DESIGN OF LINEAR FEEDBACK FOR BILINEAR CONTROL SYSTEMS

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Sufficient conditions for the conditional stability of trivial solutions for quadratic systems of ordinary differential equations are obtained. These conditions are then used to design linear control laws on the output for a bilinear system of any order. In the case of a homogeneous system, a domain of the conditional stability is also indicated (it is a cone). Some examples are given.

Keywords: system of ordinary quadratic differential equations, bilinear control system, linear control law, cone of stability, feedback, closed-loop system

1. Introduction

Consider a quadratic control system the state equation of which is

$$\dot{\mathbf{x}}(t)=\Big(\mathbf{A}_0+\sum_{i=1}^{n}x_i(t)\mathbf{A}_i+\sum_{i=n+1}^{m+n}u_{i-n}(t)\mathbf{A}_i\Big)\mathbf{x}(t), \quad (1)$$

where

$$\mathbf{x}(t) = \big(x_1(t),\dots,x_n(t)\big)^T \in \mathbb{R}^n,$$

$$\mathbf{u}(t) = \big(u_1(t),\dots,u_m(t)\big)^T \in \mathbb{R}^m,$$

and the observation equation has the form

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t), \quad \mathbf{y}(t) = (y_1(t),\dots,y_p(t))^T \in \mathbb{R}^p,$$

$$\mathbf{y}(0) = (y_{10},\dots,y_{p0})^T. \quad (2)$$

Here $\mathbb{R}^n,\mathbb{R}^m,\mathbb{R}^p$ are real vector spaces of column vectors, $\mathbf{x}(t),\mathbf{u}(t),\mathbf{y}(t)$ are vectors of states, inputs and outputs, respectively, $\mathbf{y}(0)$ is a vector of initial values, $\mathbf{A}_i:\mathbb{R}^n\to\mathbb{R}^n$ and $\mathbf{C}:\mathbb{R}^n\to\mathbb{R}^p$ are real linear mappings of appropriate real spaces, $i=0,\dots,n+m$. (If $\mathbf{A}_i=0,\ \forall\, i\in\{1,\dots,n\}$, then the system (1) is called a bilinear control system.)

In what follows, we shall continue to study the problem, the research on which was started earlier. Therefore, for the reader's convenience, we shall recall some results from the paper (Belozyorov, 2001).

Definition 1. If $\mathbf{A}_0=0$, then the system (1) is called *homogeneous*. Otherwise, it is called *non-homogeneous*.

Fixing bases in spaces $\mathbb{R}^n$ and $\mathbb{R}^p$, we denote the matrices of operators $\mathbf{A}_i$ and $\mathbf{C}$ in the selected bases as A_i and $C=(c_1,\dots,c_n)$, respectively. Here $c_1,\dots,c_n$ are columns of the matrix $\mathbf{C}$; $i=0,\dots,n+m$. For arbitrary column vectors $\mathbf{a}$ and $\mathbf{b}$, we denote by $(\mathbf{a},\mathbf{b})$ their scalar product; besides, we denote by $\|\mathbf{x}\| = \sqrt{(\mathbf{x},\mathbf{x})}$ the Euclidean norm of any vector $\mathbf{x}\in\mathbb{R}^n$. Let us recall the definition of the conditional stability of solutions to a system of differential equations (Demidovich, 1967).

Definition 2. The trivial solution $\mathbf{x}(t)\equiv 0$ of the system of differential equations

$$\dot{\mathbf{x}}(t) = \mathbf{F}\big(t,\mathbf{x}(t)\big),$$

with the vector of initial values $\mathbf{x}(0)=(x_{10},\dots,x_{n0})^T$, where $\mathbf{F}(t,\mathbf{x}) = (F_1(t,x_1,\dots,x_n),\dots,F_n(t,x_1,\dots,x_n))^T\in\mathbb{R}^n$ is a vector function, is called *conditionally stable* if there exists a variety of initial values $\mathbf{\Theta}\subset\mathbb{R}^n$ such that for any solution $\mathbf{x}(t)$ satisfying the conditions

$$\mathbf{x}(0)\in\mathbf{\Theta} \text{ and } \|\mathbf{x}(0)\|<\delta(\epsilon),$$

the inequality

$$\|\mathbf{x}(t)\|<\epsilon$$

is satisfied for $t>0$. If also

$$\lim_{t\to\infty}\|\mathbf{x}(t)\|=0,$$

then the solution $\mathbf{x}(t)\equiv 0$ is called *conditionally asymptotically stable*. (Here ϵ and δ are positive numbers, where ϵ is given and $\delta=\delta(\epsilon)$ is a function of ϵ.)

In what follows, the structure of a variety $\mathbf{\Theta}$ is not investigated. Note only that we shall deal with varieties of two types: it will be either an open sphere or an open cone with its top at the origin.

Now for the system (1), (2) let us formulate the following problem of mathematical control theory.

Problem of the synthesis of a static feedback law: Construct a matrix $\mathbf{K} = (k_1^T, \dots, k_m^T)^T \in \mathbb{R}^{m\times p}$ of a linear control law $\mathbf{u}(t) = \mathbf{K}\mathbf{y}(t)$, where $k_1, \dots, k_m$ are row vectors, such that the trivial solution of the closed-loop system

$$\dot{\mathbf{x}}(t) = \Bigg(\mathbf{A}_0 + \sum_{i=1}^{n} x_i(t)\mathbf{A}_i + \sum_{i=1}^{n}\sum_{j=n+1}^{n+m} x_i(t)(k_{j-n}, c_i)\mathbf{A}_j\Bigg)\mathbf{x}(t), \quad (3)$$

with the vector of initial values $\mathbf{x}_0 = \{x_{10}, \dots, x_{n0}\} \in \mathbf{\Theta}$ such that $\mathbf{y}_0(t) = \mathbf{C}\mathbf{x}_0$, would be asymptotically stable (at least conditionally).

Now, two practical examples of bilinear systems are given.

Control problem by the nuclear reactor on thermal neutrons: The kinetic equations of such a reactor can be presented in the following form (Bowen and Masters,1959):

$$\begin{cases} \dfrac{\mathrm{d}r_1}{\mathrm{d}t} = k_2\beta_1\dfrac{N}{l} - \lambda_1 r_1, \\ \quad\vdots \\ \dfrac{\mathrm{d}r_6}{\mathrm{d}t} = k_2\beta_6\dfrac{N}{l} - \lambda_6 r_6, \\ \dfrac{\mathrm{d}N}{\mathrm{d}t} = k_1\dfrac{N}{l} - k_2\dfrac{N}{l}\displaystyle\sum_{i=1}^{6}\beta_i + \sum_{i=1}^{6}\lambda_i r_i. \end{cases} \quad (4)$$

Here N is the density of neutrons, λ_i is the disintegration constant for the nuclei of group i (there exist six such groups), r_i is the density of the nuclei of group i, l is the average effective time of the life of neutrons, β_i is part of lagging neutrons originating from a nucleus of group i, k_1 is the excess reproduction coefficient, characterizing the affixed perturbation, and k_2 is the effective reproduction coefficient.

Usually, it is considered that coefficients k_1 and k_2 are linear functions of the movements of graphite rods in the reactor, which play the role of controls. In other words, $k_1 = b_1v_1 + \dots + b_sv_s$, $k_2 = d_1v_1 + \dots + d_sv_s$, where s is the number of rods in the reactor, v_i is the magnitude of the movement of the i-th rod, b_i and d_i are some numerical coefficients, $i = 1, \dots, s$.

With the help of controls $v_1, \dots, v_s$ it is required to stabilize the work of the reactor in a neighbourhood of some nominal values of variables $N_0, r_{10}, \dots, r_{60}$.

Introduce the notation $\beta = \sum_{i=1}^{6}\beta_i, k_1 = u_1, k_2 = u_2, N = x_7, r_i = x_i, i = 1, \dots, 6$. Then we will obtain the system (1), in which $n = 7, m = 2$ and

$$A_0 = \begin{pmatrix} -\lambda_1 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & -\lambda_6 & 0 \\ \lambda_1 & \cdots & \lambda_6 & 0 \end{pmatrix},$$

$$A_1 = \begin{pmatrix} 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & 1/l \end{pmatrix},$$

$$A_2 = \begin{pmatrix} 0 & \cdots & 0 & -\beta_1/l \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & -\beta_6/l \\ 0 & \cdots & 0 & -\beta/l \end{pmatrix}.$$

Problem of a navigation officer: Any space curve γ in a fixed coordinate system *OXYZ* can be given by means of the variable radius vector $\mathbf{r} = \mathbf{r}(s)$, where s is the magnitude of the movement along the curve from the origin. (The representation of the radius vector in the form $\mathbf{r} = \mathbf{r}(s)$ is called the natural parametrization of the curve γ.) Let $P \in \gamma$ be any point on this curve with radius vector $\mathbf{r}$. Let us denote by $\mathbf{n}, \mathbf{t}$ and $\mathbf{b}$ the unit vectors which are normal, tangent and binormal to curve γ, outgoing from the point P and having the same orientation as coordinate axes X, Y, Z, respectively. These vectors satisfy the differential equations

$$\frac{\mathrm{d}\mathbf{t}}{\mathrm{d}s} = k\mathbf{n}, \quad \frac{\mathrm{d}\mathbf{n}}{\mathrm{d}s} = -k\mathbf{t} - \tau\mathbf{b}, \quad \frac{\mathrm{d}\mathbf{b}}{\mathrm{d}s} = \tau\mathbf{n},$$

which are known as Frenet's formulae. Here k is the curvature of the curve γ at the point P, and τ is the torsion of the curve γ at the point P.

Let us look at the point P as at some flight vehicle, whose barycentre is located at the point P and the control is realized in the plane $(\mathbf{t}, \mathbf{b})$ (the pitch) and in the plane $(\mathbf{n}, \mathbf{t})$ (the yaw). Set $k = u_1$, $\tau = u_2$, $\mathbf{x}_1 = \mathbf{n}$, $\mathbf{x}_2 = \mathbf{t}$, $\mathbf{x}_3 = \mathbf{b}$ and $\mathbf{x} = (\mathbf{x}_1^T, \mathbf{x}_2^T, \mathbf{x}_3^T)^T$. Then Frenet's equations will turn into the bilinear system (1), for which $n = 9$, $m = 2$ and

$$A_0 = 0, \quad A_1 = \begin{pmatrix} 0 & -I_3 & 0 \\ I_3 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

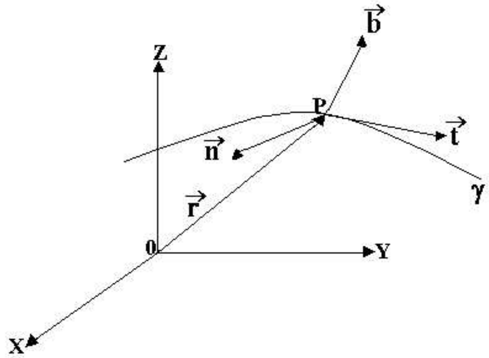

Fig. 1. Coordinate axes for the problem of the navigation officer.

$$A_2 = \begin{pmatrix} 0 & 0 & -I_3 \\ 0 & 0 & 0 \\ I_3 & 0 & 0 \end{pmatrix}.$$

(Here I_3 is the identity matrix of the third order.)

At any time moment the orientation of axes $\mathbf{n}$, $\mathbf{t}$ and $\mathbf{b}$ at the point P is assumed to be known: $\mathbf{n} = \mathbf{x}_1(s)$, $\mathbf{t} = \mathbf{x}_2(s)$ and $\mathbf{b} = \mathbf{x}_3(s)$. (Here $s = s(t)$ is a known function of time.) It is necessary to stabilize the motion of a flight vehicle via linear feedback in a neighbourhood of the nominal values $\mathbf{x}_{i0}$ of vectors $\mathbf{x}_i(s)$, $i = 1, 2, 3$.

2. Some Generic Properties of Solutions of Homogeneous Quadratic Systems

It is obvious that the system of differential equations (3) can be rewritten as follows:

$$\begin{cases} \dot{x}_1(t) \;=\; \displaystyle\sum_{j=1}^{n} d_{1j} x_j(t) + \mathbf{x}^T(t) B_1 \mathbf{x}(t), \\ \qquad\vdots \\ \dot{x}_n(t) \;=\; \displaystyle\sum_{j=1}^{n} d_{nj} x_j(t) + \mathbf{x}^T(t) B_n \mathbf{x}(t). \end{cases} \tag{5}$$

Here $D = (d_{ij}), B_1, \dots, B_n \in \mathbb{R}^{n\times n}$ are real matrixes and $B_1, \dots, B_n$ are also symmetric.

Definition 3. The system of equations (5) is called *quadratic*; if $D = 0$, then we call (5) *homogeneous quadratic*.

Definition 4. The homogeneous quadratic system (5) is called *regular* if there are no real constants $\tau_1, \dots, \tau_n$ (at least one being non-zero) such that $\forall\, \mathbf{x} \in \mathbb{R}^n$ $\mathbf{x}^T(\tau_1 B_1 + \cdots + \tau_n B_n)\mathbf{x} = 0$. Otherwise, (5) is called a *singular* or a *special* system.

In this section we will study regular homogeneous quadratic systems of order n:

$$\begin{cases} \dot{x}_1(t) \;=\; \mathbf{x}^T(t) B_1 \mathbf{x}(t), \\ \qquad\vdots \\ \dot{x}_n(t) \;=\; \mathbf{x}^T(t) B_n \mathbf{x}(t), \end{cases} \tag{6}$$

with the vector of initial values $\mathbf{x}^T(0) = (x_{10}, \dots, x_{n0})$.

Consider the matrix $\rho_1 B_1 + \cdots + \rho_n B_n \in \mathbb{R}^{n\times n}$, where $\rho_1, \dots, \rho_n$ are arbitrary real parameters. Introduce basic symmetric functions for this matrix (Gantmacher, 1990): $\sigma_1(\rho_1, \dots, \rho_n) = \mathrm{tr}\,(\rho_1 B_1 + \cdots + \rho_n B_n) = \{$it is the sum of all principal minors of the first order$\}$, $\sigma_2(\rho_1, \dots, \rho_n) = \{$it is the sum of all principal minors of the second order$\}, \dots, \sigma_n(\rho_1, \dots, \rho_n) = \det(\rho_1 B_1 + \cdots + \rho_n B_n)$.

Consider the set of equations

$$\sigma_1(\rho_1, \dots, \rho_n) = r, \quad \sigma_2(\rho_1, \dots, \rho_n) = 0, \\ \dots, \quad \sigma_n(\rho_1, \dots, \rho_n) = 0, \tag{7}$$

with respect to the unknowns $\rho_1, \dots, \rho_n$ and a known arbitrary non-zero constant $r \in \mathbb{R}$.

It is easy to show (Gantmacher, 1990) that, for generic matrices $B_1, \dots, B_n$, the system (7) has n linearly independent solutions

$$\mathbf{f_1} = (\rho_{11}, \rho_{12}, \dots, \rho_{1n}), \quad \mathbf{f_2} = (\rho_{21}, \rho_{22}, \dots, \rho_{2n}), \\ \dots, \quad \mathbf{f_n} = (\rho_{n1}, \rho_{n2}, \dots, \rho_{nn})$$

(generally speaking, they are complex).

Let us find these solutions and form the non-singular matrix $F^{-1} = (\mathbf{f_1}^T, \dots, \mathbf{f_n}^T)^T \in \mathbb{C}^{\,n\times n}$, and then introduce into (6) the new variable $\mathbf{v}(t) = (v_1(t), \dots, v_n(t))^T \in \mathbf{C}^n$ using the formula $\mathbf{x}(t) = F\mathbf{v}(t)$. Then, as shown by Belozyorov (2001), the system (6) can be presented as

$$\begin{pmatrix} \dot{v}_1(t) \\ \vdots \\ \dot{v}_n(t) \end{pmatrix} = \begin{pmatrix} (p_{11}v_1 + \cdots + p_{1n}v_n)^2 \\ \dots\dots\dots\dots\dots\dots\dots \\ (p_{n1}v_1 + \cdots + p_{nn}v_n)^2 \end{pmatrix},$$

where p_{ij} are complex numbers; $\mathbf{v}(0) = F^{-1}\mathbf{x}(0) = (v_{10}, \dots, v_{n0})^T$.

After the change of variables $\mathbf{v}(t) = P\mathbf{z}(t)$, where $P = (p_{ij}) \in \mathbf{C}^{n\times n}$, the last system takes the form

$$\begin{pmatrix} \dot{z}_1(t) \\ \dot{z}_2(t) \\ \vdots \\ \dot{z}_n(t) \end{pmatrix} = \begin{pmatrix} -\beta_{11} & \beta_{12} & \dots & \beta_{1n} \\ \beta_{21} & -\beta_{22} & \dots & \beta_{2n} \\ \dots & \dots & \dots & \dots \\ \beta_{n1} & \beta_{n2} & \dots & -\beta_{nn} \end{pmatrix} \begin{pmatrix} z_1^2(t) \\ z_2^2(t) \\ \vdots \\ z_n^2(t) \end{pmatrix}, \tag{8}$$

where β_{ij} are complex numbers, $\mathbf{z}(0) = (FP)^{-1}\mathbf{x}(0) = (z_{10}, \dots, z_{n0})^T$.

Definition 5. The regular system (6) is called a *system without invariant submanifolds* (WIS system) if there exists no non-singular transformation $S \in \mathbb{C}^{n\times n}$ such that, after the replacement $\mathbf{x} = S\mathbf{w}$, where $\mathbf{w} = (w_1,\dots,w_n)^T$, the system (6) takes the particular form (8):

$$\dot{\mathbf{w}} = \begin{pmatrix} \dot w_1 \\ \vdots \\ \dot w_k \\ \dot w_{k+1} \\ \vdots \\ \dot w_n \end{pmatrix} = S^{-1}\begin{pmatrix} \mathbf{w}^T S^T B_1 S\mathbf{w} \\ \dots\dots\dots\dots \\ \mathbf{w}^T S^T B_n S\mathbf{w} \end{pmatrix}$$

$$= \begin{pmatrix} \sum\limits_{j=1}^{n} \alpha_{1j} w_j^2 \\ \vdots \\ \sum\limits_{j=1}^{n} \alpha_{kj} w_j^2 \\ \sum\limits_{j=k+1}^{n} \alpha_{k+1,j} w_j^2 \\ \vdots \\ \sum\limits_{j=k+1}^{n} \alpha_{nj} w_j^2 \end{pmatrix}.$$

Here $\alpha_{11},\dots,\alpha_{nn}$ are complex numbers.

Denote by Ψ the set of all homogeneous quadratic systems of order n. In Appendix it will be shown that the set of all WIS systems contains open subset everywhere dense in Ψ. Thus, the systems (6) being WIS systems are generic.

Denote by $a_1, a_2, \dots, a_k$ all real singular points such as the pole of some solution of the regular WIS system (8). Let d_i be the multiplicity of the point a_i, $i = 1,\dots,k$.

Theorem 1. *Let (8) be a regular WIS system. Then all real singular points of any solution of such a system coincide with $a_1, a_2, \dots, a_k$, where the multiplicity of the point a_i is $d_i, i = 1,\dots,k$.*

Proof. For simplicity, assume that $n = 2$ and all poles of the solution $z_1(t)$ are equal to $a_1,\dots,a_l$, and all poles of the solution $z_2(t)$ coincide with points $a_{l+1},\dots,a_k$.

Also assume that $z_1(t) = f_1(t)/(t-a_1)^d$ is the pole of multiplicity d, $f_1(a_1) \neq 0$ and the point a_1 is not a pole of $z_2(t)$. Then, as $t \to a_1$, the second equation of the system (8) can be rewritten as $\lim_{t\to a_1}(t-a_1)^{2d}\dot z_2(t) = \beta_{21} f_1^2(a_1) - \beta_{22}\lim_{t\to a_1}(t-a_1)^{2d} z_2^2(t)$. Since (8) is a WIS system, we have $\beta_{21} \neq 0$. Therefore from the last relation it follows that either $f_1(a_1) = 0$ or $\lim_{t\to a_1}(t-a_1)^{2d}(\dot z_2(t) + \beta_{22} z_2^2(t)) = \mathrm{const} \neq 0$. The first expression contradicts the assumption and the second is equivalent to the relation $\dot z_2(t) + \beta_{22} z_2^2(t) = g(t)/(t-a_1)^{2d}$, where $g(a_1) \neq 0$. From this it follows that $z_2(t) = g_1(t)/(t-a_1)^d$, where again $g_1(a_1) \neq 0$.

Repeating the same process for points $a_2,\dots,a_l$, we can prove that the points are poles corresponding to the ordinals of the function $z_2(t)$. It is obvious that a similar statement holds true for function $z_1(t)$, with poles at the points $a_{l+1},\dots,a_k$.

Generally, let a be the pole of solutions $z_t,\dots, z_{n-1}(t)$. Then $z_i(t) = f_i(t)/(t-a)^d$, where $f_i(a) \neq 0$, $i = 1,\dots,n-1$. Substituting these relations into the last equation of the system (8) and passing to the limit as $t \to \infty$, we obtain $\lim_{t\to a}(t-a)^{2d}(\dot z_n(t) + \beta_{nn} z_n^2(t)) = \mathrm{const} \neq 0$. The general case of $n \neq 2$ can be considered in much the same way. The proof is thus completed. ■

Rewrite the system of equations (8) in the following form:

$$\dot{\mathbf{x}}(t) = BX(t)\mathbf{x}(t), \tag{9}$$

where

$$B = \begin{pmatrix} -\beta_{11} & \beta_{12} & \dots & \beta_{1n} \\ \beta_{21} & -\beta_{22} & \dots & \beta_{2n} \\ \dots & \dots & \dots & \dots \\ \beta_{n1} & \beta_{n2} & \dots & -\beta_{nn} \end{pmatrix},$$

$$X(t) = \begin{pmatrix} x_1(t) & 0 & \dots & 0 \\ 0 & x_2(t) & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & x_n(t) \end{pmatrix}.$$

Estimate the solution to the system (9), using the Taylor expansion. In the sequel, in order to denote the time derivative, symbols '$\cdot$' or '$\prime$' will be used.

Note that $\forall\ k \in \mathbb{Z}^+$, $\|X^{(k)}\| = \|\mathbf{x}^{(k)}\|$. Then from (9) we have

$$\|\mathbf{x}'\| \le \|B\|\|\mathbf{x}\|^2,$$

$$\begin{aligned} \mathbf{x}'' &= (\dot{\overline{BX}})\mathbf{x} + (BX)\dot{\mathbf{x}} = B\dot X\mathbf{x} + (BX)(BX)\mathbf{x} \\ &= BX\dot{\mathbf{x}} + (BX)^2\mathbf{x} = 2!(BX)^2\mathbf{x}, \end{aligned}$$

$$\|\mathbf{x}''\| \le 2!\|B\|^2\|\mathbf{x}\|^3,$$

$$\mathbf{x}''' = 2(B\dot X)(BX)\mathbf{x} + 2(BX)(B\dot X)\mathbf{x} + 2(BX)^2\dot{\mathbf{x}},$$

$$\|\mathbf{x}'''\| \le 3!\|B\|^3\|\mathbf{x}\|^4, \quad \dots.$$

It is obvious that for arbitrary $k \in \mathbb{Z}^+$ we have

$$\|\mathbf{x}^{(k)}\| \le k!\|B\|^k\|\mathbf{x}\|^{k+1}.$$

Represent formally the function $\|\mathbf{x}(t)\|$ as a Taylor series and estimate it:

$$\begin{aligned}\|\mathbf{x}(t)\| &\le \|\mathbf{x}(t_0)\| + \|B\|\|\mathbf{x}(t_0)\|^2(t-t_0) + \cdots \\ &\quad + \|B\|^k\|\mathbf{x}(t_0)\|^{k+1}(t-t_0)^k + \cdots \\ &= (1 + \|B\|\|\mathbf{x}(t_0)\|(t-t_0) + \cdots \\ &\quad + \|B\|^k\|\mathbf{x}(t_0)\|^k(t-t_0)^k + \cdots)\|\mathbf{x}(t_0)\|. \quad (10)\end{aligned}$$

As is well known, the series (10) converges for all t satisfying the condition $\|B\|\|\mathbf{x}(t_0)(t-t_0)\| < 1$ or for any $t \in [t_0, t_0 + (\|B\|\|\mathbf{x}(t_0)\|)^{-1})$. If the last restriction is satisfied, then the series on the right-hand side of (10) converges; this sum is calculated using the formula for the geometric series and the estimate (10) takes the form

$$\|\mathbf{x}(t)\| \le \frac{\|\mathbf{x}(t_0)\|}{1 - \|B\|\|\mathbf{x}(t_0)\|t}. \quad (11)$$

Theorem 2. *Let (6) be a regular WIS system. Then one of the following statements holds true: (a)* $\forall k \in \{1,\dots,n\}$ $\lim_{t\to\infty} x_k(t) = 0$*, (b) for all* $k \in \{1,\dots,n\}$ $\lim_{t\to a} |x_k(t)| = \infty$*, where* a *is some positive pole.*

Proof. (a) Assume that for any $k \in \{1,\dots,n\}$ we have $\lim_{t\to\infty} x_k(t) = c_k$, where at least one $c_k \ne 0$. Then the system (6) can be rewritten as

$$\begin{cases} \mathbf{x}^T(\infty)B_1\mathbf{x}(\infty) = 0, \\ \qquad\vdots \\ \mathbf{x}^T(\infty)B_n\mathbf{x}(\infty) = 0. \end{cases}$$

It is known (Fulton, 1984) that a system of equations which consists of linearly independent forms has only a trivial solution. Therefore we should have $c_1 = \cdots = c_k = 0$, which proves the first statement of Theorem 2.

(b) Again, for simplicity, assume that $n = 2$ and let $\lim_{t\to a} x_1(t) = \infty$ and $\lim_{t\to a} x_2(t) = c_2 = \text{const}$. Since (6) is a WIS system, the second equation takes the form $0 = b_{11}x_1^2(a) + b_{12}x_1(a)x_2(a) + b_{22}x_2^2(a)$ as $t \to \infty$, where $b_{11} \ne 0$ or $b_{12} \ne 0$. In this case $x_1(a)/x_2(a)$ is a finite non-zero number. From the condition $\lim_{t\to a} x_2(t) = c_2 \ne \infty$ it follows that $\lim\limits_{t\to a} x_1(t) = c_1 \ne \infty$ holds true. The last relation contradicts the assumption of the second part of Theorem 1.

Generally, the proof proceeds as follows. According to Theorem 1, any solution (6) can be represented as

$$x_i(t) = \frac{f_i(t)}{(t-a_1)^{d_1}\cdots(t-a_k)^{d_k}},$$

where poles $a_1,\dots,a_k$ and their multiplicities $d_1,\dots,d_k$ are the same for all solutions. These poles depend on an initial vector $\mathbf{x}_0$ (the so-called moving poles).

It is obvious that if some equation $t - a_i(\mathbf{x}_0) = 0$ has a solution in the interval $[0,\infty)$, then $\forall j$, $\lim_{t\to\infty} |x_j(t)| = \infty$. If none of these solutions belongs to the indicated interval, then from the formula (11) it follows that $\lim_{t\to\infty} x_k(t) = 0$. ■

In what follows, for an arbitrary non-negative integer k we write $\mathbf{x}(t_k) = \mathbf{x}_k, X(t_k) = X_k$. Construct a formal expansion of some vector function $\mathbf{v}(t)$ in the Taylor series in a neighbourhood of the time point t_k:

$$\begin{aligned}\mathbf{v}(t) &= \big[E + (BX_k)(t-t_k) \\ &\quad + \cdots + (BX_k)^n(t-t_k)^n + \cdots\big]\mathbf{x}_k. \quad (12)\end{aligned}$$

Assume now that $k = 1$. Then $\mathbf{x}(t_1) = \mathbf{x}_1$, where t_1 is selected taking account of the unique restriction

$$t_0 \le t_1 < t_0 + \frac{1}{\|BX_0\|}.$$

It is obvious that in this case the series (12) converges for all t satisfying the condition

$$t_1 \le t < t_1 + \frac{1}{\|BX_1\|}.$$

If we continue this procedure further, then, finally, we derive that $\forall\ k \in \mathbb{Z}^+$, the series (12) converges $\forall t \in [t_k, t_k + (\|BX_k\|)^{-1})$, and the next value t_{k+1} is selected from the range

$$t_k \le t_{k+1} < t_k + \frac{1}{\|BX_k\|}.$$

In the case of the convergence of the series (12), the sum of the series is computed using the following well-known formula from functional analysis:

$$\mathbf{v}(t) = \big[E - (BX_k)(t-t_k)\big]^{-1}\mathbf{x}_k. \quad (13)$$

It is easy to check that in the case of the convergence for the function $\mathbf{v}(t)$, the same estimate (11) as for the function $\mathbf{x}(t)$ is correct:

$$\|\mathbf{v}(t)\| \le \frac{\|\mathbf{x}(t_0)\|}{1 - \|B\|\|\mathbf{x}(t_0)\|t}.$$

Again, we will search for the solution of the system (9) using the Taylor expansion in the vector form. For this purpose, we estimate the limit values of solutions of the system (9) at critical points $a_1,\dots,a_k$ and as $t \to \infty$. Here the following result is required.

Theorem 3. *Assume that a regular WIS system is reduced to the form (9). Let* ξ *be one from singular points* $a_1,\dots,a_k$ *or the symbol* $\pm\infty$*. Then*

$$\lim_{t\to\xi} \dot{X}(t)BX(t) = \lim_{t\to\xi} X(t)B\dot{X}(t).$$

Proof. It is obvious that the last equality is equivalent to the set of equations $\lim_{t\to\xi}(\dot{x}_i(t)x_j(t)-x_i(t)\dot{x}_j(t))=0$, $i,j=1,\dots,n$. According to Theorems 1 and 2, either $\forall i\in\{1,\dots,n\},\lim_{t\to\xi}x_i(t)=0$, or $\forall i\in\{1,\dots,n\},\lim_{t\to\xi}|x_i(t)|=\infty$. In both these cases, according to the L'Hospital rule, we have

$$\lim_{t\to\xi}\frac{\dot{x}_i(t)}{\dot{x}_j(t)}=\lim_{t\to\xi}\frac{x_i(t)}{x_j(t)}=c_{ij}\neq 0.$$

But then we have $\lim_{t\to\xi}\dot{x}_i(t)=c_{ij}\lim_{t\to\xi}\dot{x}_j(t)$, $\lim_{t\to\xi}x_i(t)=c_{ij}\lim_{t\to\xi}x_j(t)$, and the limit system $\lim_{t\to\xi}(\dot{x}_i(t)x_j(t)-x_i(t)\dot{x}_j(t))=0$, $i,j=1,\dots,n$, is satisfied. ∎

Starting from Theorem 3, we have $\lim t\to\xi\mathbf{x}''=\lim_{t\to\xi}((\dot{BX})\mathbf{x}+(BX)\dot{\mathbf{x}})=\lim_{t\to\xi}(B\dot{X}\mathbf{x}+(BX)(BX)\mathbf{x})=\lim_{t\to\xi}(BX\dot{\mathbf{x}}+(BX)^2\mathbf{x})=2!\lim_{t\to\xi}(BX)^2\mathbf{x}$; $\lim_{t\to\xi}\mathbf{x}'''=\lim_{t\to\xi}(4(BX)B\dot{X}\mathbf{x}+2(BX)^2\dot{\mathbf{x}})=\lim_{t\to\xi}(4(BX)^3\mathbf{x}+2(BX)^2(BX)\mathbf{x})=3!\lim_{t\to\xi}(BX)^3\mathbf{x},\dots$. It is obvious that for any $n\in\mathbb{Z}^+$ we have $\lim_{t\to\xi}\mathbf{x}^{(n)}=n!\lim_{t\to\xi}(BX)^n\mathbf{x}$.

It is clear that the formal expansion of the solution $\mathbf{x}(t)$ in the Talor series, in a neighbourhood of the point $t_k=\xi$, has the form (12) and the convergence of this series is guaranteed by the above-mentioned conditions.

Thus we have $\lim_{t\to\xi}\|\mathbf{v}(t)-\mathbf{x}(t)\|=0$ and the function $\mathbf{x}(t)$ is asymptotically equivalent to the function $\mathbf{v}(t)$ (Demidovich, 1967). Therefore it is possible to study the behaviour of $\mathbf{x}(t)$ for $t\to\xi$ via the function $\mathbf{v}(t)$.

From the above deliberations one can conclude that $0\le t_0<t_1<\dots<t_m<\cdots$ and hence the sequence $\{t_m,m=0,1,\dots\}$ is monotonically increasing. Therefore there exists a (finite or infinite) limit $\lim_{t\to\xi}t_m=t_s$ of this sequence. It is obvious that if $t_s=a_i$ for some $i\in\{1,\dots,k\}$, then t_s is a singular point of the solution of the system (9), so that $\lim_{t\to t_s}\|\mathbf{x}(t)\|=\infty$. Otherwise, if $t_s=\infty$, then $\lim\limits_{t\to t_s}\|\mathbf{x}(t)\|=0$. Indeed, it is possible to show that the values of the function $\mathbf{v}(t_m)$ at the point t_m can be calculated using the formula

$$\mathbf{v}(t_m)=\prod_{i=1}^{m}\left[E-(BX_{i-1})(t_i-t_{i-1})\right]^{-1}\mathbf{x}(0).$$

Then, from the definition of the inverse matrix, it follows that the degree of the numerator in (13) with respect to the variable t is less than the degree of the denominator. It also reduces to the last limit.

Theorem 4. *Let $B\in\mathbb{R}^{n\times n}$ and let all the coordinates of the vector of the initial data $\mathbf{x}_0=(x_{10},\dots,x_{n0})^T$ be positive. Then for the conditional asymptotic stability of the system (9) it is sufficient that for $\forall\,\lambda\ge 0$ all the elements of the inverse matrix $[E-(BX_0)(\lambda)]^{-1}$ be nonnegative.*

Corollary 1. *For the conditional asymptotic stability of the system (9) it is necessary that the polynomial $f(\lambda)=\det(E-(BX_0)\lambda)$ have only negative real roots.*

Proof. Let us investigate the behaviour of solutions to the system (9) as $t\to\infty$. So, form positive differences $\Delta t_k=t_{k+1}-t_k,k=0,1,\dots$. Then the formula (13) shows that each term of the sequence $\mathbf{x}_k$ is a rational function with the denominator

$$f(\lambda)=1+(\beta_{11}x_{k1}+\dots+\beta_{nn}x_{kn})\Delta t_k+\dots+(-1)^n(\det B)(x_{k1}\cdots x_{kn})(\Delta t_k)^n.$$

It is obvious that to satisfy conditions of Theorem 4 it is sufficient that the function $f(\lambda)$ be positive simultaneously with all the cofactors of the matrix $[E-(BX_0)(\lambda)]^{-1}$. If these conditions are fulfilled, then the proof of the stability of solutions follows from Theorem 2 in (Belozyorov, 2001). The proof of the corollary is then straightforward. ∎

3. Construction of Domains of Conditional Stability for Homogeneous Quadratic Systems of the Second Order

Theorem 4 can be strengthened for $n=2$. For that purpose we take advantage of the asymptotic equivalence of functions $\mathbf{x}(t)$ and $\mathbf{v}(t)$. Let $t-t_0=\Delta t$. (Here $t_0\neq\xi$, where ξ is a singular point or symbol ∞.) Then on the interval $[t_0,\xi)$, where the magnitude $|t_0-\xi|\neq 0$ is small enough, the coordinates of the function (13) are given by the formulae

$$v_1(t)=\frac{x_{1t}+(x_{1t}x_{2t}\beta_{22}+x_{2t}^2\beta_{12})\Delta t}{1+(\beta_{11}x_{1t}+\beta_{22}x_{2t})\Delta t+(\beta_{11}\beta_{22}-\beta_{12}\beta_{21})x_{1t}x_{2t}(\Delta t)^2},\quad(14)$$

$$v_2(t)=\frac{x_{2t}+(x_{1t}x_{2t}\beta_{11}+x_{1t}^2\beta_{21})\Delta t}{1+(\beta_{11}x_{1t}+\beta_{22}x_{2t})\Delta t+(\beta_{11}\beta_{22}-\beta_{12}\beta_{21})x_{1t}x_{2t}(\Delta t)^2}.\quad(15)$$

Here x_{1t} and x_{2t} are coordinates of the solution $\mathbf{x}(t)$ at instant $t\neq\xi$.

It is obvious that from the point of view of stability, the most desirable situation is when the denominators of the functions $v_1(t)$ and $v_2(t)$ are not equal to zero. In other words, on the interval $[t_0,\xi)$ the function $f(\lambda)$ does not have real roots (or $f(\lambda)$ has only negative roots on the real axis). According to the Routh-Hurwitz criterion, the last restriction is achieved in the case of $(\beta_{11}x_{1t}+\beta_{22}x_{2t})>0$ and $(\beta_{11}\beta_{22}-\beta_{12}\beta_{21})x_{1t}x_{2t}>0$.

With no loss of generality it is possible to set $\beta_{11}=\beta_{22}=1$ in (9). (This can always be achieved via a suitable change of variables.) We introduce the notation $\beta_{12}=p$ and $\beta_{21}=q$. Then, using Corollary 1, we will obtain the following result.

Theorem 5. *Assume that in (9) we have $n = 2$ and $\beta_{11}\beta_{22} - \beta_{12}\beta_{21} = 1 - pq > 0$, $p > 0$, $q > 0$. Then any of the conditions:*

(a) $x_{10} > 0$, $x_{20} > 0$;

(b) $x_{10} > 0$, $x_{20} < 0$, $x_{20} + qx_{10} > 0$,
$$\det\begin{pmatrix} x_{10} & x_{20} \\ \dot{x}_{10} & \dot{x}_{20} \end{pmatrix} = -x_{20}(-x_{10}^2 + px_{20}^2) + x_{10}(-x_{20}^2 + qx_{10}^2) > 0;$$

(c) $x_{10} < 0$, $x_{20} > 0$, $x_{10} + px_{20} > 0$,
$$\det\begin{pmatrix} x_{10} & x_{20} \\ \dot{x}_{10} & \dot{x}_{20} \end{pmatrix} = -x_{20}(-x_{10}^2 + px_{20}^2) + x_{10}(-x_{20}^2 + qx_{10}^2) > 0$$

is sufficient for the conditional stability of (9).

Proof. If (a) is true, the proof of the stability of the system (9) is reduced to the proof given by Belozyorov (2001).

Consider a solution to (9) in a small neighbourhood δ of any singular point ξ. Assume now that $x_{1t} > 0$ and $x_{2t} < 0$ if $|t - \xi| < \delta$. Then, according to Theorem 1 of (Belozyorov, 2001), a solution to (9) is conditionally stable if for some $t^* > 0$ we have $x_2(t^*) \geq 0$. Indeed, in this case $x_1(t) > 0$ and $x_2(t) > 0$ for any $t > t^*$ and we have a situation described by condition (a). For this condition, it is obvious enough that in (15) the magnitude

$$\Delta t = t - t^* = -\frac{x_{2t}}{x_{1t}x_{2t} + qx_{1t}^2}$$

is positive. In turn, this inequality is equivalent to $x_{2t} + qx_{1t} > 0$. In addition, it is necessary for the denominator of the function (15) to be positive on the analysed interval $[t^*, \xi)$:

$$\begin{aligned} 1 &+ (\beta_{11}x_{1t} + \beta_{22}x_{2t})\Delta t \\ &+ (\beta_{11}\beta_{22} - \beta_{12}\beta_{21})x_{1t}x_{2t}(\Delta t)^2 \\ &= 1 - \frac{(x_{1t} + x_{2t})x_{2t}}{x_{1t}x_{2t} + qx_{1t}^2} \\ &\quad + (1 - pq)x_{1t}x_{2t}\frac{x_{2t}^2}{(x_{1t}x_{2t} + qx_{1t}^2)^2} > 0. \end{aligned}$$

(This guarantees that the convergence conditions of the series (12) are satisfied.)

Thus, after a transformation of the last inequality, we arrive at the system of inequalities

$$\begin{aligned} &x_{2t} + qx_{1t} > 0, \\ &-x_{2t}(-x_{1t}^2 + px_{2t}^2) + x_{1t}(-x_{2t}^2 + qx_{1t}^2) > 0. \end{aligned} \tag{16}$$

Assume that $u = x_{2t}/x_{1t}$. Then, from (16), we obtain the system

$$0 > u > -q, \quad -pu^3 - u^2 + u + q > 0.$$

At the beginning, consider equation $g(u) = -pu^3 - u^2 + u + q = 0$. According to the Descartes Theorem (Demidovich and Maron, 1966), it has one positive root. Further, for a sufficiently small negative u we have $g(u) > 0$, and if $u = -q$, we have $g(-q) = q^2(pq-1) < 0$. Thus, because $g(u)$ is a polynomial of the third degree, we come to the conclusion that there are two negative roots of this polynomial. Let us denote by $\lambda_{\max}$ (resp. $\lambda_{\min}$) the greater (resp. the smaller) of these roots.

Thus, if for some $t^* \in [t, \xi)$ inequalities (16) are fulfilled, $x_1(t^*) > 0$ and $x_2(t^*) > 0$ and therefore the singular point $t = \xi$ does not exist. Repeating similar reasoning for all singular points, including the first positive a_{1+}, we arrive at the conclusion that this point does not exist if a point t^{**} such that $x_1(t^{**}) > 0$ is found and $x_2(t^{**}) > 0$. Therefore it is possible to set $x_{1t} = x_{10}$ and $x_{2t} = x_{20}$ in (14) and (15).

Note that taking advantage of the L'Hospital rule, the equation $g(u) = 0$ can be also obtained from the limit

$$\lim_{t \to a_{1+}} \frac{\dot{x}_2(t)}{\dot{x}_1(t)} = \lim_{t \to a_{1+}} \frac{x_2(t)}{x_1(t)} = \lim_{t \to a_{1+}} \frac{qx_1^2(t) - x_2^2(t)}{-x_1^2(t) + px_2^2(t)},$$

with

$$u = \lim_{t \to a_{1+}} \frac{x_2(t)}{x_1(t)}.$$

As the equation $g(u) = 0$ has one positive root, the segment $[0, a_{1+})$ belongs to the domain of the convergence of the series (12), and for $t \in [0, a_{1+})$ we have $w = x_2(t)/x_1(t) \in (\lambda_{\max}, 0]$ and $-pw^3 - w^2 + w + q > 0$. (In particular, this is also true for $x_{20}/x_{10} \in (\lambda_{\max}, 0]$.) From the previous analysis it is clear that $\lambda_{\min} < -q < \lambda_{\max}$. This completes the proof of Case (2a) and Theorem 5 if we take into account that the proof of Case (2b) (using (14)) reduces to the same result. ∎

Denote by λ_q the maximum negative solution of $-pu^3 - u^2 + u + q = 0$ and by λ_p the maximum negative solution of $-qv^3 - v^2 + v + p = 0$. Then, from Theorem 5, it is possible to derive the following result.

Theorem 6. *Let in (9) $n = 2$, $\beta_{11}\beta_{22} - \beta_{12}\beta_{21} = 1 - pq > 0$, $p > 0$, $q > 0$. Then in the plane x_1x_2 the domain of the conditional stability Ω of (9) represents a cone, which is the geometric place of the points described by*

$$\Omega = \{x_1 - \lambda_p x_2 \geq 0\} \cap \{x_2 - \lambda_q x_1 \geq 0\}.$$

In addition, the apex angle of the cone Ω does not exceed π.

4. Domain of the Conditional Stability of Regular Homogeneous Quadratic Systems

Let

$$h_1(v_1,\dots,v_n)=0,\quad \dots,\quad h_n(v_1,\dots,v_n)=0 \quad (17)$$

be a regular system of n algebraic equations with respect to n unknowns $v_1,\dots,v_n$. (This system is called regular if its Jacobi determinant is not identically zero.) As is known from elimination theory (Fulton, 1984), using new variables $z_1 = w_1(v_1,\dots,v_n),\dots,z_n = w_n(v_1,\dots,v_n)$ and equivalent transformations of the initial system, it is possible to get one equation concerning one unknown (e.g., z_1):

$$\xi_0 z_1^k + \xi_1 z_1^{k-1} + \dots + \xi_k = 0,$$

where the coefficients $\xi_i,\ i=1,\dots,k$ are complex numbers. Thus all the remaining unknowns $z_i,\ i=2,\dots,n$ are polynomials in z_1. It is obvious that in general the number of solutions to (17) will equal k. The set Φ of all solutions to (17) is called an algebraic variety. The number k of all elements of this set is called its degree. The degree of the algebraic variety is denoted by $k = \deg_{\mathbf{C}}\mathbf{\Phi}$ (Fulton, 1984).

Consider the following system of real quadratic equations with respect to the unknown vector $\mathbf{v} = (v_1,\dots,v_n)^T$:

$$-v_1 = \mathbf{v}^T B_1 \mathbf{v},\quad \dots,\quad -v_n = \mathbf{v}^T B_n \mathbf{v}. \quad (18)$$

Let $\mathbf{W}\subset\mathbb{C}^n$ be the algebraic variety of all solutions to (18). Its degree equals $\deg_{\mathbf{C}}\mathbf{W}$.

Definition 6. The system of equations (18) is called complete if $\deg_{\mathbf{C}}\mathbf{W} = \deg_{\mathbf{C}}\mathbf{V}$, where $\mathbf{V}\subset\mathbb{C}^n$ is the variety of all solutions to (18), for which it is supposed that all $n^2(n+1)/2$ of the elements of matrices $B_1,\dots,B_n$ are not numbers but independent parameters.

Theorem 7. *Every complete system (18) has at least one nontrivial real solution.*

Proof. Assume that one from among variables $v_1,\dots,v_n$ (e.g., v_n) is not equal to zero. Then the system (18) can be represented as

$$\frac{w_1}{w_n} = \frac{\mathbf{w}^T B_1 \mathbf{w}}{\mathbf{w}^T B_n \mathbf{w}},\quad \dots,\quad \frac{w_{n-1}}{w_n} = \frac{\mathbf{w}^T B_{n-1} \mathbf{w}}{\mathbf{w}^T B_n \mathbf{w}}$$

or, equivalently, as

$$f_1(w_1,\dots,w_n)=0,\quad \dots,\quad f_{n-1}(w_1,\dots,w_n)=0, \quad (19)$$

where, by virtue of the completeness of the system (18), all forms $f_1,\dots,f_{n-1}$ are cubic with respect to n variables. Again, by virtue of completeness, numbers $\beta_i\in\mathbb{R}$ can be always found such that

$$\sum_{i=1}^{n-1}\beta_i f_i(w_1,\dots,w_n) = \gamma_1 w_1^3 + \dots + \gamma_n w_n^3 + Q(w_1,\dots,w_n). \quad (20)$$

Here the degree of any variable included in the form Q does not exceed 2. (Note that in (20) all $\gamma_i \neq 0$.) Then from (Fulton, 1984) it follows that the system (19) (and, consequently, (18)) has at least one nontrivial real solution. ■

Let, e.g., the system (6) be regular and complete. Then it is easy to check that for $n=2$ we have $\deg_{\mathbf{C}}\mathbf{W}=3$. On the other hand, if for $n = 2\mathbf{v}^T B_1\mathbf{v} = (\delta_1 v_1 + \delta_2 v_2)(\nu_1 v_1 + \nu_2 v_2)$ and $\mathbf{v}^T B_2 \mathbf{v} = (\delta_1 v_1 + \delta_2 v_2)(\xi_1 v_1 + \xi_2 v_2)$, where both forms have a common linear factor and $\delta_1,\delta_2,\xi_1,\xi_2,\nu_1,\nu_2\in\mathbb{R}$, then the system (6) is incomplete; for this case $\deg_{\mathbf{C}}\mathbf{W}=2$ and a nontrivial real solution cannot exist.

In what follows, we will need the following trivial corollary of Theorem 2 taken from (Belozyorov, 2001).

Theorem 8. *Assume that for a regular system (6) the following conditions are fulfilled:*

(a) initial values $x_{i0}\geq 0$*;*

(b) forms $(x_1,\dots,x_{i-1},0,x_{i+1},\dots,x_n)B_i(x_1,\dots,x_{i-1},0,x_{i+1},\dots,x_n)^T$ *are positive definite;*

(c) positive numbers r_i *can be found such that the form* $\mathbf{x}^T\sum_{i=1}^n r_i B_i)\mathbf{x}$ *is negative definite,* $i,j=1,\dots,n$.

Then any solution to (6) is conditionally asymptotically stable.

Assume that (6) is reduced to the form (9), where the matrix B is real.

Corollary 2. *Assume that for a regular system (9) the following conditions are fulfilled:*

(a) initial values $x_{i0}\geq 0$*;*

(b) $b_{ij}\geq 0$ *(*$i\neq j$*) and* $b_{ii}>0$*;*

(c) the elements of the matrix $(-B)^{-1}$ *are nonnegative,* $i,j=1,\dots,n$.

Then any solution to (9) is conditionally asymptotically stable.

Set $\mu_1 = w_1/w_n, \dots, \mu_{n-1} = w_{n-1}/w_n$ *and transform (19) to the form*

$$g_1(\mu_1, \dots, \mu_{n-1}) = 0, \quad \dots,$$
$$g_{n-1}(\mu_1, \dots, \mu_{n-1}) = 0. \tag{21}$$

The following result establishes one important property of the solutions to (21).

Theorem 9. *Under the conditions of Theorem 8 the system (21) has a solution* $\mu = (\mu_1, \dots, \mu_n)$ *such that* $\mu_i > 0,\ i = 1, \dots, n-1$.

Proof. From Theorem 8 it follows that in (6) we have $\lim_{t\to\infty} \mathbf{x}(t) = \mathbf{0}$. Then the system (21) is obtained from (6) via dividing all equations by the last one and passing to the limit as $t \to \infty$. Since in (6) it follows that $x_i(t) > 0,\ \forall\, t \geq 0$, it is clear that the limit relations $w_1/w_n, \dots, w_1/w_n$ being solutions to (21), also possess this property.

Assume now that $x_{n0} > 0$. Then under the conditions of Theorem 8 we have $x_n(t) > 0$. It is clear that the limit $\lim_{t\to\infty} x_i(t)/x_n(t),\ i = 1, \dots, n-1$ is a solution to (21). Therefore if $\mu_i > 0$, then $x_i(t) > 0$. ■

Theorem 10. *Under the conditions of Theorem 8, among the coordinates* $v_1, \dots, v_n$ *of the solution* $\mathbf{v}$ *to (18) there is at least one positive.*

Proof. Let as take advantage of Condition 3 of Theorem 8. It is possible to find as much collections of positive numbers $r_1, \dots, r_n$ as desired, such that the form $\mathbf{x}^T(\sum_{i=1}^n r_i B_i)\mathbf{x}$ is negative definite. In other words, there is a solution $\mathbf{v} = (v_1, \dots, v_n)^T$ to the system of inequalities

$$\begin{pmatrix} r_{11}, \dots, r_{1n} \\ \dots\dots\dots\dots \\ r_{n1}, \dots, r_{nn} \end{pmatrix} \mathbf{v} > \mathbf{0},$$

where $r_{ij} \geq 0$ and the rows of the last matrix are linearly independent for $i, j = 1, \dots, n$. Hence there is no solution $\mathbf{v} = (v_1, \dots, v_n)^T$ such that $\forall\, i \in \{1, \dots, n\}$, $v_i \leq 0$. ■

Theorem 11. *Under conditions (b) and (c) of Theorem 8 there are* $2^n - 1$ *vector solutions to (18). What is more, all these solutions are real.*

Proof. Let the coordinates of the vector of initial data $\mathbf{x}_0$ satisfy the conditions $x_{i_1 0} > 0, \dots, x_{i_r 0} > 0,\ x_{i_{r+1} 0} < 0, \dots, x_{i_n 0} < 0$. (Here $\{i_1, \dots, i_n\} \in \{1, \dots, n\}$ is a permutation and r is the number of the positive coordinates of the initial vector.)

According to Theorem 8, if $x_{j0} \geq 0$, then $x_j(t) \geq 0$. If $x_{j0} < 0$, there are two possibilities: either a moment t^* can be found such that $x_j(t^*) \geq 0$ for any $t \geq t^*$, or for any $t > 0$ we have $x_j(t) < 0$. Thus, if $x_{j0} < 0$ and the number $|x_{j0}| > 0$ is large enough, then $x_j(t) < 0$ for $t < a_1$, where a_1 is the first positive pole of $x_j(t)$.

Fix the initial data so that the magnitudes $|x_{i_{r+1}}|, \dots, |x_{i_n}|$ are large enough. Then the sign of $x_i(t)$ coincides with the sign of x_{i0} for any $t < a_1$, $i = 1, \dots, n$. (In other words, the solution to (6) does not fall beyond the limits of the appropriate orthant.)

Consider the limits

$$\lim_{t\to\xi} \frac{x_i(t)}{x_j(t)}, \quad i, j = 1, \dots, n, \quad i \neq j,$$

where the variable ξ runs through all real poles of functions $x_i(t)$ and ∞. (Since we consider only WIS systems, then all the coordinates of vector solutions to (6) have the same poles.) It is obvious that all these limits are defined by the equations of (18), and therefore they do not depend on the initial data, but only on the forms of matrices $B_1, \dots, B_n$. Then, according to Theorem 10, there are as many real vector solutions to (21) as numbers of orthants in the n-dimensional system of coordinates minus 1, i.e. $2^n - 1$. (No solution $\mathbf{v}$ to (18) exists such that $v_i < 0,\ i = 1, \dots, n$.)

It is possible to obtain the proof of this theorem for (9) in a more straightforward manner. (For $n = 2$ the proof is given in Theorem 5.)

Rewrite (18) in the form

$$(-B)^{-1} \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} = - \begin{pmatrix} v_1^2 \\ v_2^2 \\ \vdots \\ v_n^2 \end{pmatrix}.$$

Let $\mathbf{v}^* = (v_1^*, \dots, v_n^*)^T \neq 0$ be a solution to (18), for which $v_i^* \geq 0,\ i = 1, \dots, n$. (According to Theorem 9, such a solution always exists.) Introduce the variable $u = v_1/v_2$ and suppose that $v_3 = v_3^*, \dots, v_n = v_n^*$. Now we will devide the first equation of this system by the second one. Then we obtain

$$\frac{v_1^2}{v_2^2} = \frac{p_{11}v_1 + p_{12}v_2 + \dots + p_{1n}v_n}{p_{21}v_1 + p_{22}v_2 + \dots + p_{2n}v_n} = u^2 = \frac{p_{11}u + q_1}{p_{21}u + q_2},$$

where $p_{11} > 0,\ p_{21} > 0,\ q_1 > 0,\ q_2 > 0$.

The last relation can be rewritten as $g(u) = 0$, where

$$g(u) = \frac{p_{11}u + q_1}{p_{21}u + q_2} - u^2. \tag{22}$$

Let $u = 0$. Then $g(u) > 0$. Denote respectively by $u_1 = -p_{12}/p_{11} < 0$ and $u_1 = -p_{22}/p_{21} < 0$ the roots of the numerator and the denominator of the fraction which is included in (22). Assume that $u_1 < u_2 < 0$. If $u_r \in (u_1, u_2)$, we obtain $g(u_r) < 0$, and a negative solution to

$$-p_{21}u^3 - q_2u^2 + p_{11}u + q_1 = 0$$

exists in the interval $(0, u_r)$.

According to the Descartes Theorem (Demidovich and Maron, 1966), the last equation has one positive root u_1^*. It was shown above that in the interval $(0, u_r)$ there is one negative root. But by virtue of the cube of (22), in reality there exist two such roots. Thus $u_1^* = (v_1/v_2)_1^* > 0, u_2^* = (v_1/v_2)_2^* < 0, u_3^* = (v_1/v_2)_3^* < 0$.

Let, e.g., $n = 3$. Then from the last section it follows that the vector solutions $\mathbf{v} = (v_1, v_2, v_3)^T$ to (18) can be distinguished as follows:

(a) $v_3^* > 0$ and the fraction v_1/v_2 is investigated; then the coordinates of the vector solutions have signs $(+,+,+)^T$, $(+,-,+)^T$, $(-,-,+)^T$ and $(-,+,+)^T$;

(b) $v_2^* > 0$ and the fraction v_1/v_3 is investigated; then the coordinates of the vector solutions have signs $(+,+,+)^T$, $(-,+,+)^T$, $(-,+,-)^T$ and $(-,+,+)^T$;

(c) $v_1^* > 0$ and the fraction v_2/v_3 is investigated; then the coordinates of the vector solutions have signs $(+,+,+)^T$, $(+,+,-)^T$, $(+,-,+)^T$ and $(+,-,-)^T$.

From these combinations of sign coordinates of the vector solutions to (18) it follows that the number of various vectors can be only $7 = 2^3 - 1$. This completes the proof. ■

Let us now go on to the construction of stability domains of (6) for an arbitrary n. Assume that r coordinates of the initial vector are positive (then the remaining $n - r$ coordinates are negative). Additionally, we shall assume that (6) is a regular WIS system.

To investigate the system (6), we will use the iterated Euler method:

$$\mathbf{x}_{i+1} = \mathbf{x}_i + \begin{pmatrix} \mathbf{x}_i^T B_1 \mathbf{x}_i \\ \vdots \\ \mathbf{x}_i^T B_n \mathbf{x}_i \end{pmatrix} \Delta t, \quad i = 0, 1 \ldots,$$

where $\Delta t > 0$ is a sufficiently small integration step.

Definition 7. Assume that $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$ are arbitrary vectors. Then $\mathbf{a} \geq \mathbf{b}$ means that $a_i - b_i \geq 0,\ i = 1, \ldots, n$.

From Theorem 8 it follows that if $\mathbf{x}(t^*) > \mathbf{0}$ for some t^*, then the solution to (6) is conditionally stable. Therefore if for some i and a sufficiently small Δt the conditions

$$\begin{cases} x_{1,i} + \mathbf{x}_i^T B_1 \mathbf{x}_i \Delta t > 0, \\ \qquad\vdots \\ x_{n,i} + \mathbf{x}_i^T B_n \mathbf{x}_i \Delta t > 0 \end{cases}$$

are satisfied, then the solution to (6) will be conditionally stable.

Thus the investigation of stability is reduced to that of set solutions for the system of inequalities

$$\begin{cases} x_1 + \lambda \mathbf{x}^T B_1 \mathbf{x} > 0, \\ \qquad\vdots \\ x_n + \lambda \mathbf{x}^T B_n \mathbf{x} > 0 \end{cases} \tag{23}$$

in the orthant given by the relations

$$\begin{aligned} x_{i_1} > 0, \quad \ldots, \quad x_{i_r} > 0, \\ x_{i_{r+1}} < 0, \quad \ldots, \quad x_{i_n} < 0. \end{aligned} \tag{24}$$

Here $(i_1, \ldots, i_n) \in \{1, \ldots, n\}$ is a permutation and r is the number of the positive coordinates of the initial vector; λ is some positive parameter.

For simplicity, assume that $x_1 > 0$ and $x_r < 0$. Then the first and the r-th inequality of (23) can be transformed into the form

$$\begin{cases} x_1 x_r + \lambda x_r \mathbf{x}^T B_1 \mathbf{x} < 0, \\ x_r x_1 + \lambda x_1 \mathbf{x}^T B_r \mathbf{x} > 0. \end{cases}$$

Hence

$$-\lambda x_1 \mathbf{x}^T B_r \mathbf{x} < x_1 x_r < -\lambda x_r \mathbf{x}^T B_1 \mathbf{x}$$

and, as $\lambda > 0$,

$$\mathbf{x}^T (x_1 B_r - x_r B_1) \mathbf{x} > 0.$$

Using similar reasoning, if $r \neq 0$ or $r \neq n$, it is possible to get the following system of inequalities:

$$\mathbf{x}^T (x_k B_l - x_l B_k) \mathbf{x} > 0, \tag{25}$$

where $k \in (i_1, \ldots, i_r), l \in (i_{r+1}, \ldots, i_n)$, and $(i_1, \ldots, i_n)$ is a permutation of n positive integers. It is obvious that, in all, there are $r(n - r)$ inequalities (25) of which only $n - 1$ are independent.

It is known that the set of solutions to a homogeneous system of inequalities is a cone. Therefore, taking account of (24), the set of solutions to (23) constitutes a cone belonging to an appropriate orthant. Since there are 2^n orthants, $r \neq 0$ and $r \neq n$, we get $d = 2^n - 2$ of such cones. Let us mark these cones as $\Omega_1, \ldots, \Omega_d$. Besides, we shall denote by Ω_0 the cone of vectors with positive coordinates.

In what follows, a central role will be played by the cone

$$\Omega = \bigcup_{i=0}^{d} \Omega_i.$$

Consider the operator $\mathbf{D} : \mathbb{R}^{n+1} \to \mathbb{R}^n$, acting as

$$\forall \mathbf{x} \in \mathbb{R}^n,\ \forall \lambda \in \mathbb{R}^+ : \quad \mathbf{D}(\mathbf{x}, \lambda) = \mathbf{x} + \lambda \begin{pmatrix} \mathbf{x}^T B_1 \mathbf{x} \\ \vdots \\ \mathbf{x}^T B_n \mathbf{x} \end{pmatrix}.$$

Theorem 12. *The cone Ω is invariant with respect to the action of the operator $\mathbf{D}$ ($\forall \mathbf{x} \in \Omega$, $\forall \lambda \in \mathbb{R}^+$, $\mathbf{D}(\mathbf{x}, \lambda) \in \Omega$).*

Proof. Suppose that the system (6) is reduced to the form (9). Besides, for simplicity, we consider the case $n = 2$. Only such an approach allows us to demonstrate all the details valid for an arbitrary n.

Let, e.g., $x_{10} < 0$ and $x_{20} > 0$, where $\mathbf{x}_0 = (x_{10}, x_{20})^T \in \Omega$. Then one of the inequalities (25) can be written down as

$$\mathbf{x}_0^T (x_{20} B_1 - x_{10} B_2) \mathbf{x}_0 > 0. \tag{26}$$

If we express it through the variable $u = x_{10}/x_{20}$, then for any $u < 0$ we obtain the inequality $g(u) > 0$, where $g(u)$ is defined in (22). The behaviour of the cubic curve $g(u)$ implies $\mu_1^- < (x_{10}/x_{20}) < \mu^+$, where μ_1^- is the maximum negative root of $g(u) = 0$, and μ^+ is the positive root of the same equation.

Return now to the inequalities (23) and construct the first iteration of Euler's process:

$$\begin{pmatrix} x_{11} \\ x_{21} \end{pmatrix} = \begin{pmatrix} x_{10} \\ x_{20} \end{pmatrix} + \Delta t \begin{pmatrix} \mathbf{x}_0^T B_1 \mathbf{x}_0 \\ \mathbf{x}_0^T B_2 \mathbf{x}_0 \end{pmatrix}. \tag{27}$$

Then from the first two equations of (27) and the inequality (26) the following inequality can be obtained:

$$x_{11} x_{20} - x_{21} x_{10} = \Delta t (\mathbf{x}_0^T (x_{20} B_1 - x_{10} B_2) \mathbf{x}_0) > 0.$$

But since $x_{20} x_{21} > 0$, from the last inequality it follows that $x_{11}/x_{21} > x_{10}/x_{20}$. Here two cases can be distinguished: either $x_{11} \geq 0$, $x_{21} > 0$ or $x_{11} < 0$, $x_{21} > 0$ and $\mu_1^- < (x_{11}/x_{21}) < \mu^+$. In the first case, we have $\mathbf{x}_1 = (x_{11}, x_{21}) \in \Omega$. In the second case, we get $\mathbf{x}_1^T (x_{21} B_1 - x_{11} B_2) \mathbf{x}_1 > 0$. Hence also $\mathbf{x}_1 \in \Omega$.

It is obvious that after the i-th iteration we get a monotonically increasing sequence

$$\frac{x_{10}}{x_{20}} < \frac{x_{11}}{x_{21}} < \cdots < \frac{x_{1,i}}{x_{2,i}}.$$

Taking into account the fact that in the interval (μ_1^-, μ^+) there are no zeros of $g(u)$, it is clear that $\forall i$, $\mu_1^- < (x_{1,i}/x_{2,i}) < \mu^+$. Considering similarly all the situations for which $x_{i,0} < 0$, we finally get the statement of the theorem. ■

Theorem 13. *If $\mathbf{x}_0 \in \Omega$, then under conditions (b) and (c) of Theorem 8 the trivial solution of the regular system (6) is conditionally asymptotically stable.*

Proof. From Theorem 12 it follows that $\lim_{i\to\infty} x_{1,i}/x_{2,i} = \mu^+ > 0$. Hence, starting from some i, we obtain $x_{2,i} > 0$ and $x_{1,i} > 0$. The satisfaction of the last inequalities guarantees conditional stability. ■

The cone of stability Ω has a nonlinear directrix. This is not suitable for practical purposes. Therefore we shall construct a cone $\Omega_\Lambda \subset \Omega$ with rectilinear directrices.

Consider the set of equations similar to the system (21):

$$\left\{ \begin{aligned} g_{i1}(\mu_{i1}, \dots, \mu_{i,j-1}, \mu_{i,j+1}, \dots, \mu_{i,n}) &= 0, \\ &\vdots \\ g_{i,k-1}(\mu_{i1}, \dots, \mu_{i,j-1}, \mu_{i,j+1}, \dots, \mu_{i,n}) &= 0, \\ g_{i,k+1}(\mu_{i1}, \dots, \mu_{i,j-1}, \mu_{i,j+1}, \dots, \mu_{i,n}) &= 0, \\ &\vdots \\ g_{i,n}(\mu_{i1}, \dots, \mu_{i,j-1}, \mu_{i,j+1}, \dots, \mu_{i,n}) &= 0, \end{aligned} \right. \tag{28}$$

where $\mu_{ij} = x_j/x_i$ for $i, j, k = 1, \dots, n$, $j \neq i$, $k \neq i$, $j - 1 > 0$, $k - 1 > 0$. (It is clear that, in all, there are n such systems. Any such system depends on $n - 1$ unknowns μ_{ij}.)

Denote by $d = \deg_{\mathbf{C}} \mathbf{W}_i$ the degree of the variety of solutions to these systems. (Above it was shown that these degrees are really identical for complete systems and they are equal to $d = 2^n - 1$.) Then there exist exactly d $(n-1)$-dimensional vectors

$$(\mu_{i1}, \dots, \mu_{i,j-1}, \mu_{i,j+1}, \dots, \mu_{i,n})_k$$

being solutions to (28), $k = 1, \dots, d$.

Delete from the set of these vectors the vector having all positive coordinates. Furthermore, any $(n-1)$-dimensional vector with coordinates μ_{ij} is substituted for the n-dimensional vector with the same coordinates, but in place of i we have 1. Then the remaining $d-1$ vectors will be placed in $d-1$ orthants, except for the orthants with only positive and only negative coordinates. Denote these orthants by Ω_0 and Ω_d. From the resulting $d-1$ n-dimensional vectors we select $n-1$ vectors such that the $(n-1)$-dimensional hyperplane spanned on them intersects orthants Ω_0 and Ω_d only at zero. There will be exactly $d-1$ such hyperplanes. The sum of these hyperplanes forms the required cone Ω_Λ.

5. Lyapunov Stability of Non-Homogeneous Quadratic Systems with Singular Linear Part

Return now to the study of the system (5). In addition, we will assume that $\det A_0 = 0$, where A_0 is the matrix of the operator $\mathbf{A}_0$. (In the case of $\det A_0 \neq 0$ the solution to the problem is well known (Khalil, 1995): for the local stability of the system (5) the matrix A_0 has to be Hurwitz.) Besides, in the given section we will assume that A_0 is diagonalizable in the field of complex numbers $\mathbb{C}$. Using a nonsingular transformation $L \in \mathbb{R}^{n\times n}$ the matrix A_0 can be reduced to the form

$$L^{-1}A_0L=\begin{pmatrix} \alpha_1 & & & & & & & & & \\ & \ddots & & & & & \mathbf{0} & & & \\ & & \alpha_r & & & & & & & \\ & & & \beta_1 & \gamma_1 & & & & & \\ & & & -\gamma_1 & \beta_1 & & & & & \\ & & & & & \ddots & & & & \\ & & \mathbf{0} & & & & \beta_s & \gamma_s & & \\ & & & & & & -\gamma_s & \beta_s & & \\ & & & & & & & & \mathbf{0} \end{pmatrix}, \quad (29)$$

where the dimension of the zero diagonal block is equal to $n-r-2s$.

Perform the change of variables $\mathbf{x} \to L\mathbf{x}$. Then the matrix A_0 will be replaced with $L^{-1}A_0L$. Below, in order to simplify the notation, we will retain the former notation A_0 for the matrix $L^{-1}A_0L$. Similarly, after the application of the transformation L, we will retain the notation B_i, $i=1,\dots,n$ for the matrices of quadratic forms of the system (5).

Introduce Lyapunov's function as the positive definite quadratic form (Khalil, 1995):

$$F = x_1^2 + \cdots + x_n^2.$$

Then the derivative $\mathrm{d}f/\mathrm{d}t$, calculated on the basis of (5), has the form

$$\begin{aligned}\frac{\mathrm{d}f}{\mathrm{d}t} &= \alpha_1x_1^2+\cdots+\alpha_rx_r^2+\beta_1x_{r+1}^2\\ &\quad+\beta_1x_{r+2}^2+\cdots+\beta_sx_{r+2s-1}^2\\ &\quad+\beta_sx_{r+2s}^2+\mathbf{x}^{\mathbf{T}}(x_1B_1+\cdots+x_nB_n)\mathbf{x}.\end{aligned}$$

Rewrite the last formula as follows:

$$\begin{aligned}\frac{\mathrm{d}f}{\mathrm{d}t} &= (\alpha_1+\sum_{i=1}^n\xi_{1i}x_i)x_1^2+\cdots+(\alpha_r+\sum_{i=1}^n\xi_{ri}x_i)x_r^2\\ &\quad+(\beta_1+\sum_{i=1}^n\tau_{1i}x_i)x_{r+1}^2\\ &\quad+(\beta_1+\sum_{i=1}^n\upsilon_{1i}x_i)x_{r+2}^2+\cdots\\ &\quad+(\beta_s+\sum_{i=1}^n\tau_{si}x_i)x_{r+2s-1}^2\\ &\quad+(\beta_s+\sum_{i=1}^n\upsilon_{si}x_i)x_{r+2s}^2\\ &\quad+(\sum_{i=1}^n\rho_{r+2s+1,i}x_i)x_{r+2s+1}^2+\cdots\\ &\quad+(\sum_{i=1}^n\rho_{n,i}x_i)x_n^2.\end{aligned}\quad (30)$$

It is clear that $\mathrm{d}f/\mathrm{d}t$ will be nonpositive in some small neighbourhood of the point 0 if the following conditions are fulfilled: $\alpha_1 < 0,\dots,\alpha_r < 0, \beta_1 < 0,\dots,\beta_s < 0$ and the $(n-r-2s)\times n$-matrix

$$J = \begin{pmatrix} \rho_{r+2s+1,1},\dots,\rho_{r+2s+1,n} \\ \dots\dots\dots\dots\dots\dots \\ \rho_{n,1},\dots,\rho_{n,n}\end{pmatrix} \quad (31)$$

is the null matrix.

The last result can be used to design stable control laws for the system (1),(2). For that purpose we introduce linear feedback $\mathbf{u} = K_L\mathbf{y}$ into the last system; then we get (3). Further, we reduce A_0 to the form (29). (It is clear that J will be linearly dependent on $K_L : J = J(K_L)$.) Finally, we obtain the system $J(K_L) = 0$, which consists of $(n-r-2s)\times n$ linear equations with respect to the $m\times p$ unknown elements of the matrix K_L. The solutions to this system will give the required gains. If the system $J(K_L) = 0$ is incompatible, there does not exist linear feedback ensuring the stability of the closed system (3) (even if all the nonzero eigenvalues of matrix A_0 have negative real parts).

Denote by $\mathrm{Ker}\,\mathbf{A}_0$ the kernel of the operator $\mathbf{A}_0$ and by $\mathbf{A}_{0f}$ the restriction operator $\mathbf{A}_0$ on the quotient space $\mathbb{R}^n/\mathrm{Ker}\,\mathbf{A}_0$.

Theorem 14. *Assume that the matrix of the operator $\mathbf{A}_{0f}$ is diagonalizable and all its eigenvalues have negative* *real parts. Then for the stability of the solutions to the system (5) it is sufficient that* $mp > (n - r - 2s) \times n$.

In the present section we will suppose that $n = 2$. Rewrite (5) in the following form:

$$\begin{cases} \dot{x}_1(t) = d_{11}x_1(t) + d_{12}x_2(t) \\ \qquad +b_{111}x_1^2(t) + b_{112}x_1(t)x_2(t) \\ \qquad +b_{121}x_2(t)x_1(t) + b_{122}x_2^2(t), \\ \dot{x}_2(t) = d_{21}x_1(t) + d_{22}x_2(t) \\ \qquad +b_{211}x_1^2(t) + b_{212}x_1(t)x_2(t) \\ \qquad +b_{221}x_2(t)x_1(t) + b_{222}x_2^2(t). \end{cases} \tag{32}$$

Here $b_{112} = b_{121}$ and $b_{212} = b_{221}$.

Let us show that for a generic system of the second order the design of control laws on the output, ensuring the stability of a closed-loop system, it is not possible.

Suppose that the matrix A_0 is diagonalizable and $\det \mathbf{A}_0 = 0$. Then the following cases are possible: (a) $mp = 2, r = 1, s = 0$, (b) $mp = 2, r = 0, s = 0$.

It is easy to show that for these cases the conditions of Theorem 14 are not fulfilled. Therefore there is a unique possibility: (c) $mp = 4,\ r = 1,\ s = 0$ (state feedback). In the remaining cases it is possible to speak only about the conditional stability of the closed-loop system.

6. Algorithms for the Design of Linear Control Laws for Bilinear Systems

6.1. Design for Homogeneous Systems of the Second Order

We will assume that in the system (1), (2) we have $n = 2$ and $A_0 = 0$.

1. Introduce the feedback $\mathbf{u} = KC\mathbf{x}$ into the system (1), where $K \in \mathbb{R}^{m\times p}$ is an unknown matrix. Thus we will get (32), in which $d_{ij} = 0$ and the coefficients b_{ijk} depend on K.

2. Set

$$U = \begin{pmatrix} b_{111} & b_{122} \\ b_{211} & b_{222} \end{pmatrix}, \quad U_1 = \begin{pmatrix} b_{111} & b_{112} \\ b_{211} & b_{212} \end{pmatrix},$$

$$U_2 = \begin{pmatrix} b_{121} & b_{122} \\ b_{221} & b_{222} \end{pmatrix}, \quad U_3 = \begin{pmatrix} b_{111} & b_{112} \\ b_{112} & b_{122} \end{pmatrix},$$

$$U_4 = \begin{pmatrix} b_{211} & b_{212} \\ b_{212} & b_{222} \end{pmatrix}, \quad U_5 = \begin{pmatrix} b_{211} & b_{212} \\ b_{112} & b_{122} \end{pmatrix}.$$

As for the invariants J_1, J_2 and J_3 (Belozyorov,2001), we have

$$\begin{aligned} J_1(K) &= \det(U^2) - 4\det(U_1U_2), \\ J_2(K) &= \det(U_1U_2 - U_2U_1), \\ J_3(K) &= \mathrm{tr}(U_2^2)(\det U_1 + \det U_4) \\ &\quad + \mathrm{tr}(U_1^2)(\det U_2 + \det U_3) \\ &\quad + \mathrm{tr}(U_1U_2 + U_2U_1)\det U_5. \end{aligned}$$

These are polynomials of the fourth degree with respect to the elements of the matrix K.

3. Form the system of inequalities

$$J_1(K) > 0, \quad J_2(K) > 0, \quad J_3(K) < 0 \tag{33}$$

and find the domain $Z \subset \mathbb{R}^{m\times p}$ (or its part), determined by this system.

The following steps of the algorithm are intended for the determination of the cone of conditional stability.

4. Fix a feedback matrix $K_0 \in Z$ and write down the system (6) for this matrix.

5. Form the equations (7) with respect to the unknowns ρ_1 and ρ_2 (it is possible to set $r = 1$).

6. Find the solutions $\mathbf{f_1} = (\rho_{11}, \rho_{12})$ and $\mathbf{f_2} = (\rho_{21}, \rho_{22})$ to (7) (they are necessarily real) and form the matrix

$$F = \begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix}^{-1}.$$

7. Use $\mathbf{x} = FP\mathbf{z}$ to reduce (6) to the form (8).

8. Once again, changing the variables $\mathbf{z} = F_1\mathbf{v}$, where $F_1 = \mathrm{diag}\,(\mathrm{sign}\,\beta_{11}, \mathrm{sign}\,\beta_{22})$ is a diagonal matrix, make all diagonal elements negative in the matrix B of the system (8). In this case the matrix

$$B = \begin{pmatrix} -\beta_{11} & \beta_{12} \\ \beta_{21} & -\beta_{22} \end{pmatrix}$$

of the system (9) will satisfy the properties (b) and (c) of Corollary 2.

9. Calculate the maximum negative root $\lambda = v_2/v_1$ of the equation $\beta_{12}\lambda^3 + \beta_{22}\lambda^2 - \beta_{11}\lambda - \beta_{21} = 0$ and the maximum negative root $\mu = v_1/v_2$ of the equation $\beta_{21}\mu^3 + \beta_{11}\mu^2 - \beta_{22}\mu - \beta_{12} = 0$.

10. Calculate generators of the stability cone; they will be the vectors $\mathbf{d_1} = (1, \lambda)^T$ and $\mathbf{d_2} = (\mu, 1)^T$.

11. Find any solution $\mathbf{w} = (w_1, w_2)^T$ to the system of the inequalities $(\mathbf{d_1}, \mathbf{w}) \geq 0, (\mathbf{d_2}, \mathbf{w}) \geq 0$.

12. Define the generators of the stability cone in the initial system of coordinates under the formulae $\mathbf{x_i} = FPF_1\mathbf{d_i}, i = 1, 2$. They divide the x_1x_2 plane into four sectors. Take the sector which contains the point $FPF_1\mathbf{w}$. It will be the stability cone $\Omega(K_0)$ depending on the feedback matrix K_0.

6.2. Design for the Non-Homogeneous System of Any Order

1. Compose the matrix L reducing the matrix A_0 to the form (29).

2. Transform the system (1) using the formula $\mathbf{x} = L\mathbf{z}$.

3. Introduce feedback into the transformed system (1) using the formula $\mathbf{u} = KCL\mathbf{z}$, where $K \in \mathbb{R}^{m\times p}$ is an unknown matrix.

4. Introduce the function f and calculate its derivative $\mathrm{d}f/\mathrm{d}t$ (it depends on K) using (30).

5. Write down the system of the linear equations $J(K_L) = 0$, where the matrix $J(K_L)$ has the form (31). (If output feedback is sought, then $K = K_L$; if state feedback is sought, then $K = K_L L^{-1}$.) Let $K_L \neq 0$ be one of nontrivial solutions. (If the system $J(K_L) = 0$ is unsolvable, then finish the design procedure.)

6. Substitute the matrix K_L into the derivative (30).

7. Let $\alpha_1 < 0, \dots, \alpha_r < 0, \beta_1 < 0, \dots, \beta_s < 0$. Using (30), form the following system of linear inequalities:

$$\sum_{i=1}^{n} \xi_{1i} z_i < |\alpha_1|, \quad \dots, \quad \sum_{i=1}^{n} \xi_{ri} z_i < |\alpha_r|,$$

$$\sum_{i=1}^{n} \tau_{1i} z_i < |\beta_1|, \quad \sum_{i=1}^{n} v_{1i} z_i < |\beta_1|, \quad \dots,$$

$$\sum_{i=1}^{n} \tau_{si} z_i < |\beta_s|, \quad \sum_{i=1}^{n} v_{si} x_i < |\beta_s|.$$

8. Using the Cauchy inequality, estimate the sizes of the domain H in which $\mathrm{d}f/\mathrm{d}t < 0$, according to the formula

$$\|\mathbf{z}\| < \min\left(\frac{|\alpha_1|}{\sqrt{\sum_{i=1}^{n}\xi_{1i}^2}}, \dots, \frac{|\alpha_r|}{\sqrt{\sum_{i=1}^{n}\xi_{ri}^2}}, \frac{|\beta_1|}{\sqrt{\sum_{i=1}^{n}\tau_{1i}^2}}, \frac{|\beta_1|}{\sqrt{\sum_{i=1}^{n}v_{1i}^2}}, \dots, \frac{|\beta_s|}{\sqrt{\sum_{i=1}^{n}\tau_{si}^2}}, \frac{|\beta_s|}{\sqrt{\sum_{i=1}^{n}v_{si}^2}} \right).$$

The sizes of H for a variable $\mathbf{x}$ are determined from $\|\mathbf{x}\| \leq \|\mathbf{z}\|/\|L^{-1}\|$. The domain of the initial values Y, for which the closed-loop system (by feedback K_0) will be stable in the Lyapunov sense, is included in H.

It is necessary to note that, generally speaking, $Y \neq H$. Therefore for the definition of Y it is necessary to take some points from H and, based on these points, to estimate sizes Y.

7. Examples

Below we shall consider some examples in which Algorithms 6.1 and 6.2 are used in a simplified form.

Example 1. Assume that a bilinear system is given by the equations

$$\begin{cases} \dot{x}_1 = -4x_1 + 6x_2 + (11x_1 - 15x_2)u_1 \\ \qquad +(-10x_1 + 19x_2)u_2, \\ \dot{x}_2 = -2x_1 + 3x_2 + (7x_1 - 10x_2)u_1 \\ \qquad +(-7x_1 + 13x_2)u_2. \end{cases}$$

It is required to construct a feedback matrix K of the linear control law $\mathbf{u} = K\mathbf{x}$, ensuring the stability of the trivial solution of the closed-loop system. (In the given example, the control law must be only by law on the state).

We start with the matrix L transforming the linear part of the system (32) into the diagonal form:

$$L = \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}, \quad L^{-1} = \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}.$$

Then, after the substitution of $\mathbf{x} = L\mathbf{z}$, the required system can be transformed into the form

$$\begin{cases} \dot{z}_1 = -z_1 + (2z_1 + 3z_2)u_1 + (z_1 + z_2)u_2, \\ \dot{z}_2 = (z_1 - z_2)u_1 + (-z_1 + 2z_2)u_2. \end{cases}$$

Let $u_1 = k_{11}z_1 + k_{12}z_2$ and $u_2 = k_{21}z_1 + k_{22}z_2$. Let us construct the Lyapunov function $f = z_1^2+z_2^2$. Then

$$\dot{f}_t = z_1^2[-1 + (2k_{11} + k_{21})z_1 + (2k_{12} + k_{22} + 4k_{11})z_2] + z_2^2[(4k_{12} - k_{11} + 2k_{21})z_1 + (2k_{22} - k_{12})z_2].$$

Thus, if the following two conditions are satisfied:

$$4k_{12} - k_{11} + 2k_{21} = 0, \quad 2k_{22} - k_{12} = 0,$$

then the matrix $J(K_L)$ is a null matrix.

In the sequel, we have

$$K_L = \begin{pmatrix} 8k_{22} + 2k_{21} & 2k_{22} \\ k_{21} & k_{22} \end{pmatrix} \neq \mathbf{0}.$$

Let, e.g., $k_{22} = 0.1$ and $k_{21} = -0.4$. Then $k_{12} = 0.2$, $k_{11} = 0$ and

$$K_L = \begin{pmatrix} 0 & 0.2 \\ -0.4 & 0.1 \end{pmatrix}.$$

Going back to the initial basis of $\mathbb{R}^2$, we obtain the feedback matrix

$$K = K_L L^{-1} = \begin{pmatrix} -0.2 & 0.4 \\ -0.9 & 1.4 \end{pmatrix}.$$

The system closed by this type of feedback is represented in the form

$$\begin{cases} \dot{x}_1 = -4x_1 + 6x_2 + 6.8x_1^2 - 23.7x_1x_2 + 20.6x_2^2, \\ \dot{x}_2 = -2x_1 + 3x_2 + 4.9x_1^2 - 13.7x_1x_2 + 14.2x_2^2. \end{cases}$$

The domain of the negative definiteness H for the function $\dot{f}_t$ is determined by the inequality

$$\|\mathbf{x}\| \leq \frac{1}{\sqrt{18}\sqrt{(16k_{22}+5k_{21})^2+(37k_{22}+8k_{21})^2}} < 0.5.$$

The behaviour of solutions to the last system for various initial data is represented in Figs. 2–5:

Thus the trivial solution is stable (but only locally). Besides, the domain of stability $Y = Y(\mathbf{x}_0)$ is determined by the inequality $\|\mathbf{x_0}\| \leq 0.4$. ♦

Example 2. Suppose that a bilinear system is given by the equations

$$\begin{cases} \dot{x}_1 = x_1u_1 + (-x_1 - x_2)u_2, \\ \dot{x}_2 = (2x_1 + x_2)u_1 + x_2u_2. \end{cases}$$

Similary to the previous example, it is required to construct a feedback matrix K of the control law $\mathbf{u} = K\mathbf{x}$ on the state. It is clear that the system closed by this type of feedback should have a nonzero domain of conditional stability. (Stability in the Lyapunov sense in the given example is impossible (even locally).)

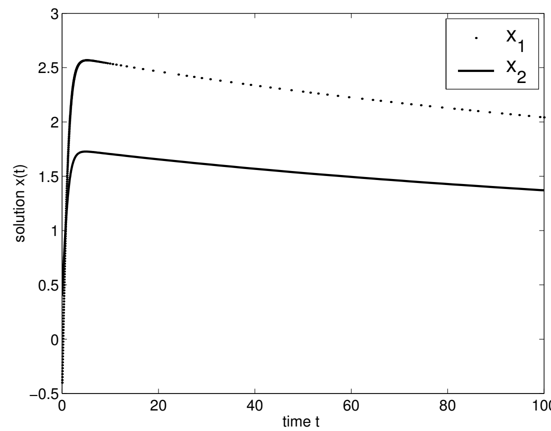

Fig. 2. Solutions of the quadratic system for $x_{10} = 0.4$ and $x_{20} = 0.4$.

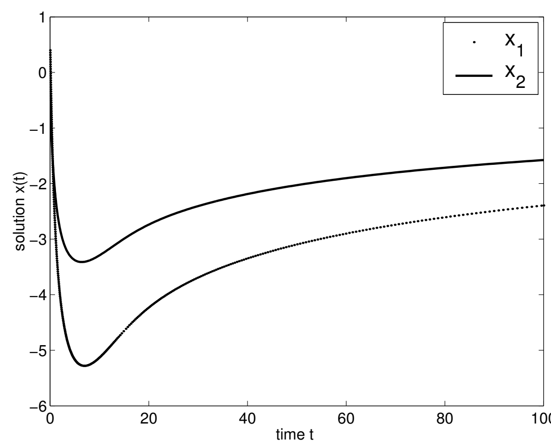

Fig. 3. Solutions of the quadratic system for $x_{10} = 0.4$ and $x_{20} = -0.4$.

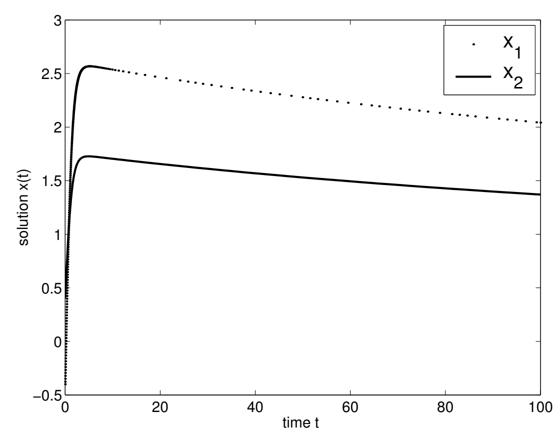

Fig. 4. Solutions of the quadratic system for $x_{10} = -0.4$ and $x_{20} = 0.4$.

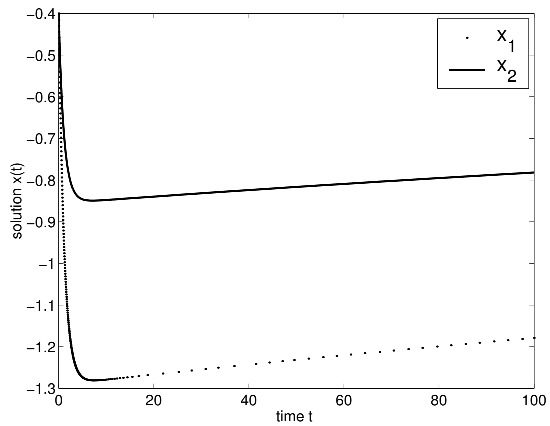

Fig. 5. Solutions of the quadratic system for $x_{10} = -0.4$ and $x_{20} = -0.4$.

The system closed by the state feedback has the form:

$$\begin{cases} \dot{x}_1 = (k_{11} - k_{21})x_1^2 + (k_{12} - k_{22} - k_{21})x_1x_2 \\ \qquad -k_{22}x_2^2, \\ \dot{x}_2 = 2k_{11}x_1^2 + (2k_{12} + k_{11} + k_{21})x_1x_2 \\ \qquad +(k_{12} + k_{22})x_2^2. \end{cases} \tag{34}$$

Thus in (32) we have $b_{111} = k_{11} - k_{21}, b_{112} = b_{121} = (k_{12} - k_{22} - k_{21})/2, b_{122} = -k_{22}, b_{211} = 2k_{11}, b_{212} = b_{221} = (2k_{12} + k_{11} + k_{21})/2, b_{222} = k_{12} + k_{22}$ (the linear part is missing).

For the solution of the design problem, the inequalities (33) are used. Assume that $k_{12} - k_{22} - k_{21} = 0$ and $2k_{12} + k_{11} + k_{21} = 0$. (Then we can avoid the analysis for polynomial inequalities of the 4-th degree.) Thus we obtain the domain given by the inequalities $k_{22} + 2k_{21} > 0$, $2k_{22} + 3k_{21} \geq 0$, $k_{22} \leq 0$, $k_{21} + k_{22} < 0$ and $k_{21}^2 + 4k_{21}k_{22} + 2k_{22}^2 < 0$.

It is easy to check that this domain is non-empty. (For example, the point $k_{11} = 0.4$, $k_{12} = -0.8$, $|, k_{21} = 1.2$, $k_{22} = -2$ belongs to the domain.) Then the system (34) closed by the feedback

$$K = \begin{pmatrix} 0.4 & -0.8 \\ 1.2 & -2 \end{pmatrix}$$

reduces to the form

$$\begin{cases} \dot{x}_1 = -0.8x_1^2 + 2x_2^2, \\ \dot{x}_2 = 0.8x_1^2 - 2.8x_2^2. \end{cases}$$

Calculating $\mu = -0.6720$ and $\lambda = -0.4762$ of Step 9 of Algorithm 6.1, we conclude that the last system is conditionally stable in the domain $\Omega = \{x_1 + 0.6720x_2 > 0\} \bigcap \{x_2 + 0.4762x_1 > 0\}$. (Here a simplified variant of Steps 4–12 of Algorithm 6.1 is used.) The computation of the invariants $J_1 = 0.4096 > 0$, $J_2 = 1.024 > 0$, $J_3 = -13.4656 < 0$ shows that for the given example the conditions (33) are fulfilled.

Solutions of the closed-loop system for various initial data are given in Figs. 6–9. For the first three cases we have $\mathbf{x}_0 \in \Omega$ and in the last case we get $\mathbf{x}_0 \notin \Omega$. It is clear that in the first three cases the solution is conditionally stable and in the fourth one it is unstable. ♦

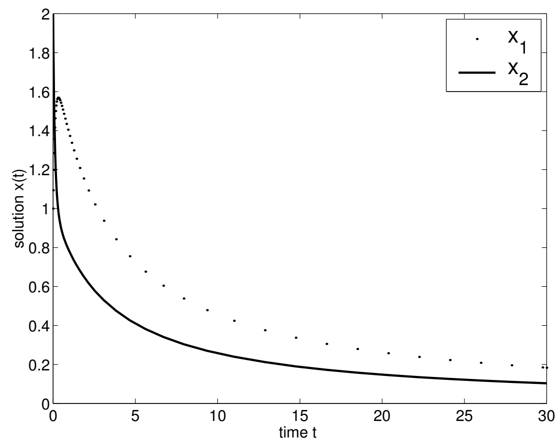

Fig. 6. Solutions of the quadratic system for $x_{10} = 1.0$ and $x_{20} = 2.0$.

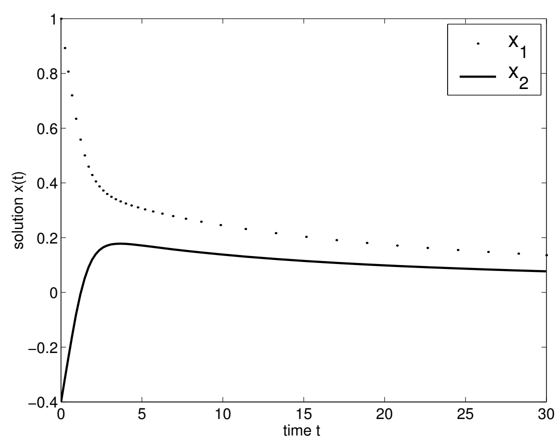

Fig. 7. Solutions of the quadratic system for $x_{10} = 1.0$ and $x_{20} = -0.4$.

Example 3. Consider an example of the construction of the stability domain for the following homogeneous system of the third order:

$$\begin{cases} \dot{x}_1 = -2x_1^2 + x_2^2, \\ \dot{x}_2 = x_1^2 - 2x_2^2 + x_3^2, \\ \dot{x}_3 = x_1^2 + x_2^2 - 2x_3^2. \end{cases}$$

(It is easy to verify that the system satisfies Conditions (b) and (c) of Theorem 8.)

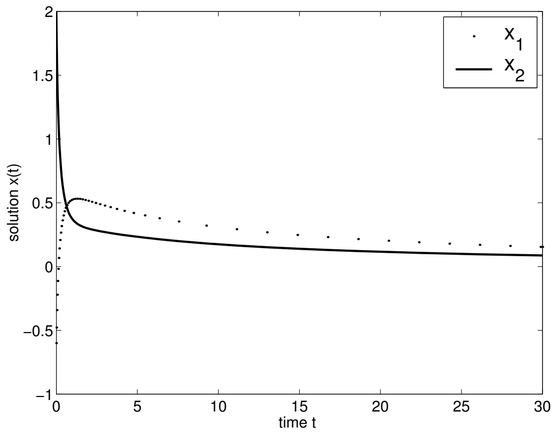

Fig. 8. Solutions of the quadratic system for $x_{10} = -0.6$ and $x_{20} = 2.0$.

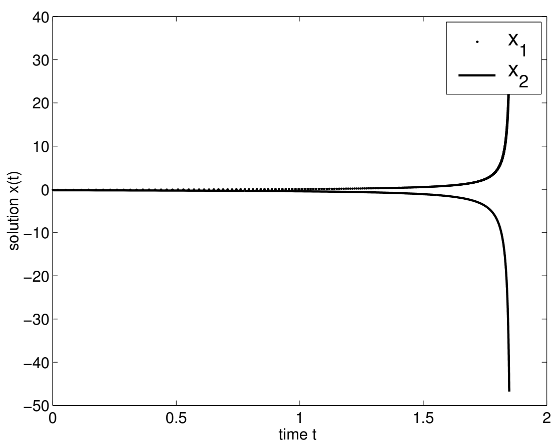

Fig. 9. Solutions of the quadratic system for $x_{10} = -0.1$ and $x_{20} = -0.2$.

Define vectors forming the stability cone. For this purpose, we introduce new variables $x_1/x_3 = u$, $x_2/x_3 = v$ and consider the system (28):

$$u = \frac{-2u^2 + v^2}{u^2 + v^2 - 2}, \quad v = \frac{u^2 - 2v^2 + 1}{u^2 + v^2 - 2}.$$

Eliminating the variable v from the last system, we obtain the following equation:

$$18u^7 + 27u^6 - 57u^5 - 39u^4 + 55u^3 + 11u^2 - 13u - 1 = 0.$$

Furthermore, the unknown v is defined by

$$v = \frac{3u^3 + 3u^2 - 3u - 1}{-3u^2 + 2}.$$

Thus we have seven vectors $\mathbf{d}_i = (u_i, v_i, 1)^T$, $i = 1, \ldots, 7$, out of which six (except for the vector with positive coordinates) give vectors forming the cone in the rectangular system of coordinates $x_1x_2x_3$:

$$\mathbf{d}_1 = \begin{pmatrix} -2.247 \\ 1.000 \\ 1 \end{pmatrix}, \quad \mathbf{d}_2 = \begin{pmatrix} -1.089 \\ -1.249 \\ 1 \end{pmatrix},$$

$$\mathbf{d}_3 = \begin{pmatrix} -0.555 \\ 1.000 \\ 1 \end{pmatrix}, \quad \mathbf{d}_4 = \begin{pmatrix} 0.891 \\ -2.175 \\ 1 \end{pmatrix},$$

$$\mathbf{d}_5 = \begin{pmatrix} 0.772 \\ -0.691 \\ 1 \end{pmatrix}, \quad \mathbf{d}_6 = \begin{pmatrix} -0.074 \\ -0.384 \\ 1 \end{pmatrix}.$$

Note that all these vectors are real.

It is required that these vectors be placed in six various orthants. (If it is not the case, then an appropriate vector is multiplied by -1. Taking into account the last remark, we obtain vectors $\mathbf{a}_1 = -\mathbf{d}_1$, $\mathbf{a}_2 = -\mathbf{d}_2$, $\mathbf{a}_3 = \mathbf{d}_3$, $\mathbf{a}_4 = -\mathbf{d}_4$, $\mathbf{a}_5 = \mathbf{d}_5$, $\mathbf{a}_6 = \mathbf{d}_6$.

Now we can construct the edges of the cone from these vectors. (It should be noted that the cone has to contain the first orthant.) Thus the edges of the cone are generated by vectors $\{\mathbf{a}_3, \mathbf{a}_6\}$(I), $\{\mathbf{a}_6, \mathbf{a}_5\}$(II), $\{\mathbf{a}_4, \mathbf{a}_2\}$(III), $\{\mathbf{a}_2, \mathbf{a}_1\}$(IV), $\{\mathbf{a}_5, \mathbf{a}_1\}$(V), $\{\mathbf{a}_3, \mathbf{a}_4\}$(VI).

From analytical geometry it is known that the plane passing through three points with coordinates (f_1, f_2, f_3), (g_1, g_2, g_3), (h_1, h_2, h_3) in the rectanguler system of coordinates $x_1x_2x_3$ is defined by the equation

$$\det \begin{pmatrix} x_1 - f_1 & x_2 - f_2 & x_3 - f_3 \\ g_1 & g_2 & g_3 \\ h_1 & h_2 & h_3 \end{pmatrix} = 0.$$

The last formula can be used to obtain equations for the edges of the cone. Assume that $(f_1, f_2, f_3) = (0, 0, 0)$, and instead of coordinates (g_1, g_2, g_3) and (h_1, h_2, h_3) we substitute coordinates of the vectors forming planes (I), ..., (VI). The required stability cone Ω_Λ can be defined by the following system of linear inequalities:

$$\begin{pmatrix} 1.385 & 0.481 & 0.287 \\ 0.307 & 0.846 & 0.348 \\ 0.926 & 1.981 & 3.483 \\ 2.249 & 1.158 & 3.896 \\ 1.691 & 3.019 & 0.781 \\ 3.175 & 1.446 & 0.316 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \geq 0.$$

♦

8. Conclusion

Conditions (33) allow us to state that when they are met (for an appropriate matrix K), there exists a domain of conditional stability for the homogeneous system (32). This domain is also defined by the set of equations (19), which (unfortunately) is not invariant with respect to the transformations of system coordinates. However, non-satisfied conditions (33) for any K mean that the appropriate closed control system cannot be made conditionally stable via linear feedback for any initial data. This involves the following problem: For what homogeneous regular complete systems (32) is the system of inequalities (33) solvable? A similar problem can be formulated for control systems of any order.

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Appendix

Any quadratic form on the right-hand side of the system (6) can be represented as follows:

$$\mathbf{x}^T B_i \mathbf{x} = (\mathbf{r}_{i1}, \dots, \mathbf{r}_{in}) \cdot (x_1\mathbf{x}^T, \dots, x_n\mathbf{x}^T)^T,$$

where $\mathbf{r}_{i1}, \dots, \mathbf{r}_{in}$ are n-dimensional row vectors of the matrix B_i, $i = 1, \dots, n$. Thus any system (6) can be presented as

$$\dot{\mathbf{x}}(t) = \mathbf{T} \cdot \big(\mathbf{x}(t) \otimes \mathbf{x}(t)\big),$$

where

$$\mathbf{T} = \begin{pmatrix} \mathbf{r}_{11}, & \dots & , \mathbf{r}_{1n} \\ \vdots & \dots & \vdots \\ \mathbf{r}_{n1}, & \dots & , \mathbf{r}_{nn} \end{pmatrix} \in \mathbb{R}^{n \times n^2} \tag{A1}$$

is a mixed tensor once contravalent and twice covalent. The new vector $\mathbf{x} \otimes \mathbf{x} = (x_1\mathbf{x}^T, \dots, x_n\mathbf{x}^T)^T$ is a tensor product of the vector $\mathbf{x}$ by itself. (Here the tensor $\mathbf{T}$ is realized as the space of matrices of sizes $n \times n^2$. The space Ψ of such tensors has dimension $n^2(n+1)/2$.)

Let $\mathrm{GL}(n, C)$ be a complete linear group of all square invertible matrices of sizes $n \times n$ with elements from the field of complex numbers $\mathbb{C}$. Introduce a new variable $\mathbf{z}$ into (A1) according to the formula $\mathbf{x}(t) = S\mathbf{z}(t)$, where $S \in \mathrm{GL}(n, \mathbf{C})$. Then we get an operation $\mathrm{GL}(n, C) : \Psi \to \Psi$ of the group on the space of tensors, acting as follows:

$$\forall\, \mathbf{T} \in \Psi,\ \forall S \in \mathrm{GL}(n, \mathbf{C}) : S(\mathbf{T}) = S^{-1} \cdot \mathbf{T} \cdot (S \otimes S),$$

where $S = (s_{ij})$ for $i, j = 1, \dots, n$ and

$$S \otimes S = \begin{pmatrix} s_{11} \cdot S, & \dots, & s_{1n} \cdot S \\ \vdots & \dots & \vdots \\ s_{n1} \cdot S, & \dots, & s_{nn} \cdot S \end{pmatrix}.$$

Note that a polynomial $g(\mathbf{T})$ is called an invariant of weight l of the group $\mathrm{GL}(n, \mathbf{C})$ if $\forall S \in \mathrm{GL}(n, \mathbf{C})$, $g(S(\mathbf{T})) = (\det S)^l \times g(\mathbf{T})$, where l is some integer.

Write

$$\Delta_i = \begin{pmatrix} \mathbf{r}_{1i} \\ \vdots \\ \mathbf{r}_{ni} \end{pmatrix} \in \mathbb{C}^{n \times n}, \quad i = 1, \dots, n.$$

Then we get $\mathbf{T} = (\Delta_1, \dots, \Delta_n) \in \mathbb{C}^{n \times n^2}$.

Construct the function

$$J_n(\mathbf{T}) = \det\Big(\sum_{1 \le j_1, \dots, j_n \le n} (-1)^{\sigma} \Delta_{j_1} \Delta_{j_2} \dots \Delta_{j_n} \Big),$$

where the summation is taken over all permutations $(j_1, \dots, j_n)$ of n numbers $1, 2, \dots, n$; σ is number of transpostions in permutation $(j_1, \dots, j_n)$. (For example, $J_2(\mathbf{T}) = \det(\Delta_1\Delta_2 - \Delta_2\Delta_1)$, $J_3(\mathbf{T}) = \det(\Delta_1\Delta_2\Delta_3 + \Delta_2\Delta_3\Delta_1 + \Delta_3\Delta_1\Delta_2 - \Delta_1\Delta_3\Delta_2 - \Delta_2\Delta_1\Delta_3 - \Delta_3\Delta_2\Delta_1)$.)

Theorem A1. *The function $J_n(\mathbf{T})$ is a relative invariant of weight n of the groups $\mathrm{GL}(n, C)$ with respect to the operation mentioned above.*

Proof. Consider the sequence of transformations

$$\begin{aligned} S(\mathbf{T}) &= S^{-1}\cdot\mathbf{T}\cdot(S\otimes S) \\ &= S^{-1}(\Delta_1,\dots,\Delta_n)\cdot(S\otimes S) \\ &= S^{-1}\Big(\sum_{i=1}^{n}\Delta_i s_{i1},\dots,\sum_{i=1}^{n}\Delta_i s_{in}\Big)S. \end{aligned}$$

The computation of the determinant of the last matrix proves our assertion, i.e., $\det S(\mathbf{T}) = (\det S)^n J_n(\mathbf{T})$. ■

Changing variables $\mathbf{x}(t) = S\mathbf{z}(t)$, we reduce (6) to the form (8). Then the tensor $\mathbf{T}$ is transformed to a new tensor $\mathbf{Q}$.

Theorem A2. $J_n(\mathbf{Q}) = (\det B)g(\mathbf{Q})$, *where* $g(\mathbf{Q})$ *is a polynomial in* $\mathbf{Q}$.

Proof. Let $\det B \neq 0$. It is obvious that

$$\mathbf{Q} = (\Delta_1(S),\dots,\Delta_n(S)),$$

where $\Delta_i(S)$ is an $(n\times n)$ matrix with a unique nonzero column $(\beta_{1i},\dots,\beta_{i-1,i},\beta_{ii},\dots,\beta_{in})^T$, $i=1,\dots,n$.

As $\det B \neq 0$, the function $(\det B)^{-1}\cdot J_n(\mathbf{Q})$ is well defined. Then taking into account the fact that $S^{-1}S = I_n$, we obtain

$$\begin{aligned} &(\det B)^{-1}\cdot J_n(\mathbf{Q}) \\ &\quad = \det\Big(\sum_{1\le j_1,\dots,j_n\le n}(-1)^{\sigma}B^{-1}\Delta_{j_1}(S)\Delta_{j_2}(S) \\ &\qquad\qquad \cdots\Delta_{j_n}(S)\Big). \end{aligned}$$

But for any $\forall\, j_1 \in \{1,\dots,n\}$ the matrix $B^{-1}\Delta_{j_1}$ has a unique nonzero element equal to unity. Thus the function $(\det B)^{-1}\cdot J_n(\mathbf{Q}) = g(\mathbf{Q})$ is a polynomial. The proof of Theorem A2 is thus completed. ■

Theorem A3. *Assume that the tensor* $\mathbf{T}_0$ *defines a regular but non-WIS system. Then* $J_n(\mathbf{T}_0) = 0$.

Proof. As $\mathbf{T}_0$ is not a WIS system, there exists a transformation S such that while passing from the system (6) to the system (8), the matrix B gets a block-triangular structure. For simplicity, assume that in this matrix we have $\beta_{21} = \cdots = \beta_{n1} = 0$. Then

$$\begin{aligned} &\sum_{1\le j_1,\dots,j_n\le n}(-1)^{\sigma}\Delta_{j_1}(S)\Delta_{j_2}(S)\dots\Delta_{j_n}(S) \\ &\qquad = M_1\Delta_1(S)+\cdots+M_n\Delta_n(S), \end{aligned}$$

where in the matrix M_i the i-th column is equal to zero and in the matrix $\Delta_i(S)$ the i-th column is not equal to zero. But the matrix $\Delta_1(S)$ has a unique nonzero element $-\beta_{11}$. Therefore $M_1\Delta_1(S) = 0$, and in the matrix $M_1\Delta_1(S)+\cdots+M_n\Delta_n(S)$ the first column is zero. This completes the proof. ■

Theorems A1, A2 and A3 imply the following result:

Corollary A1. *Let* $\mathbf{M}\subset\Psi$ *be the set of all tensors such that if* $\mathbf{T}\in\mathbf{M}$, *then* $J_n(\mathbf{T})\neq 0$. *Then* $\mathbf{M}$ *is an open and everywhere dense invariant subset in* Ψ. *Consequently, any system from* $\mathbf{M}$ *is a regular WIS system.*

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