\star + B^\star G^\star S^\star(T-\cdot)\big)C^\star\theta\big\|_{L^2(0,T;\,\mathcal{U}')}.$$

Consequently, there exists $\gamma = M(\gamma_1)^2 > 0$ such that

$$\|(-G^\star S^\star(T-\cdot)A^\star + G^\star S^\star(T-\cdot))C^\star\theta\|^2_{L^2(0,T;L^2(\Gamma))}$$
$$\le \gamma\|(-B^\star G^\star S^\star(T-\cdot)A^\star + B^\star G^\star S^\star(T-\cdot))C^\star\theta\|^2_{L^2(0,T;\,\mathcal{U}')},$$

and the result follows from Proposition 1. ■

The converse is not true. This is illustrated with the following example:

Example 1. Let Ω be a bounded open subset of $\mathbb{R}^n$, with a sufficiently regular boundary $\Gamma = \partial\Omega$, $X = L^2(\Omega)$ and $Az = \Delta z$ for $z \in D(A) = H^2(\Omega) \cap H^1_0(\Omega)$. Consider the system

$$(S) \begin{cases} \dfrac{\partial z}{\partial t}(x,t) = \Delta z(x,t) & \text{in} \quad \Omega\times]0,T[, \\ \dfrac{\partial z}{\partial \nu}(\xi,t) = u(\xi,t) & \text{on} \quad \Gamma\times]0,T[, \\ z(x,0) = 0 & \text{in} \quad \Omega. \end{cases}$$

(S) is augmented by the output equation

$$(E) \quad y(\cdot,t) = Cz(\cdot,t).$$

In this case, $B = I$ and (S) is not exactly controllable on $L^2(\Omega)$ (El Jai and Pritchard, 1988). However, (11) is satisfied on $L^2(\Omega)$, so that $(S_P) + (E)$ is exactly remediable for any output operator C. ♦

Proposition 12. *If* (S) *is weakly controllable on* $[0,T]$*, then* $(S_P) + (E)$ *is weakly remediable on* $[0,T]$*.*

Proof. From (15) and (17), we deduce that $(S_P) + (E)$ is weakly remediable if and only if $\ker(H^\star C^\star) = \ker(R^\star)$, or equivalently, $\ker(H^\star C^\star) \subset \ker(R^\star)$. Then, for $\theta \in \ker(H^\star C^\star)$, we have $H^\star C^\star\theta = 0$, and hence $C^\star\theta = 0$, because $\ker(H^\star) = \{0\}$. Since $\ker(C^\star) \subset \ker(R^\star)$, we have $\theta \in \ker(R^\star)$. ■

Remark 5. In case $C = I$ and A generates an s.c.s.g. given by (19), remediability is equivalent to controllability, and the characterizations are the same.

In multi-actuator and multi-sensor cases, we have the following result:

Corollary 6. *Strategic actuators are necessarily efficient.*

The converse is not true (cf. Section 5).

5. Applications

Let Ω be an open and bounded subset of $\mathbb{R}^n$, with a sufficiently regular boundary $\partial\Omega$. We consider the diffusion system

$$(S)\begin{cases} \dfrac{\partial z}{\partial t}(x,t) = \Delta z(x,t) & \text{in} \quad \Omega\times]0,T[, \\ \dfrac{\partial z}{\partial \nu}(\xi,t) = \displaystyle\sum_{i=1}^{p} g_i(\xi)u_i(t) & \text{on} \quad \Gamma\times]0,T[, \\ z(x,0) = 0 & \text{in} \quad \Omega. \end{cases}$$

(S) is augmented by the output equation

$$(E) \quad y = Cz = \big(\langle h_1, z\rangle_\Omega, \dots, \langle h_q, z\rangle_\Omega\big)^{\mathrm{tr}}.$$

If the system is disturbed on its boundary by a term $f \in L^2(0,T;L^2(\Gamma))$, we have

$$(S_P)\begin{cases} \dfrac{\partial z}{\partial t}(x,t) = \Delta z(x,t) & \text{in} \quad \Omega\times]0,T[, \\ \dfrac{\partial z}{\partial \nu}(\xi,t) = f(\xi,t) + \displaystyle\sum_{i=1}^{p} g_i(\xi)u_i(t) & \text{on} \quad \Gamma\times]0,T[, \\ z(x,0) = 0 & \text{in} \quad \Omega. \end{cases}$$

5.1. Case of a Rectangle $\Omega=]0,\alpha[\times]0,\beta[$

In this setting, the eigenvectors of Δ are defined by

$$\psi_{m,n}(x,y) = \frac{2}{\sqrt{\alpha\beta}} \cos\left(\frac{m\pi x}{\alpha}\right) \cos\left(\frac{n\pi y}{\beta}\right).$$

The associated eigenvalues are

$$\lambda_{m,n} = -\left(\frac{m^2}{\alpha^2} + \frac{n^2}{\beta^2}\right)\pi^2.$$

It is known (Berrahmoune, 1984; El Jai, 1991; El Jai and Pritchard, 1988) that

(i) If $\alpha^2/\beta^2 \notin Q$, then the eigenvalues are simple, and hence a single actuator (Γ_1, g_1) with $\Gamma_1 = \mathrm{supp}\,(g_1) \subset \Gamma$ is enough to have weak controllability. Indeed, an actuator (Γ_1, g_1) is strategic if and only if

$$\langle g_1, \psi_{mj,nj}\rangle_{\Gamma_1} \neq 0, \quad \forall\, m,n \geq 1. \tag{53}$$

(ii) If $\alpha = \beta = 1$, i.e. in the case of a square domain Ω, we have $\lambda_{m,n} = -(m^2 + n^2)\pi^2$, $\sup_{m,n\geq 1} r_{m,n} = \infty$, and then we cannot have weak controllability by a finite number of boundary actuators.

On the other hand, using Corollary 5, $(S_P)+(E)$ is weakly remediable if for every $m,n\geq 1$, vectors $(L_{m,nj})_{j=1:r_{m,n}}$ are linearly independent, and there exist $m_0,n_0\geq 1$ such that

$$\operatorname{rank}(M_{m_0n_0}G^{\mathrm{tr}}_{m_0n_0})=q.$$

Thus, in the case of one sensor (h_1,Ω_1), $(q=1)$, one actuator is enough for any α and β.

Since $(L_{m,nj})_{\substack{n\geq 1\\ j=1:r_{m,n}}}$ are linearly independent vectors, (Γ_1,g_1) is efficient if there exist $m_0,n_0\geq 1$ such that

$$\sum_{j=1}^{r_{m_0,n_0}}\langle g_1,\psi_{m_0j,n_0j}\rangle_{\Gamma_1}\langle h_1,\psi_{m_0j,n_0j}\rangle_{\Omega_1}\neq 0. \tag{54}$$

If $\alpha^2/\beta^2\notin Q$, we have $r_{m,n}=1,\forall\, m,n\geq 1$, and hence the condition (54) becomes

$$\langle g_1,\psi_{m_0,n_0}\rangle_{\Gamma_1}\langle h_1,\psi_{m_0,n_0}\rangle_{\Omega_1}\neq 0. \tag{55}$$

In this case, we have the following possibilities:

- If $\Gamma_1=[a_1-l_1,a_1+l_1]\times\{0\}$, there exists $m_1\geq 1$ such that $2a_1m_1/\alpha_1$ is odd and $h_1=\psi_{m_0,n_0}$ with $m_0\neq m_1$. Then an actuator (Γ_1,g_1) such that g_1 is symmetric with respect to the straight line $x_1=a_1$ is efficient, but not strategic.
- If $\Gamma_1=[0,a_1+l_1]\times\{0\}\cup\{0\}\times[0,a_2+l_2]$, there exist $m_1,n_1\geq 1$ such that $2a_1m_1/\alpha_1$ and $2a_2n_1/\alpha_2$ are odd, and $h_1=\psi_{m_0,n_0}$ with $m_0\neq m_1$ and $n_0\neq n_1$. Then an actuator (Γ_1,g_1) such that g_1 is symmetric with respect to the straight lines $x_1=a_1$ and $x_2=a_2$ is efficient, but not strategic.

Let us note that, in the case of a square domain, one boundary actuator can be efficient, e.g. for $h=\psi_{m_0j_0,n_0j_0}$ and $g_1\in L^2(\Gamma_1)$ such that $\langle g_1,\psi_{m_0,n_0}\rangle_{\Gamma_1}\neq 0$, the actuator (Γ_1,g_1) is efficient, but a finite number of actuators cannot be strategic.

5.2. Case of Ω Being a Disc $D(0,1)$

The Laplacian in polar coordinates is given by

$$\Delta z=\frac{\partial^2 z}{\partial\rho^2}+\frac{1}{\rho^2}\frac{\partial z}{\partial\rho}+\frac{1}{\rho^2}\frac{\partial^2 z}{\partial\theta^2}$$

with $0\leq\rho<1$ and $0\leq\theta<2\pi$. The eigenvalues of Δ are given by

$$\lambda_{nm}=-{\beta_{nm}}^2,\quad\forall\, n\geq 0\ \text{ and }\ m\geq 1,$$

where β_{nm} are non-zero roots of Bessel functions J_n defined by

$$J_n(t)=\frac{1}{2\pi}\int_{-\pi}^{\pi}e^{it\sin(\theta)}e^{-in\theta}\,\mathrm{d}\theta,\quad n\geq 0.$$

The eigenfunctions are given by

$$\begin{aligned} \varphi_{0m}(\rho,\theta) &= J_0(\beta_{0m}\rho), && m \geq 1, \\ \varphi_{nm1}(\rho,\theta) &= J_n(\beta_{nm1}\rho)\cos(n\theta) && n \geq 1, \ \ m \geq 1, \\ \varphi_{nm2}(\rho,\theta) &= J_n(\beta_{nm2}\rho)\sin(n\theta) && n \geq 1, \ \ m \geq 1. \end{aligned}$$

In this case, the multiplicity orders are defined as follows:

$$\begin{aligned} r_{nm} &= 2, \quad \forall\, n, m \geq 1, \\ r_{0m} &= 1, \quad \forall\, m \geq 1. \end{aligned}$$

System (S) cannot be weakly controllable with a single actuator (Berrahmoune, 1984; El Jai, 1991; El Jai and Pritchard, 1988). For $p = 2$, the actuators (Γ_1, g_1) and (Γ_2, g_2), with $\Gamma_i = \operatorname{supp}(g_i) \subset \Gamma$ for $i = 1, 2$ and $\Gamma_1 \bigcap \Gamma_2 = \emptyset$, are strategic if and only if

$$\langle g_1, \psi_{0m}\rangle^2_{\Gamma_1} + \langle g_2, \psi_{0m}\rangle^2_{\Gamma_2} \neq 0, \quad \forall\, m \geq 1 \tag{56}$$

and

$$\begin{vmatrix} \langle g_1, \psi_{nm1}\rangle_{\Gamma_1} & \langle g_1, \psi_{nm2}\rangle_{\Gamma_1} \\ \langle g_2, \psi_{nm1}\rangle_{\Gamma_2} & \langle g_2, \psi_{nm2}\rangle_{\Gamma_2} \end{vmatrix} \neq 0, \quad \forall\, n\,,\ m \geq 1. \tag{57}$$

On the other hand, in a single-sensor case (h_1, Ω_1), using Corollary 5, an actuator (Γ_1, g_1) is efficient if and only if there exist n_0 and m_0 such that

$$\sum_{j=1}^{r_{n_0,m_0}} \langle g_1, \psi_{n_0,m_0j}\rangle_{\Gamma_1} \langle h_1, \psi_{n_0,m_0j}\rangle_{\Omega_1} \neq 0. \tag{58}$$

Let us remark that for $n_0 = 0$ and $n_0 \neq 0$, (58) respectively becomes

$$\langle g_1, \psi_{0,m_0}\rangle_{\Gamma_1} \langle h_1, \psi_{0,m_0}\rangle_{\Omega_1} \neq 0 \tag{59}$$

and

$$\sum_{j=1}^{2} \langle g_1, \psi_{n_0,m_0j}\rangle_{\Gamma_1} \langle h_1, \psi_{n_0,m_0j}\rangle_{\Omega_1} \neq 0. \tag{60}$$

Remark 6. In the case of a Dirichlet boundary condition, System (S_P) becomes

$$(S) \begin{cases} \dfrac{\partial z}{\partial t}(x,t) = \Delta z(x,t) & \text{in} \quad \Omega \times]0, T[, \\ z(x,t) = f(x,t) + Bu(t) & \text{on} \quad \Gamma \times]0, T[, \\ z(x,0) = 0 & \text{in} \quad \Omega. \end{cases}$$

The state of (S_P) at final time T is not necessarily in $L^2(\Omega)$, for $u \in L^2(0, T; \mathcal{U})$ and $f \in L^2(0, T; L^2(\Gamma))$. But for $u \in L^d(0, T; \mathcal{U})$ and $f \in L^r(0, T; L^2(\Gamma))$ with $d > 4$

and $r > 4$, we have $z_{u,f}(T) \in L^2(\Omega)$ (Berrahmoune, 1984; El Jai and Pritchard, 1988). Up to some technical details, the results are analogous to those obtained in the case of the Neumann boundary condition, with the controllability matrix

$$M_n = \left(\langle g_i, \frac{\partial \varphi_{nj}}{\partial \nu} \rangle_{L^2(\Gamma)} \right)_{\substack{i=1:p \\ j=1:r_n}} .$$

Similarly, we show that the actuators $(\Gamma_i, g_i)_{i=1:p}$ are efficient if and only if

$$\bigcap_{n \geq 1} \ker(M_n G_n^{\mathrm{tr}}) = \bigcap_{n \geq 1} \ker[(G_n a_n)^{\mathrm{tr}}],$$

where

$$G_n = \big(\langle h_i, \varphi_{nj} \rangle \big)_{\substack{i=1:q \\ j=1:r_n}}$$

and

$$a_n = \left(\frac{\partial \varphi_{n1}}{\partial \nu}, \dots, \frac{\partial \varphi_{nr_n}}{\partial \nu} \right)^{\mathrm{tr}} .$$

6. Conclusion

In the case of systems subjected to Neumann boundary conditions, we defined and characterized the notion of remediability, which consists for a given output equation (sensors) in studying a possibility of finite-time space compensation of any boundary disturbance using boundary actions. We showed how to find convenient boundary actuators ensuring this compensation (efficient actuators). Then we studied the relationship between the notions of remediability and controllability with boundary actuators. More precisely, we showed that remediability is a weaker notion than controllability, and hence that strategic actuators are necessarily efficient, the converse not being true. Then we demonstrated (in the last section) that the number condition ($p \geq \sup r_n$) for boundary actuators to be strategic is not necessary for them to be efficient. We also determined the set of remediable boundary disturbances, and for a boundary disturbance, we gave an optimal control guaranteeing its compensation. We also indicated how to extend the results to a Dirichlet boundary condition. Finally, let us note that this approach and the results obtained can be also extended to the case of internal actuators or boundary sensors, and that other aspects of remediability can be studied and other systems can be considered.

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