UPPER AND LOWER SET FORMULAS: RESTRICTION AND MODIFICATION OF THE DEMPSTER-PAWLAK FORMALISM†

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A modification of Dempster's and Pawlak's constructs forms a new foundation for the identification of upper and lower sets formulas. Also, in this modified Dempster-Pawlak construct we require that subsets of the power set be restricted to the well-known information granules of the power set. An aggregation of upper information granules amongst each other and lower information granules amongst each other determine upper and lower set formulas for both crisp and fuzzy sets. The results are equivalent to the Truth Table derivation of FDCF and FCCF, Fuzzy Disjunctive Canonical Forms and Fuzzy Conjunctive Canonical Forms, respectively. Furthermore, they collapse to $\mathrm{DNF} \equiv \mathrm{CNF}$, i.e., the equivalence of Disjunctive Normal Forms and Conjunctive Normal Forms, in the combination of concepts once the LEM, LC and absorption, idempotency and distributivity axioms are admitted into the framework. Finally, a proof of the containment is obtained between FDCF and FCCF for the particular class of strict and nilpotent Archimedian t-norms and t-conorms.

Keywords: upper and lower set formulas, information granules, Dempster-Pawlak modification, fuzzy canonical formulas

1. Introduction

We first reinterpret and modify the Dempster-Pawlak formalism with information granules to identify upper and lower sets that determine upper and lower set formulas for both crisp and fuzzy sets. The aim is to provide an answer to a question that most skeptics of fuzzy theory have been asking for a long time.

One of the questions that non-fuzzy researchers usually ask is: "Why are the two-valued formulas applied in fuzzy theory without any modification in the combination of concepts when propositions and their denotations are fuzzy?". That is, they suspect that there ought to be a re-assessment of the formulas in combination of fuzzy sets with "AND", "OR", "IMP", etc.

In other words, they wonder why there are no new formulas since the axiomatic foundation of fuzzy set theory relaxes the Law of Excluded Middle, LEM, $A \cup c(A) = I$, to be $A \cup c(A) \subseteq I$ and its dual Law of Contradiction, LC, $A \cap c(A) = \emptyset$, to be $A \cap c(A) \supseteq \emptyset$, where $c(A)$ is the complement of A. Furthemore, it is known that the axioms of absorption, idempotency and distributivity are no longer applicable for the general class of t-norms and t-conorms. Thus, they question the use of the two-valued formulas and ask that they be re-examined since LEM, LC and axioms such as distributivity, idempotency and absorption are no longer applicable to the general class of connectives known as t-norms and t-conorms.

A related question is: "What happens to the equivalence between Disjunctive and Conjunctive normal forms, i.e., is $\mathrm{DNF} \equiv \mathrm{CNF}$, if the formulas of the two-valued theory are fuzzified by a direct substitution of fuzzy values in these formulas?".

In the classical, two-valued set theory, we always compute the formula for "A AND B" $= A \cap B$, which is $\mathrm{DNF}\,(A \text{ AND } B)$, and for "$A$ OR B" $= A \cup B$, which is $\mathrm{CNF}\,(A \text{ OR } B)$. They are the usual formulas used in applications. There are, however, other formulas, such as

$$\text{“}A \text{ AND } B\text{”} = (A \cup B) \cap \big(c(A) \cup B\big) \cap \big(A \cup c(B)\big),$$

which is $\mathrm{CNF}\,(A \text{ AND } B)$, and

$$\text{“}A \text{ OR } B\text{”} = (A \cap B) \cup \big(c(A) \cap B\big) \cup \big(A \cap c(B)\big),$$

which is $\mathrm{DNF}\,(A \text{ OR } B)$.

The shorter forms are used because it is known and easy to show that

$$A \cap B \equiv (A \cup B) \cap \big(c(A) \cup B\big) \cap \big(A \cup c(B)\big)$$

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and

$$A \cup B \equiv (A \cap B) \cup \big(c(A) \cap B\big) \cup \big(A \cap c(B)\big).$$

These equivalences are essentially due to the fact that the classical, two-valued, set theory axioms contain LEM, its dual LC, and idempotency, distributivity and absorption.

For example, in the demonstration of

$$A \cap B \overset{?}{\equiv} (A \cup B) \cap \big(c(A) \cup B\big) \cap \big(A \cup c(B)\big),$$

we first apply idempotency and get $(A \cup B) \cap (A \cup B) \cap (c(A) \cup B) \cap (A \cup c(B))$. Next, we apply commutativity and distributivity, to obtain $(A \cap c(A)) \cup B$ from the first and third terms and get $(B \cap c(B)) \cup A$ from the second and fourth terms.

As a result, we obtain

$$\big[\big(A \cap c(A)\big) \cup B\big] \cap \big[\big(B \cap c(B)\big) \cup A\big] .$$

Next, with the application of LC, $A \cap c(A) = \emptyset$ and $B \cap c(B) = \emptyset$, we get $B \cap A$. Then, using commutativity, we finally get $A \cap B \equiv A \cap B$.

Because of this equivalence, the shorter form amongst the DNF and CNF is used in computations and applications. Clearly, since axioms of distributivity, idempotency, absorption and LEM and/or LC are no longer applicable in fuzzy theory, we suspect that, in general,

$$A \cap B \neq (A \cup B) \cap \big(c(A) \cup B\big) \cap \big(A \cup c(B)\big).$$

In fact, it was shown (Türkşen, 1986) that

$$\mathrm{DNF}\,(A \ \text{AND} \ B) \subseteq \mathrm{CNF}\,(A \ \text{AND} \ B),$$

where

$$\mathrm{DNF}\,(A \text{ AND } B) = A \cap B,$$

$$\mathrm{CNF}\,(A \text{ AND } B) = (A \cup B) \cap \big(c(A) \cup B\big) \cap \big(A \cup c(B)\big)$$

for some t-norms and t-conorms.

Also, this is shown to be true for certain well-known t-norms and t-conorms of fuzzy theory (Türkşen, 1999; 2001).

It should be recalled that all of the 16 combinations of any two crisp sets have DNF and CNF, Disjunctive Normal Form and Conjunctive Normal Form, respectively, as shown in Table 1.

Furthermore, it is well known that we have

$$\mathrm{DNF}\,(\cdot) \equiv \mathrm{CNF}\,(\cdot)$$

for all the 16 possible combinations shown in Table 2 for the two-valued set and logic theory, whose axioms are shown in Table 3.

Table 1. Classical Disjunctive Normal and Fuzzy Disjunctive Canonical Forms, DNF and FDCNF, and Classical Conjunctive Normal and Fuzzy Conjunctive Canonical Forms, CNF and FCCF, where $\cap$ is a conjunction, $\cup$ is a disjunction and c is a complementation operator.

No.	Fuzzy Disjunctive Canonical Forms/Disjunctive Normal Forms
1	$(A\cap B)\cup\big(A\cap c(B)\big)\cup\big(c(A)\cap B\big)\cup\big(c(A)\cap c(B)\big)$
2	ϕ
3	$(A \cap B) \cup \big(A \cap c(B)\big) \cup \big(c(A) \cap B\big)$
4	$\big(c(A) \cap c(B)\big)$
5	$\big(A\cap c(B)\big) \cup \big(c(A)\cap B\big) \cup \big(c(A)\cap c(B)\big)$
6	$(A \cap B)$
7	$(A \cap B) \cup \big(c(A) \cap B\big) \cup \big(c(A) \cap c(B)\big)$
8	$\big(A \cap c(B)\big)$
9	$(A \cap B) \cup \big(A \cap c(B)\big) \cup \big(c(A) \cap c(B)\big)$
10	$\big(c(A) \cap B\big)$
11	$(A \cap B) \cup \big(c(A) \cap c(B)\big)$
12	$\big(A \cap c(B)\big) \cup \big(c(A) \cap B\big)$
13	$(A \cap B) \cup \big(A \cap c(B)\big)$
14	$\big(c(A) \cap B\big) \cup \big(c(A) \cap c(B)\big)$
15	$(A \cap B) \cup \big(c(A) \cap B\big)$
16	$\big(A \cap c(B)\big) \cup \big(c(A) \cap c(B)\big)$

No.	Fuzzy Conjunctive Canonical Forms/Conjunctive Normal Forms
1	I
2	$(A\cup B)\cap\big(A\cup c(B)\big)\cap\big(c(A)\cup B\big)\cap\big(c(A)\cup c(B)\big)$
3	$(A \cup B)$
4	$\big(A\cup c(B)\big) \cap \big(c(A)\cup B\big) \cap \big(c(A)\cup c(B)\big)$
5	$\big(c(A)\cup c(B)\big)$
6	$(A \cup B) \cap \big(A \cup c(B)\big) \cap \big(c(A) \cup B\big)$
7	$\big(c(A) \cup B\big)$
8	$(A \cup B) \cap \big(A \cup c(B)\big) \cap \big(c(A) \cup c(B)\big)$
9	$\big(A \cup c(B)\big)$
10	$(A \cup B) \cap \big(c(A) \cup B\big) \cap \big(c(A) \cup c(B)\big)$
11	$\big(A \cup c(B)\big) \cap \big(c(A) \cup B\big)$
12	$(A \cup B) \cap \big(c(A) \cup c(B)\big)$
13	$(A \cup B) \cap \big(A \cup c(B)\big)$
14	$\big(c(A) \cup B\big) \cap \big(c(A) \cup c(B)\big)$
15	$(A \cup B) \cap \big(c(A) \cup B\big)$
16	$\big(A\cup c(B)\big) \cap \big(c(A)\cup c(B)\big)$

There are Truth Table derivations of DNF and CNF expressions for all the 16 combinations of concepts with the application of the Normal Form (Canonical Form) Derivation Algorithm (see Appendix A). An example of the Truth Table for "A AND B" is shown in Table 4.

Table 2. Meta-Linguistic Expressions of Combined Concepts for any A and B which may be crisp or fuzzy.

Number	Meta-linguistic expressions
1	UNIVERSE
2	EMPTY SET
3	A OR B
4	NOT A AND NOT B
5	NOT A OR NOT B
6	A AND B
7	A IMPLIES B
8	A AND NOT B
9	A OR NOT B
10	NOT A AND B
11	A IF AND ONLY IF B
12	A EXCLUSIVE OR B
13	A
14	NOT A
15	B
16	NOT B

Table 3. Axioms of classical set and logic operations.

Involution	$c\big(c(A)\big) = A$
Commutativity	$A \cup B = B \cup A$
	$A \cap B = B \cap A$
Associativity	$(A \cup B) \cup C = A \cup (B \cup C)$
	$(A \cap B) \cap C = A \cap (B \cap C)$
Distributivity	$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
	$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
Idempotence	$A \cup A = A$
	$A \cap A = A$
Absorption	$A \cup (A \cap B) = A$
	$A \cap (A \cup B) = A$
Absorption by X and l	$A \cup X = X$
	$A \cap \phi = \phi$
Identity	$A \cup \phi = A$
	$A \cap X = A$
Law of contradiction	$A \cap c(A) = \phi$
Law of excluded middle	$A \cup c(A) = X$
De Morgan's laws	$c(A \cap B) = c(A) \cup c(B)$
	$c(A \cup B) = c(A) \cap c(B)$

Table 4. Classical truth table interpretations of "A AND B".

Truth assignments to classical meta-linguistic variables		Truth assignments to the meta-linguistic expression	Primary conjunctions
A	B	"A AND B"	
$T(A)$	$T(B)$	$T(A$ AND $B)$	$A \cap B$
$T(A)$	$F(B)$	$F(A$ AND $B)$	$A \cap c(B)$
$F(A)$	$T(B)$	$F(A$ AND $B)$	$c(A) \cap B$
$F(A)$	$F(B)$	$F(A$ AND $B)$	$c(A) \cap c(B)$

The early investigations (Türkşen, 1986) showed that the substitution of fuzzy values in the interval $[0,1]$ into the classical DNF and CNF formulas, instead of crisp values of the lattice $\{0,1\}$, leads to the result that $\mathrm{DNF}(\cdot) \subseteq \mathrm{CNF}(\cdot)$ for some well-known class of t-norms and t-conorms.

However, there remained the crucial question: "Could we derive Fuzzy Disjunctive Canonical Forms, FDCF, and Fuzzy Conjunctive Canonical Forms, FCCF, directly from Fuzzy Truth Tables in a manner analogous to the classical Truth Table derivation of DNF and CNF?", using the Normal Form (Canonical Form) Derivation Algorithm (see Appendix A).

This concern was investigated in several papers (Türkşen, 1999; 2001) with a positive constructive result. It was shown that DNF is equivalent in form only to FDCF and CNF is equivalent in form only to FCCF, as shown in Table 1, for all the 16 possible combination concepts shown in Table 2.

In this paper, in order to support and generalize these earlier results, we first show that there is a connection with both Dempster's upper and lower bounds on sets (Dempster, 1967) and Pawlak's upper and lower approximations on sets (Pawlak, 1991). Secondly, we modify the Dempster-Pawlak construction schema and restrict the subsets of the power set to a particular subset of information granules in the power set.

Thirdly, the formation of FDCF and FCCF from the lower and upper set approximations, respectively, of the Dempster-Pawlak constructs should establish the natural containment of FDCF in FCCF due to the construction schema. However, the question of whether FDCF is always contained in FCCF is resolved with a proof based on generator functions (see Appendix B). That is, $\mathrm{FDCF}(\cdot) \subseteq \mathrm{FCCF}(\cdot)$ holds by the construction of these upper and lower sets for all of the 16 combination of concepts and for all connectives, i.e., t-norms and t-conorms.

Recall that in previous papers, this inclusion relationship was shown to hold only for the well-known t-norms and t-conorms, such as $(\mathrm{Max}, \mathrm{Min}, -)$, i.e., $(\vee, \wedge, -)$ (algebraic sum, product, $-$), i.e., $(\oplus, \odot, -)$, and Łukasiewicz operators or bold operators, i.e., $(B_\oplus, B_\odot, -)$, where "$-$" stands for standard complementation.

But at the time it was not possible to show the containment in general for all t-norms and t-conorms. In this sense, this paper generalizes the previous results, as well as establishes a connection with Dempster's and Pawlak's constructs.

2. Dempster's and Pawlak's Formulations

In order to handle the uncertainty associated with imprecise sources of information in the identification of sets, Dempster and Pawlak propose alternative schemas with perspectives and contents fitting their own agenda. With a multi-valued mapping, Dempster (1967) proposes the construction of upper and lower probabilities. In order to formulate upper and lower probabilities, he first proposes a multi-valued mapping from a space to another. On the other hand, Pawlak (1991) approaches the same concern from the perspective of rough sets. We shall briefly summarize Dempster's and Pawlak's proposals, and then show a correspondence between these two construction schemas.

2.1. Dempster's Construction

Let X and S be two spaces, and consider a multi-valued mapping Γ, Γ: $X \to (S)$, where $P(S)$ is the power set of S, i.e., $P(S) = 2^{|S|}$. Furthermore, let T be a target set, $T \subseteq S$, i.e., $T \in P(S)$. Then the upper and lower sets of this multi-valued mapping, Γ, are defined for any target set T as follows:

$$T^* = \{x \mid x \in X, \quad \Gamma(x) \cap T \neq \emptyset\}, \tag{1}$$

and

$$T_* = \{x \mid x \in X, \quad \Gamma(x) \subseteq T, \quad \Gamma(x) \neq \emptyset\}, \tag{2}$$

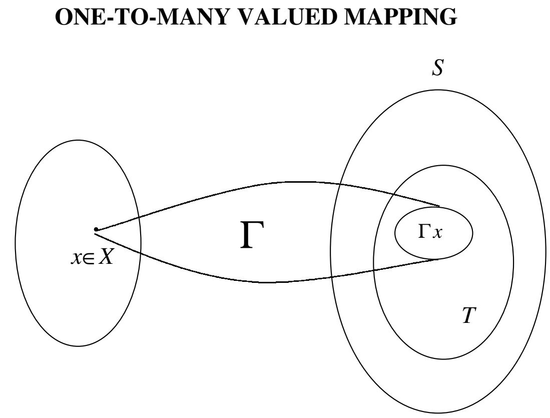
respectively, where, for a target set T, we are interested in determining the upper and lower sets, T^* and T_*, respectively, under the multi-valued mapping Γ (Dempster, 1967).

Clearly, we have $T_* \subseteq T^*$ from the construction. Dempster's construction is motivated by concern for the transformation of probabilities from the space X with a probability measure to the space S via a multi-valued mapping in order to define upper and lower probability estimates for an unsharp information source. However, prior to these probabilities, Dempster's upper and lower sets T^* and T_* identify upper and lower sets, respectively, for the target set T constructed with the subsets of $P(S)$ identified by $\Gamma(x)$.

For further clarity of the expressions (1) and (2), we provide a graphical representation of the expressions (1) and (2) in Figs. 1 and 2.

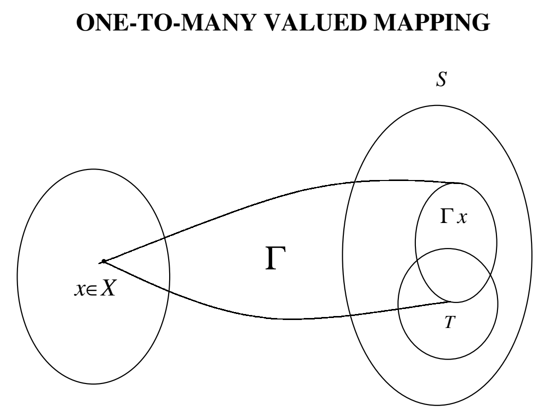
2.2. Pawlak's Construction

Let X be a given universal set and R an equivalence relation on X. The relation R induces a partition on X. Let $[x]_R$ be the equivalence class containing $x \in X$. It is to be noted that each $[x]_R$ is simply a subset of X. That



$$T \subseteq S, \;\; T \in P(S) = 2^{|S|}$$
$$T_* = \{x \in X, \;\; \Gamma x \subseteq T, \;\; \Gamma(x) \neq \emptyset\}$$

Fig. 1. Lower set representation of Dempster's formalism.



$$T \subseteq S, \;\; T \in P(S) = 2^{|S|}$$
$$T^* = \{x \in X, \;\; \Gamma x \cap T \neq \emptyset\}$$

Fig. 2. Upper set representation of Dempster's formalism.

is, if x is a typical element of X, then "the equivalence class of x" is $[x] = \{t\colon\ tRx\}$. Given a set X and an equivalence relation R, we define a new set X/R, the set of equivalence classes of X modulo R.

Thus, the set of all $[x]_R$ will form a partition of X, denoted by X/R. Next, let $A \subseteq X$. Then upper and lower estimates of A, based on X/R, are defined as follows:

$$A^*_R = \cup\{[x]_R \mid [x]_R \cap A \neq \emptyset\},$$

and

$$A_{*R} = \cup\{[x]_R \mid [x]_R \subseteq A\}.$$

Clearly, we have $A_{*R} \subseteq A^*_R$. Pawlak's rough set constructs A_{*R} and A^*_R identify an approximate cover of A with the information granules of X/R (Pawlak, 1991).

2.3. Restatement of Dempster's and Pawlak's Constructs

Both Dempster's and Pawlak's constructions rely on a set inclusion for the lower estimate and a non-empty intersection for the upper estimate.

In fact, the two approaches are analogous and each can be expressed in the language of the syntax and semantics of the other.

2.3.1. Pawlak to Dempster Transformation

Consider Pawlak's construct with the universe of discourse U, and the equivalence relation R on U. In Dempster's notation, let X be U/R and S be U. Let Γ of Demspter be the multi-valued mapping that causes the transformation of the equivalence classes to the corresponding set; i.e., Γ maps each equivalence class or member of the partition to all the elements that are in it. Thus we have $\Gamma\colon\ U/R \to P(U)$ and $\Gamma([u]_R) = \{t \mid tRu,\ t \in U\} \in P(U)$.

Let $A \subseteq U$. Then $A_{*R} = \cup\{[x]_R \mid [x]_R \subseteq A\}$ in Pawlak's notation, and $A_* = \{[x]_R \mid [x]_R \in U/R,\ \Gamma([x]_R) \subseteq A\}$ in Dempster's notation. Note that $\Gamma([x])_R)$ is just the set $[x]_R$. Thus A_{*R} and A_* are composed of the same information granules from U/R, and capture the same information. A similar argument can be written for the correspondence between the upper estimates.

A minor structural difference between these two approaches is that the union of these equivalence classes is taken in Pawlak's approach, whereas a set of these classes is collected in Dempster's approach. Note that in Dempster's approach, probability measures are computed and *added* over the upper and lower sets in order to arrive at the upper and lower probabilities.

2.3.2. Dempster to Pawlak Transformation

Now let us recall the two spaces X and S and the multi-valued mapping $\Gamma\colon\ X \to P(S)$ in Dempster's approach. Let a relation R on X be defined as xRy if and only if $\Gamma(x) = \Gamma(y)$. That is, two elements $x, y \in X$ are considered equivalent if they map to exactly the same set of values in S. Naturally, this is an equivalence relation on X and it induces the partition X/R. Let $T \subseteq S$. In Dempster's approach, we have $T_* = \{x \mid x \in X,\ \ \Gamma(x) \subseteq T,\ \ \Gamma(x) \neq \emptyset\}$. Note that if $x \in T_*$ and yRx, then $y \in T_*$, because $\Gamma(x) = \Gamma(y)$. Therefore T_* can be written as the union of partiton members or classes in X/R, and thus expressed in Pawlak's notation. Naturally, one can state a similar argument for the correspondence of the upper estimates.

Since it is shown that there is a natural correspondence between Pawlak's and Dempster's notations, we shall refer to them as the Dempster-Pawlak notation, or as the D-P schema, in deriving the upper and lower set formulas in the combination of concepts depending on whether they are represented by crisp or fuzzy sets.

3. Sets and Logic Constructs

Let a proposition P be expressed by a predicate A which could be either a crisp or a fuzzy set depending on how A is obtained or defined. In previous works (Türkşen, 1999; 2001), such propositions and their predicates were labeled as "descriptive" propositions and words, respectively.

Suppose that values x are taken from a universe of discourse X. For a given value $x \in X$ with the membership value $\mu_A(x)$ let T be true in the Type 1 fuzzy theory. In this way, the truth of a descriptive proposition P is equated to the set A being true with the membership degree of $\mu_A(x)$.

Next, let two linguistic concepts, P_1 and P_2, i.e., propositions, or their expressions be represented by two sets A and B, respectively. These two propositions and hence their set representations can be combined in 16 possible meta-linguistic expressions. These 16 expressions are shown in Table 5 together with their usual target set expressions written in a set notation (crisp or fuzzy).

Table 5. 16 meta-linguistic expressions together with their "usual target set expressions" (crisp or fuzzy).

Number	Meta-linguistic expressions	Usual target set expression
1	Universe	I
2	Empty set	$\emptyset$
3	A OR B	$A \cup B$
4	NOT A AND NOT B	$c(A) \cap c(B)$
5	NOT A OR NOT B	$c(A) \cup c(B)$
6	A AND B	$A \cap B$
7	A IMPLIES B	$c(A) \cup B$
8	A AND NOT B	$A \cap c(B)$
9	A OR NOT B	$A \cup c(B)$
10	NOT A AND B	$c(A) \cap B$
11	A IF AND ONLY IF B	$(A \cap B) \cup \big(c(A) \cap c(B)\big)$
12	A EXCLUSIVE OR B	$(A \cup B) \cap \big(c(A) \cup c(B)\big)$
13	A	A
14	NOT A	$c(A)$
15	B	B
16	NOT B	$c(B)$

3.1. Restriction and Modification

We propose a restriction and a modification of the D-P formalism:

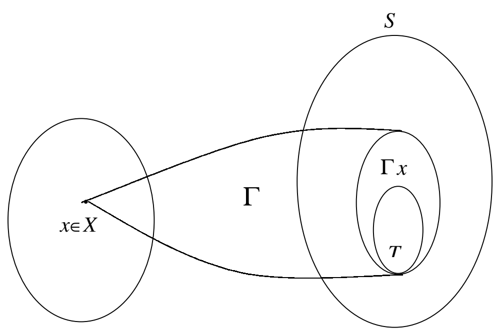
1. Restrict the subset of the power set to the family of information granules $\boldsymbol{G}_n$, where

$$\boldsymbol{G}_n = \{A_1 \cap A_2 \cap \cdots \cap A_n, \ldots, c(A_1) \cup c(A_2) \cup \cdots \cup c(A_n)\}$$

   for all possible combinations of subsets A_i, $i = 1, \ldots, n$ in a universe of discourse by $\cup$, $\cap$, and $c(\cdot)$.

2. Modify the condition for the upper set identification from a "non-empty set intersection" to "set inclusion" (this is shown in Fig. 3).

ONE-TO-MANY VALUED MAPPING



$$T \subseteq S,\ \ T \in G \subseteq P(S) = 2^{|S|}$$
$$T^* = \{x \in X,\ \ T \subseteq \Gamma x\}$$

Fig. 3. Modification of Dempster's formalism.

It is to be noted that:

1. The restriction of the subsets of the power set to the family $\boldsymbol{G}$ is formed by the primitive information granules that are all directly identified by the set operators on subsets in a universe of discourse and their combinations *cover* the set of all meta-linguistic expressions for both crisp and fuzzy cases.

2. The modification, however, strengthens the upper set condition to "set inclusion" from "non-empty set intersection", which provides a conceptual and intuitive appeal.

In other words, let E be the target set of a meta-linguistic expression. Then $\boldsymbol{G}$ covers E from below, because there exists at least one $G \in \boldsymbol{G}$, such that $G \subseteq E$. Because E must contain as many information granules as there are contained in it, i.e., $G \subseteq E$, it makes a semantic sense that E contains as many information granules that are true as the *disjunction* of all such Gs. Thus the disjunction of those information granules forms a lower set formula for E. This is in agreement with Dempster's formalism but requires a disjunctive aggregation of the information granules so obtained in Pawlak's formalism.

$\boldsymbol{G}$ also covers E from above, because there is at least one $G \in \boldsymbol{G}$ such that $E \subseteq G$. Furthermore, because any such G contains E, it makes a semantic sense that we take the *conjunction* of all such Gs that contain E. Thus, the conjunction of these information granules forms an upper set formula for E. It is to be noted that this is a further modification to the D-P formalism. That is, we take the conjunctive aggregation of the information granules so obtained after the proposed change from "non-empty set intersection" to "set inclusion".

In summary, the disjunction of the proposed collection of information granules $G \subseteq E$ will give us a lower set formula for E, and the conjunction of the proposed collection of information granules G, $E \subseteq G$ will give us an upper set formula for E with the modified D-P formalism of E. It should be noted that both the upper and lower set formulas are defined so far with inclusion, i.e., "$\subseteq$". This creates anomalies in the construction schema for the identification of granules $G \in \boldsymbol{G}$. This issue will be resolved in the next section, where it depends on particular classes of five meta-linguistic expressions, dependent on the linguistic operator "AND", and five meta-linguistic expressions dependent on the linguistic operator "OR", and the remaining six special cases that include bi-conditional, exclusive-or and confirmation and negation of two singletons.

4. Upper and Lower Canonical Forms

Let us now consider again two linguistic concepts A and B, i.e., the two predicates (words) A and B and the 16 meta-linguistic expressions that can be generated for them. We start with the usual target set expressions shown in Table 5 as the target sets in the sense of Dempster and Pawlak and the sets whose upper and lower set formulas are to be determined by the proposed modification of the Dempster-Pawlak formalism.

Our aim is to demonstrate how we could generate FDCF and FCCF expressions directly with the use of information granules and with the proposed modification of the Dempster-Pawlak approach stated above without resorting to the Truth Table derivation that was discussed in our previous papers (Türkşen, 1999; 2001).

Let E be any of these 16 expressions, and $\boldsymbol{G}_2$ be the family of information granules that are the eight possible combinations of two predicates A and B under conjunction, disjunction and complementation operations:

$$\boldsymbol{G}_2 = \{G_1, G_2, \dots, G_8\},$$

where $G_1 = A \cap B, \dots, G_8 = c(A) \cup c(B)$, such that

$$G_2 = \big\{A \cap B, c(A) \cap B, A \cap c(B), c(A) \cap c(B), \\ A \cup B, c(A) \cup B, A \cup c(B), c(A) \cup c(B)\big\}.$$

It should be noted that the first four of these information granules, $G_1, \dots, G_4$, i.e., $A \cap B$, $c(A) \cap B$, $A \cap c(B)$ and $c(A) \cap c(B)$, form a disjoint partition of the universe, whereas the last four of these information granules, $(G_5, \dots, G_8)$, i.e., $A \cup B$, $c(A) \cup B$, $A \cup c(B)$ and $c(A) \cup c(B)$, have overlaps.

As a result of our discussions and the proposed modifications stated in Section 3 for each target set E, we define the upper and lower subset formulas to be one of the information granules as follows:

$$\ell(E) = \big\{G \mid G \in \boldsymbol{G},\ \ G \subseteq E\big\}, \\ u(E) = \big\{G \mid G \in \boldsymbol{G},\ \ E \subseteq G\big\}. \tag{3}$$

It is to be observed that each element of $\ell(E)$ comes from the subsets of the conjunctive information granules, i.e., $A \cap B$, $c(A) \cap B$, $A \cap c(B)$ and $c(A) \cap c(B)$, whereas each element of $u(E)$ comes from the disjunctive information granules, i.e., $A \cup B$, $c(A) \cup B$, $A \cup c(B)$ and $c(A) \cup c(B)$.

With these lower and upper subsets, we determine the lower and upper set formulas of the target set E in the proposed modification of the Dempster-Pawlak formalism as

$$L(E) = \cup \ell(E), \quad U(E) = \cap u(E). \tag{4}$$

It is to be noted that the disjunction of $\ell(E)$s is taken to form the lower set $L(E)$ since they are all contained in the target set and they are disjoint among themselves. Thus $L(E)$ forms the greatest lower bound.

But the conjunction of $u(E)$s is taken to form the upper set, $U(E)$, since they all contain the target set and are not disjointed. Thus $U(E)$ forms the least upper bound.

At a first glance, it appears that $L(E) \subseteq E \subseteq U(E)$ by the construction schema. However, the inclusion relation $L(E) \subseteq U(E)$ requires an investigation in the t-norm and conorm space. It can be shown that $L(E) \subseteq U(E)$ for strict and nilpotent Archimedean t-norms and t-conorms, as well as $(\mathrm{Max}, \mathrm{Min})$ (Bilgic, 1995) (Appendix B).

Since $L(E)$ turns out to be equal to FDCF and $U(E)$ turns out to be equal to FCCF, by the construction, we have determined that $\mathrm{FDCF} \subseteq \mathrm{FCCF}$ not only for the well-known specific t-norms and t-conorms, but for all cases of t-norms and t-conorms that are strict and nilpotent Archimedians.

Now let us return to the task at hand. That is, we are to drive the upper and lower set formulas that represent each of the 16 meta-linguistic expressions. In each case, the selections of information granules are taken from $\boldsymbol{G}$. It should be noted that the selection of each information granule that makes up the sets $\ell(E)$s and $u(E)$s is made with definitions given by (3) and(4), i.e., $\ell(E)$ is the set of information granules that are contained in E and $u(E)$ is the set of Es that are contained by information granules $G \in \boldsymbol{G}$, where E is the usual target set in common use. Furthermore, we determine the lower set formula by the disjunctive aggregation of $\ell(E)$s, i.e., $L(E) = \cup L(E)$ and the upper set formula by the conjunctive aggregation of $u(E)$s, i.e., $U(E) = \cap u(E)$ as indicated by (3) and (4).

However, there is yet another issue to be clarified, as has been pointed out earlier. That is, in eqns. (3) we observe that $G \subseteq E$ for $\ell(E)$ and $E \subseteq G$ for $u(E)$ and every $G \in \boldsymbol{G}$. Thus there are possible Gs that may belong to both $l(E)$ and $u(E)$. That is, we need to identify which G's are taken for the equality and which are taken for inclusion with respect to linguistic operators, "AND" and "OR" and other special cases. This will be sorted out and clarified in Sections 4.1 and 4.2 below because they depend on whether a meta-linguistic concept is formed by "AND" or "OR" or other schemas. Thus we next develop the formulas for the cases of "AND" and "OR", and other schemas, and then we tabulate the results of all the 16 expressions in Table 6.

4.1. Five Meta-Linguistic Expressions That Have "AND" Composition

The usual, commonly used target set E is "$A \cap B$" for "A AND B" (see Table 5, row 6). With the discussion and the proposed modification of the D-P formalism stated in Sections 3 and 4 above, we identify the set of information granules that are contained in E as $\ell(E) = \{A \cap B\}$, and thus the lower set formula is $L(E) = \{A \cap B\} = \ell(E)$.

The set of information granules that contain E is $u(E) = \{A \cup B, c(A) \cup B, A \cup c(B)\}$ and thus the upper set formula is $U(E) = \cap u(E) = (A \cup B) \cap (c(A) \cup B) \cap (A \cup c(B))$.

It is to be observed that $\ell(E) = \{A \cap B\}$, which is the target set itself, and thus $L(E) = \ell(E)$, i.e., $L(E) = \{G \mid G \in \boldsymbol{G},\ \ G = E\}$. We generalize this for

Table 6. Lower and upper set expression.

ID number of meta-linguistic expression	Usual target set expression E	Lower set expression $L(E) = \text{FDCF}$ Fuzzy Disjunctive Canonical Form
1	I	$(A \cap B) \cup \big(c(A) \cap B\big) \cup \big(A \cap c(B)\big) \cup \big(c(A) \cap c(B)\big)$
2	$\emptyset$	$\emptyset$
3	$A \cup B$	$(A \cap B) \cup \big(c(A) \cap B\big) \cup \big(A \cap c(B)\big)$
4	$c(A) \cap c(B)$	$c(A) \cap c(B)$
5	$c(A) \cup c(B)$	$\big(c(A) \cap B\big) \cup \big(A \cap c(B)\big) \cup \big(c(A) \cap c(B)\big)$
6	$A \cap B$	$A \cap B$
7	$c(A) \cup B$	$(A \cap B) \cup \big(c(A) \cap B\big) \cup \big(c(A) \cap c(B)\big)$
8	$A \cap c(B)$	$A \cap c(B)$
9	$A \cup c(B)$	$(A \cap B) \cup \big(c(A) \cap B\big) \cup \big(c(A) \cap c(B)\big)$
10	$c(A) \cap B$	$c(A) \cap B$
11	$(A \cap B) \cup \big(c(A) \cap c(B)\big)$	$(A \cap B) \cup \big(c(A) \cap c(B)\big)$
12	$(A \cup B) \cap \big(c(A) \cup c(B)\big)$	$\big(c(A) \cap B\big) \cup \big(A \cap c(B)\big)$
13	A	$(A \cap B) \cup \big(A \cap c(B)\big)$
14	$c(A)$	$\big(c(A) \cap B\big) \cup \big(c(A) \cap c(B)\big)$
15	B	$(A \cap B) \cup \big(c(A) \cap B\big)$
16	$c(B)$	$\big(A \cap c(B)\big) \cup \big(c(A) \cap c(B)\big)$

ID number of meta-linguistic expression	Usual target set expression E	Lower set expression $U(E) = \text{FCCF}$ Fuzzy Conjuctive Canonical Form
1	I	I
2	$\emptyset$	$(A \cup B) \cap \big(c(A) \cup B\big) \cap \big(A \cup c(B)\big) \cap \big(c(A) \cup c(B)\big)$
3	$A \cup B$	$A \cup B$
4	$c(A) \cap c(B)$	$\big(c(A) \cup B\big) \cap \big(A \cup c(B)\big) \cap \big(c(A) \cup c(B)\big)$
5	$c(A) \cup c(B)$	$c(A) \cup c(B)$
6	$A \cap B$	$(A \cup B) \cap \big(c(A) \cup B\big) \cap \big(A \cup c(B)\big)$
7	$c(A) \cup B$	$c(A) \cup B$
8	$A \cap c(B)$	$(A \cup B) \cap \big(A \cup c(B)\big) \cap \big(c(A) \cup c(B)\big)$
9	$A \cup c(B)$	$A \cup c(B)$
10	$c(A) \cap B$	$(A \cup B) \cap \big(c(A) \cup B\big) \cap (c(A) \cup c(B))$
11	$(A \cap B) \cup \big(c(A) \cap c(B)\big)$	$\big(c(A) \cup B\big) \cap \big(A \cup c(B)\big)$
12	$(A \cup B) \cap \big(c(A) \cup c(B)\big)$	$(A \cup B) \cap \big(c(A) \cup c(B)\big)$
13	A	$(A \cup B) \cap \big(A \cup c(B)\big)$
14	$c(A)$	$\big(c(A) \cup B\big) \cap \big(c(A) \cup c(B)\big)$
15	B	$(A \cup B) \cap \big(c(A) \cup B\big)$
16	$c(B)$	$\big(A \cup c(B)\big) \cap \big(c(A) \cup c(B)\big)$

all the five out of 16 combination of concepts that admit the linguistic “AND” connective in their meta-linguistic combination, i.e., the meta-linguistic expressions 2, 4, 6, 8 and 10 in Table 5. Therefore, the rule is $\ell(E) = \{G \mid G = E\}$ for the targets sets $\emptyset$, $c(A) \cap c(B)$, $A \cap B$, $A \cap c(B)$ and $c(A) \cap B$, where the usual target set itself forms the greatest lower bound.

This, in turn, entailes the rule for $u(E)$ to be $U(E) = \cap\{G \mid G \in \boldsymbol{G},\ E \subset G\}$ for these five out of 16 combinations of concepts that admit the linguistic “AND” connective in their meta-linguistic concept combination, i.e., meta-linguistic expressions 2, 4, 6, 8 and 10 in Table 5. This clarification resolves the anomaly generated by eqns. (3) and (4) for the cases of the linguistic “AND” connective.

It is to be noted that $L(E) = \mathrm{FDCF}\,(A\ \text{ AND }\ B)$, and $U(E) = \mathrm{FCCF}\,(A\ \text{ AND }\ B)$, and therefore we have $\mathrm{FDCF}\,(A\ \text{ AND }\ B) \subseteq \mathrm{FCCF}\,(A\ \text{ AND }\ B)$. This fact holds true for both the crisp and fuzzy sets and for all the t-norms and t-conorms due to the construction of $L(E) \subseteq U(E)$.

It is to be noted that if we apply the law of contradiction after the axiom of commutativity and distributivity, we get $\mathrm{FDCF}\,(A\ \text{ AND }\ B) = \mathrm{FCCF}\,(A\ \text{ AND }\ B)$, as demonstrated in Section 1.

Thus we obtain

$$\mathrm{DNF}\,(A\ \text{ AND }\ B) = \mathrm{CNF}\,(A\ \text{ AND }\ B) = A \cap B$$

in the two-valued set and logic theory. In fuzzy set and logic theory, FDCF and FCCF provide lower and upper set formulas, respectively, for the “AND” combination of two fuzzy concepts, A and B, i.e., fuzzy predicates. That is, we get $\mathrm{FDCF}\,(A\ \text{ AND }\ B) \subseteq \mathrm{FCCF}\,(A\ \text{ AND }\ B)$.

4.2. Five Meta-Linguistic Expressions That Have “OR” Composition

The usual, commonly used target set E is “$A \cup B$” for “A OR B” (see Table 5, row 3). Again, with the discussion and the proposed modification of the D-P formalism stated in Section 3, we identify the set of information granules that are contained in E as $\ell(E) = \{A \cap B, c(A) \cap B, A \cap c(B)\}$, and thus the lower set formula is $L(E) = \cup\ell(E) = (A \cap B) \cup (c(A) \cap B) \cup (A \cap c(B))$.

The set of information granules that contain E is $u(E) = \{A \cup B\}$, and thus the upper set formula is $U(E) = \{A \cup B\} = u(E)$.

In an analogous manner, we observe that $u(E) = \{A \cup B\}$, which is the target set itself, and thus $U(E) = \{G \mid G \in G,\ E = G\}$. We also generalize this for all the five cases out of the 16 combinations of concepts that admit the linguistic “OR” connective in their combination. Therefore the rule is $u(E) = \{G \mid G \in \boldsymbol{G},\ E = G\}$ for the target sets I, $A \cup B$, $c(A) \cup c(B)$, $c(A) \cup B$ and $A \cup c(B)$, i.e., the meta-linguistic expressions 1, 3, 5, 7 and 9 in Table 5, where the usual target set forms the least upper band.

This, in turn, entailes the rule for $L(E)$ to be $L(E) = \cup\{G \mid G \in \boldsymbol{G},\ G \subset E\}$ for these five out of the 16 cases that admit the “OR” connective in their meta-linguistic expressions, i.e., expressions 1, 3, 5, 7 and 9 in Table 5. Again, this clarification resolves the anomaly introduced by eqns. (3) and (4) for the cases of “OR” connective.

Again, it is to be noted that

$$L(E) = \mathrm{FDDCF}\,(A\ \text{ OR }\ B),$$

$$U(E) = \mathrm{FCCF}\,(A\ \text{ OR }\ B),$$

and therefore we have

$$\mathrm{FDCF}\,(A\ \text{ OR }\ B) \subseteq \mathrm{FCCF}\,(A\ \text{ OR }\ B).$$

This fact again holds true for both the crisp and fuzzy sets and for all t-norms and t-conorms due to the construction of $L(E) \subseteq U(E)$.

Again, it is to be noted that if we apply the commutativity and distributivity first, and then the Law of the Excluded Middle, LEM, we get $\mathrm{FDCF}\,(A\ \text{ OR }\ B) = \mathrm{FCCF}\,(A\ \text{ OR }\ B)$, and thus we obtain

$$\mathrm{DNF}\,(A\ \text{ OR }\ B) = \mathrm{CNF}\,(A\ \text{ OR }\ B) = A \cup B$$

in the two-valued set and logic theory. In fuzzy set and logic theory, FDCF and FCCF provide lower and upper set formulas, respectively, for the “OR” combination of two fuzzy concepts, A and B, i.e., fuzzy predicates. That is, we get $\mathrm{FDCF}\,(A\ \text{ OR }\ B) \subseteq \mathrm{FCCF}\,(A\ \text{ OR }\ B)$.

4.3. Other Six Meta-Linguistic Expressions

The remaining six meta-linguistic expressions, i.e., 11, 12, 13, 14, 15 and 16, are treated in a slightly different manner. Let us investigate, e.g., the meta-linguistic expression 11, i.e., the biconditional, “A IF AND ONLY IF B”. Its usual target set is symbolically “$A \leftrightarrow B$”. It is clear that

$$\ell(E) = \{A \cap B, c(A) \cap c(B)\}$$

with the property $\ell(E) = \{G \mid G \in \boldsymbol{G},\ G \subset E\}$. Thus we have $L(E) = \cup\ell(E) = (A \cap B) \cup (c(A) \cap c(B))$.

Also, it is clear that

$$u(E) = \{c(A) \cup B, A \cup c(B)\},$$

with the property $u(E) = \{G \mid G \in \boldsymbol{G},\ E \subset G\}$. Thus, we have $U(E) = \cap u(E) = (c(A) \cup B) \cap (A \cup c(B))$.

Therefore the rules for this remaining six meta-linguistic expressions, i.e., 11, 12, 13, 14, 15 and 16 in Table 5, are that

$$L(E) = \cup\{G \mid G \in \boldsymbol{G},\ G \subset E\}$$

and

$$U(E) = \cap\{G \mid G \in \boldsymbol{G},\ E \subset G\}.$$

Thus, in this category we have upper and lower set formulas for the special cases of $A \leftrightarrow B$, A XOR B, A, $c(A)$, B and $c(B)$.

5. Generalization

The schema developed for the determination of upper and lower set definitions for any two sets A and B, crisp or fuzzy, can be generalized to n sets.

Suppose that there are n concepts that are represented by n predicates $A_1, \dots, A_n$, crisp or fuzzy. We can write $\boldsymbol{G}_n$ with the formation of 2^{n+1} primitives, i.e., information granules, derived from conjunction, disjunction and complementation of these n concepts as

$$G_n = \{A_1, \cap A_2 \cap \dots \cap A_n, \dots,$$
$$c(A_1) \cup c(A_2) \cup \dots \cup c(A_n)\}.$$

Then we can apply the same method developed in Sections 3 and 4 as shown in its application to two sets A and B in Section 4 to determine the upper and lower set formulas of any meta-linguistic expression made up of these n concepts.

It is clear that with $\boldsymbol{G}$ we can derive upper and lower set formulas for any meta linguistic expression, $\mathcal{MLE}$. But these formulas must be developed carefully in three categories, i.e., (a) those that are formed with the linguistic "AND" connective, (b) these that are formed with the linguistic "OR" connective, and (c) others that are more complex and are singletons. Thus the lower set formula will be $\mathrm{FDCF}\,(\mathcal{MLE})$ and the upper set formula will be $\mathrm{FCCF}\,(\mathcal{MLE})$.

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Appendices

A. Normal Form (or Canonical Form) Derivation Algorithm

(a) First, assign truth values $T(\cdot)$, $F(\cdot)$ to the meta-linguistic values (labels, variables) A and B and then assign truth values $T(\cdot)$, $F(\cdot)$ to the meta-linguistic expression of concern, say "A AND B", in order to define its meaning as shown in Table 4.

(b) Next, construct primary conjunctions of the set symbols A, B, corresponding to linguistic values such that in a given row we have

  (i) If a "$T(\cdot)$" appears, then take the set affirmation symbol of that meta-linguistic variable;

  (ii) If an "$F(\cdot)$" appears, then take the set complementation symbol of that meta-linguistic variable;

  (iii) Finally, conjunct the two symbols.

For example, in the second row of Table 4, we have a $T(A)$ under A and an $F(B)$ under B. Therefore, we get $A \cap c(B)$ as the primary conjunction corresponding to the second row entry of Table 4.

(c) Construct the disjunctive normal form of the meta-linguistic expression of concern:

  (i) First, take the conjunctions corresponding to the $T(\cdot)$s of the truth assignment made under the column of the meta-linguistic expression, such as "A AND B" (Table 4);

  (ii) Then combine these conjunctions with disjunctions.

Thus one gets (see Table 4) DNF (or FDCF) $(A\ \text{AND}\ B) = A \cap B$. Note that in this case there is only one conjunctive term in DNF (or FDCF.)

(d) Next, construct the conjunctive normal form of the meta-linguistic expression of concern:

(i) First, take the conjunctions corresponding to $F(\cdot)$s of the truth assignment made under the column of the meta-linguistic expression, such as "A AND B" (Table 4);

(ii) Then combine these conjunctions with disjunctions;

(iii) Next, take the complement of these disjuncted conjunctions.

Thus one gets (cf. Table 4)

$$\begin{aligned}&\mathrm{CNF}\ (\text{or FCCF})(A \ \text{ AND } \ B)\\ &\quad= c\big\{\big(A \cap c(B)\big) \cup \big(c(A) \cap c(B)\big)\big\}\\ &\quad= \big(c(A) \cup B\big) \cap \big(A \cup c(B) \cap (A \cup B).\end{aligned}$$

B. Inclusion for Continuous Archimedean t-Norms

Theorem 1. *If $\langle \Delta, \nabla, n\rangle$ is a strong De Morgan triple, then $FDCD(\cdot) \subseteq FCCF(\cdot)$ for the 16 combinations of concepts, i.e., basic protoforms for CWW.*

Theorem 2. *If $\langle \Delta, \nabla, n\rangle$ is a strict De Morgan triple, then $FDCD(\cdot) \subseteq FCCF(\cdot)$ for all 16 combinations of concepts, i.e., basic protoforms for CWW.*

Generating Functions

For Theorem 1, we apply

$$\begin{aligned}\Delta(a,b) &= g^{-1}\left(\min\left\{g(a)+g(b), g(0)\right\}\right),\\ \nabla(a,b) &= g^{-1}\left(\max\left\{g(a)+g(b)-g(0), 0\right\}\right),\\ n(a) &= g^{-1}\big(g(0)-g(a)\big).\end{aligned}$$

For Theorem 2, we apply

$$\begin{aligned}\Delta(a,b) &= \phi^{-1}\big(\phi(a)\phi(b)\big),\\ \nabla(a,b) &= \phi^{-1}\big(\phi(a)+\phi(b)-\phi(a)\phi(b)\big),\\ n(a) &= \phi^{-1}\big(1-\phi(a)\big).\end{aligned}$$

Here $g(\cdot)$ is a continuous and strictly decreasing function such that

$$g\colon\ [0,1] \to \mathbb{R}^{+}$$

with $g(1) = 0$ and $g(0) < +\infty$, and $\phi^{-}(\cdot)$ is the automorphism of the unit interval.

Proof.

$$\nabla\big[\Delta(a,b), \Delta\big(a, n(b)\big), \Delta\big(n(a), b\big)\big] \le \nabla(a,b),$$

$$\nabla\big[\Delta(a,b), \Delta\big(n(a), n(b)\big)\big] \le \Delta\big[\nabla\big(a, n(b)\big), \nabla\big(n(a), b\big)\big],$$

$$\nabla\left[\Delta(a,b), \Delta\big(a, n(b)\big)\right] \le \Delta\left[\nabla(a,b), \nabla\big(a, n(b)\big)\right].$$

It is sufficient to show that one of the following dependences holds:

$$\begin{aligned}\nabla\left[\Delta(a,b), \Delta\big(a, n(b)\big)\right] &\le a,\\ \nabla\left[\Delta(a,b), \Delta\big(n(a), b\big)\right] &\le a,\\ \nabla\left[\Delta\big(a, n(b)\big), \Delta\big(n(a), b\big)\right] &\le a.\end{aligned}$$

Thus, for example, one shows that

$$S1\colon\ \nabla\left[\Delta(a,b), \Delta\big(a, n(b)\big)\right] \le a,$$

$$S2\colon\ \nabla\big[\Delta(a,b), \Delta\big(a, n(b)\big)\big] \le \Delta\big[\nabla(a,b), \nabla\big(a, n(b)\big)\big],$$

$$\begin{aligned}S3\colon\ &\nabla\left[\Delta(a,b), \Delta\big(n(a), n(b)\big)\right]\\ &\qquad \le \Delta\left[\nabla\big(a, n(b)\big), \nabla\big(n(a), b\big)\right].\end{aligned}$$