THE RELATIONSHIP BETWEEN THE INFINITE EIGENVALUE ASSIGNMENT FOR SINGULAR SYSTEMS AND THE SOLVABILITY OF POLYNOMIAL MATRIX EQUATIONS

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Two related problems, namely the problem of the infinite eigenvalue assignment and that of the solvability of polynomial matrix equations are considered. Necessary and sufficient conditions for the existence of solutions to both the problems are established. The relationships between the problems are discussed and some applications from the field of the perfect observer design for singular linear systems are presented.

Keywords: assignment, infinite eigenvalue, singular, polynomial matrix equation, system, relationship

1. Introduction

It is well known (Dai, 1989; Kaliath, 1980; Wonham, 1979; Kaczorek, 1993; Kučera, 1981) that if the pair (A, B) of a standard linear system $\dot{x} = Ax + Bu$ is controllable, then there exists a state-feedback gain matrix K such that $\det[I_n s - A + BK] = p(s)$, where $p(s) = s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0$ is a given arbitrary n-degree polynomial. By changing K we can modify arbitrarily only the coefficients $a_0, a_1, \ldots, a_{n-1}$, but we are not able to change the degree n of the polynomial which is determined by the matrix $I_n s$. In singular linear systems we are also able to change the degree of closed-loop characteristic polynomials by a suitable choice of the state-feedback matrix K. The problem of finding a state-feedback matrix K such that $\det[Es - A + BK] = \alpha \neq 0$ (α is independent of s) was considered in (Kaczorek, 2002b; Chu and Ho, 1999).

The polynomial equation approach to linear control systems has been considered in many papers and books (Kučera, 1972; Kučera, 1981; Kučera, 1979; Kaczorek, 1993). In this paper a new approach to solve the problems will be proposed. The problem of the infinite eigenvalue assignment is closely related to that of finding a solution $X = I_n$, $Y = K$ to the polynomial matrix equation $[Es - A]X + BY = U(s)$ for an unimodular matrix $U(s)$ with $\det U(s) = \alpha$. Necessary and sufficient conditions for the existence of a solution (X, Y) to the polynomial matrix equation are established. The relationship between the problems is discussed and some applications from the field of the perfect observer design for singular linear systems are presented.

2. Problem Formulation

Let $\mathbb{R}^{n\times m}$ be the set of $n \times m$ real matrices and $\mathbb{R}^n = \mathbb{R}^{n\times 1}$. Consider the continuous-time linear system

$$E\dot{x} = Ax + Bu, \tag{1}$$

where $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$ are respectively the semistate and input vectors, $\dot{x} = \mathrm{d}x/\mathrm{d}t$, and $E, A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$. The system (1) is called singular if $\det E = 0$, and it is called standard when $\det E \neq 0$.

It is assumed that $\operatorname{rank} E = r < n$, $\operatorname{rank} B = m$ and the pair (E, A) is regular, i.e.

$$\det[Es - A] \neq 0 \tag{2}$$

for some $s \in \mathbb{C}$ (the field of complex numbers).

Let us consider the system (1) with the state-feedback

$$u = v - Kx, \tag{3}$$

where $v \in \mathbb{R}^m$ is a new input and $K \in \mathbb{R}^{m\times n}$ is a gain matrix. From (1) and (3) we have

$$E\dot{x} = (A - BK)x + Bv. \tag{4}$$

Problem 1. *Given the matrices E, A, B of (1) and a nonzero scalar α (independent of s), find $K \in \mathbb{R}^{m\times n}$ such that*

$$\det\,[Es - A + BK] = \alpha. \tag{5}$$

Let $\mathbb{R}^{n\times m}[s]$ be the set of $n\times m$ polynomial matrices in s with real coefficients and $U(s) \in \mathbb{R}^{n\times n}[s]$ be a unimodular matrix such that $\det U(s) = \alpha$. Then (5) can be written as

$$\det\left\{\big[Es - A, B\big]\begin{bmatrix} I_n \\ K \end{bmatrix}\right\} = \det U(s), \tag{6}$$

where I_n stands for the identity matrix, and

$$\big[Es - A\big]X + BY = U(s), \tag{7}$$

in which

$$X = I_n, \quad Y = K. \tag{8}$$

Therefore, the following problem associated with Problem 1 can be formulated:

Problem 2. *Given the matrices $Es-A, B$ and $U(s)$ with $\det U(s) = \alpha$, find a solution X, Y of the polynomial matrix equation (7) satisfying (8).*

In this paper necessary and sufficient conditions for the existence of solutions to both the problems are established and procedures for the computation of K are proposed. The relationships between the problems are also discussed.

3. Solution of Problem 1

It is well known (Dai, 1989; Kaczorek, 1993) that the system (1) is completely controllable if and only if

$$\operatorname{rank}\big[Es - A, B\big] = n \tag{9a}$$

for all finite $s \in \mathbb{C}$, and

$$\operatorname{rank}\,[E, B] = n. \tag{9b}$$

The solution of Problem 1 is based on the following lemma (Chu and Ho, 1999).

Lemma 1. *If the condition (2) is satisfied, then orthogonal matrices U and V exist such that*

$$U[Es - A]V = \begin{bmatrix} E_1 s - A_1 & * \\ 0 & E_0 s - A_0 \end{bmatrix},$$

$$UB = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}, \tag{10a}$$

where $E_1, A_1 \in \mathbb{R}^{n_1\times n_1}$, $E_0, A_0 \in \mathbb{R}^{n_0\times n_0}$, $B_1 \in \mathbb{R}^{n_1\times m}$, the subsystem (E_1, A_1, B_1) is completely controllable, the pair (E_0, A_0) is regular, E_1 is upper triangular and '$*$' denotes some unimportant matrix. Moreover, the matrices E_1, A_1 and B_1 are of the forms

$$E_1 s - A_1 = \begin{bmatrix} E_{11}s - A_{11} & E_{12}s - A_{12} & \cdots & E_{1,k-1}s - A_{1,k-1} & E_{1k}s - A_{1k} \\ -A_{21} & E_{22}s - A_{22} & \cdots & E_{2,k-1}s - A_{2,k-1} & E_{2k}s - A_{2k} \\ 0 & -A_{32} & \cdots & E_{3,k-1}s - A_{3,k-1} & E_{3k}s - A_{3k} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & -A_{k,k-1} & E_{kk}s - A_{kk} \end{bmatrix},$$

$$B_1 = \begin{bmatrix} B_{11} \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \tag{10b}$$

where $E_{ij}, A_{ij} \in \mathbb{R}^{\bar{n}_i\times\bar{n}_j}$, $i, j = 1, \dots, k$ and $B_{11} \in \mathbb{R}^{\bar{n}_i\times m}$, $\sum_{i=1}^{n} \bar{n}_i = n_1$ with $B_{11}, A_{21}, \dots, A_{k,k-1}$ of full row ranks and nonsingular $E_{22}, \dots, E_{kk}$.

Theorem 1. *Let the condition (2) be satisfied and the matrices E, A, B of (1) be transformed into the forms (10). A matrix K satisfying (5) exists if and only if*

(i) the subsystem (E_1, A_1, B_1) is singular, i.e.

$$\det E_1 = 0, \tag{11a}$$

(ii) if $n_0 > 0$, then the degree of the polynomial $\det\,[E_0 s - A_0]$ is zero, i.e.

$$\deg\det\,[E_0 s - A_0] = 0 \quad \textit{for} \quad n_0 > 0. \tag{11b}$$

Proof. (cf. Chu and Ho, 1999)

(Necessity) From (5) and (10a) we have

$$\begin{aligned} \det\,[Es - A + BK] &= \det U^{-1} \det V^{-1} \\ &\quad \times \det\,[E_1 s - A_1 + B_1\bar{K}] \\ &\quad \times \det\,[E_0 s - A_0] = \alpha, \end{aligned} \tag{12}$$

where $\bar{K} = KV \in \mathbb{R}^{m\times n}$ and $\det\,[E_0 s - A_0] = 1$ if $n_0 = 0$. From (12) it follows that the condition (5) holds only if the conditions (11) are satisfied.

(Sufficiency) Consider first the single-input case $(m = 1)$. We have

$$E_1 = \begin{bmatrix} e_{11} & e_{12} & \cdots & e_{1n_1} \\ 0 & e_{22} & \cdots & e_{2n_1} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & e_{n_1 n_1} \end{bmatrix},$$

$$A_1 = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1,n_1-1} & a_{1n_1} \\ a_{21} & a_{22} & \cdots & a_{2,n_1-1} & a_{2n_1} \\ 0 & a_{31} & \cdots & a_{3,n_1-1} & a_{3n_{1a}} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & a_{n_1,n_1-1} & a_{n_1 n_1} \end{bmatrix},$$

$$B_1 = b_1 = \begin{bmatrix} b_{11} \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \tag{13}$$

where $e_{ii} \neq 0$, $a_{i,i-1} \neq 0$ for $i = 2, \ldots, n_1$ and $b_{11} \neq 0$.

The condition (11a) implies $e_{11} = 0$. Premultiplying the matrix $[E_1 s - A_1, b_1]$ by an orthogonal matrix of row operations P_1, it is possible to make the entries $e_{12}, e_{13}, \ldots, e_{1n_1}$ of E_1 zero since $e_{ii} \neq 0$, $i = 2, \ldots, n_1$. By this reduction only the entries of the first row of A_1 are modified. We get

$$\bar{E}_1 = P_1 E_1 = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & e_{22} & \cdots & e_{2n_1} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & e_{n_1 n_1} \end{bmatrix},$$

$$\bar{A}_1 = P_1 A_1 = \begin{bmatrix} \bar{a}_{11} & \bar{a}_{12} & \cdots & \bar{a}_{1,n_1-1} & \bar{a}_{1n_1} \\ a_{21} & a_{22} & \cdots & a_{2,n_1-1} & a_{2n_1} \\ 0 & a_{31} & \cdots & a_{3,n_1-1} & a_{3n_1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & a_{n_1,n_1-1} & a_{n_1 n_1} \end{bmatrix},$$

$$\bar{b}_1 = P_1 b_1 = b_1. \tag{14}$$

Let

$$\bar{k}_1 = \frac{1}{b_{11}} \left[-\bar{a}_{11}, -\bar{a}_{12}, \ldots, -\bar{a}_{1,n_1-1}, 1 - \bar{a}_{1n_1} \right]. \tag{15}$$

Using (12), (14) and (15), we obtain

$$\begin{aligned} &\det \left[\bar{E}_1 s - \bar{A}_1 + \bar{b}_1 \bar{k}_1 \right] \\ &= \begin{vmatrix} 0 & 0 & \cdots & 0 & 1 \\ -a_{21} & e_{22}s - a_{22} & \cdots & e_{2,n_1-1}s - a_{2,n_1-1} & e_{2n_1}s - a_{2n_1} \\ 0 & -a_{31} & \cdots & e_{3,n_1-1}s - a_{3,n_1-1} & e_{3n_{1a}}s - a_{3n_{1a}} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & -a_{n_1,n_1-1} & e_{n_1 n_1}s - a_{n_1 n_1} \end{vmatrix} \\ &= a_{21} a_{31} \cdots a_{n_1,n_1-1} = \bar{\alpha}, \end{aligned} \tag{16}$$

where $\bar{\alpha} = \alpha \det U \det V \det P_1 \det [E_0 s - A_0]^{-1}$.

The above can be easily extended to multi-input systems, i.e. $m > 1$. In this case the matrix P_1 of orthogonal row operations is chosen so that all the entries of the first row of $\bar{E}_1 = P_1 E_1$ be zero. By this reduction only the entries of A_{1i}, $i = 1, \ldots, k$ and B_{11} will be modified. The modified matrices will be denoted by $\bar{A}_{1i}$, $i = 1, \ldots, k$ and $\bar{B}_{11}$, respectively.

Let

$$\bar{K} = \bar{B}_1^{-1} \{ [\bar{A}_{11}, \bar{A}_{12}, \ldots, \bar{A}_{1k}] + \hat{E} \}. \tag{17}$$

The matrix $\hat{E} \in \mathbb{R}^{m \times n}$ in (17) is chosen so that

$$\begin{aligned} &\bar{E}_1 s - \bar{A}_1 + \bar{B}_1 \bar{K} \\ &= \begin{bmatrix} 0 & 0 & \cdots & 0 & (-1)^{l+1} h \\ \bar{a}_{21} & * & \cdots & * & * \\ 0 & \bar{a}_{32} & \cdots & * & * \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \bar{a}_{l,l-1} & * \end{bmatrix}, \end{aligned} \tag{18}$$

where ‘$*$’ denotes unimportant entries,

$$h = \frac{\alpha (-1)^{l+1}}{\bar{a}_{21} \bar{a}_{32} \ldots \bar{a}_{l,l-1} c},$$

$$c = \det U^{-1} \det V^{-1} \det P_1^{-1} \det [E_0 s - A_0].$$

Using (12), (17) and (18), it is easy to verify that

$$\det [Es - A + BK] = c \det [\bar{E}_1 s - \bar{A}_1 + \bar{B}_1 \bar{K}] = \alpha. \tag{19}$$

■

Remark 1. For $m > 1$ there exist many different matrices K satisfying the condition (5).

Remark 2. If the system order is not high, elementary row and column operations can be used instead of orthogonal operations.

Example 1. For the singular system (1) with

$$E = \begin{bmatrix} 0 & 2 & 1 & 0 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$A = \begin{bmatrix} 1 & -1 & 0 & 1 \\ 0 & 1 & 2 & 0 \\ -1 & 0 & 1 & -1 \\ 0 & 0 & 2 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \tag{20}$$

we wish to find a gain matrix $K \in \mathbb{R}^{2\times 4}$ such that the condition (5) is satisfied for $\alpha = 1$.

In this case the pair (E, A) is regular since

$$\det[Es - A] = \begin{vmatrix} -1 & 2s+1 & s & -1 \\ 0 & s-1 & -s-2 & 2s \\ 1 & 0 & s-1 & 1-s \\ 0 & 0 & -2 & s-1 \end{vmatrix} = (1-2s)(s-1)^2.$$

The matrices (20) have already the desired forms (10) with $E_1 = E$, $A_1 = A$, $B_1 = B$, $n_1 = n = 4$, $\bar{n}_1 = 2$, $\bar{n}_2 = \bar{n}_3 = 1$, $m = 2$ and

$$E_{11} = \begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix}, \quad E_{12} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad E_{13} = \begin{bmatrix} 0 \\ 2 \end{bmatrix},$$

$$E_{22} = [1], \quad E_{23} = [-1], \quad E_{33} = [1]$$

$$A_{11} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}, \quad A_{12} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \quad A_{13} = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

$$A_{21} = [-10], \quad A_{22} = [1], \quad A_{23} = [-1], \quad A_{32} = [2],$$

$$A_{33} = [1], \quad B_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Using elementary row and column operations (Kaczorek, 1993; Kaliath, 1980), we obtain

$$P_1 = \begin{bmatrix} 1 & -2 & -3 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and

$$[\bar{E}_1 s - \bar{A}_1, \bar{B}_1] = P_1[Es - A, B] = \begin{bmatrix} -4 & 3 & 5 & -5 & 1 & -2 \\ 1 & s-1 & -1 & 2 & 0 & 1 \\ 1 & 0 & s-1 & 1-s & 0 & 0 \\ 0 & 0 & -2 & s-1 & 0 & 0 \end{bmatrix}.$$

Taking into account that in this case

$$\hat{E} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -0.5 \end{bmatrix},$$

$$[\bar{A}_{11}, \bar{A}_{12}, \bar{A}_{13}] = \begin{bmatrix} 4 & -3 & -5 & 5 \\ -1 & 1 & 1 & -2 \end{bmatrix},$$

$$\bar{B}_1 = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$$

and using (17), we obtain

$$K = \bar{K} = \bar{B}_1^{-1}\{[\bar{A}_{11}, \bar{A}_{12}, \bar{A}_{13}] + \hat{E}\} = \begin{bmatrix} 2 & -2 & -5 & 0 \\ -1 & 1 & 1 & -2.5 \end{bmatrix}.$$

It is easy to check that

$$\det[Es - A + BK] = \begin{vmatrix} 1 & 2s-1 & s-3 & -1 \\ -1 & s & -s & 2s-2.5 \\ 1 & 0 & s-1 & -s+1 \\ 0 & 0 & -2 & s-1 \end{vmatrix} = 1.$$

If a matrix K satisfying (5) exists, then it can be also computed with the use of the following procedure: Calculate

$$\det[Es - A + BK] = a_r s^r + a_{r-1} s^{r-1} + \cdots + a_1 s + a_0, \quad r < \operatorname{rank} E, \tag{21}$$

where the coefficients $a_i = a_i(K)$, $i = 0, 1, \dots, r$ depend on the entries of K. Equating the coefficients related to the same powers of s in (21) and (5) yields

$$a_0(K) = \alpha, \quad a_i(K) = 0, \quad i = 1, \dots, r. \tag{22}$$

Solving (22), we can determine the entries of K. ♦

4. Solution of Problem 2 and Relationship between Both Problems

Theorem 2. *Problem 2 has a solution only if*

$$\operatorname{rank}[Es - A, B] = n \tag{23}$$

for all finite $s \in \mathbb{C}$, *and*

$$D = Es - U(s) \tag{24}$$

is a real matrix independent of s.

Proof. From the equality

$$Es - A + BK = [Es - A, B] \begin{bmatrix} I_n \\ K \end{bmatrix} \tag{25}$$

it follows that (5) implies (23). From (7) and (8) we have

$$Es - U(s) = A - BK = D \in \mathbb{R}^{n \times n}. \tag{26}$$

Therefore (7) has a solution (8) only if (24) is satisfied. ■

Example 2. Consider the problems for

$$E = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

$$B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \alpha = 1 \tag{27}$$

and the following two situations:

Case A:

$$U(s) = \begin{bmatrix} 1 & s \\ 0 & \alpha \end{bmatrix},$$

Case B:

$$U(s) = \begin{bmatrix} s & 1 \\ -\alpha & 0 \end{bmatrix}.$$

Problem 1 has a solution since for $K = [k_1 k_2]$ we have

$$\det[Es - A + BK] = \begin{vmatrix} s + k_1 & k_2 - 1 \\ -1 & 0 \end{vmatrix} = k_2 - 1 = \alpha$$

for $k_2 = 1 + \alpha = 2$ and arbitrary k_1. Problem 2 in Case A has no solution since the condition (24) is not satisfied.

The matrix

$$D = Es - U(s) = \begin{bmatrix} s - 1 & -s \\ 0 & -\alpha \end{bmatrix}$$

is a polynomial matrix (not a real matrix).

For Case B the condition (24) is satisfied since the matrix

$$D = Es - U(s) = \begin{bmatrix} 0 & 1 \\ -\alpha & 0 \end{bmatrix}$$

is real. Problem 2 has the solution $K = [0\ 2]$ since

$$Es - A + BK = \begin{bmatrix} s + k_1 & k_2 - 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} s & 1 \\ -\alpha & 0 \end{bmatrix}$$

and from a comparison of the corresponding entries we obtain $k_1 = 0$ and $k_2 = 2$. ■

Let the matrices E, A and B of (7) satisfy the conditions (23) and (24). If the system (E, A, B) is completely controllable, then by Lemma 1 there exist orthogonal matrices P and Q such that

$$\tilde{E} = PEQ = \begin{bmatrix} \tilde{E}_{11} & \tilde{E}_{12} & \cdots & \tilde{E}_{1k} \\ 0 & \tilde{E}_{22} & \cdots & \tilde{E}_{2k} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \tilde{E}_{kk} \end{bmatrix},$$

$$\tilde{A} = PAQ = \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} & \cdots & \tilde{A}_{1,k-1} & \tilde{A}_{1k} \\ \tilde{A}_{21} & \tilde{A}_{22} & \cdots & \tilde{A}_{2,k-1} & \tilde{A}_{2k} \\ 0 & \tilde{A}_{32} & \cdots & \tilde{A}_{3,k-1} & \tilde{A}_{3k} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \tilde{A}_{k,k-1} & \tilde{A}_{kk} \end{bmatrix},$$

$$\tilde{B} = PB = \begin{bmatrix} \tilde{B}_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{28}$$

with $B_{11} \in \mathbb{R}^{\hat{n}_1 \times m}$, $A_{i,i-1} \in \mathbb{R}^{\tilde{n}_i \times \tilde{n}_{i-1}}$, $i = 2, \dots, k$ of full row ranks and nonsingular $E_{11} \in \mathbb{R}^{\tilde{n}_i \times \tilde{n}_i}$, $i = 2, \dots, k$.

Premultiplying (7) by the matrix P, postmulplying the result by Q and using (28), we obtain

$$P[Es - A]Q + PBKQ = \tilde{E}s - \tilde{A} + \tilde{B}\tilde{K} = \tilde{U}(s), \tag{29}$$

where $\tilde{K} = KQ$ and $\tilde{U}(s) = PU(s)Q$. From the equality

$$P[Es - U(s)]Q = PDQ = \tilde{D} = \tilde{E}s - \tilde{U}(s)$$

it follows that if D is a real matrix, then so is $\tilde{D}$.

Let

$$\tilde{D} = \begin{bmatrix} \tilde{D}_1 \\ \tilde{D}_2 \end{bmatrix}, \quad \tilde{A} = \begin{bmatrix} \tilde{A}_1 \\ \tilde{A}_2 \end{bmatrix},$$

where $\tilde{D}_1, \tilde{A}_1 \in \mathbb{R}^{\tilde{n}_1 \times n}, \tilde{D}_2, \tilde{A}_2 \in \mathbb{R}^{(n-\tilde{n}_1)\times n}$. From (28) and (29) we see that

$$\begin{bmatrix} \tilde{D}_1 \\ \tilde{D}_2 \end{bmatrix} = \begin{bmatrix} \tilde{A}_1 \\ \tilde{A}_2 \end{bmatrix} - \begin{bmatrix} \tilde{B}_1 \\ 0 \end{bmatrix} \tilde{K}$$

and

$$\tilde{D}_1 = \tilde{A}_1 - \tilde{B}_1 \tilde{K}, \tilde{D}_2 = \tilde{A}_2. \tag{30}$$

Therefore we have the following result:

Theorem 3. *Let the matrices E, A, B satisfy the assumptions (9) and (26) and let them be transformed into the forms (28). Equation (7) has a solution X, Y satisfying (8) if and only if*

$$\tilde{D}_2 = \tilde{A}_2. \tag{31}$$

Proof. The necessity of (31) follows immediately from (30). If the assumption (26) is satisfied, then D is a real matrix and so is $\tilde{D}$. The matrix $\tilde{B}_1$ is nonsingular, and from (30) we obtain

$$\tilde{K} = \tilde{B}_1^{-1}[\tilde{A}_1 - \tilde{D}_1]$$

and

$$X = K = \tilde{K}Q^{-1} = \tilde{B}_1^{-1}[\tilde{A}_1 - \tilde{D}_1]Q^{-1}. \tag{32}$$

∎

Remark 3. From a comparison of Theorems 2 and 3 and Example 2 it follows that the solvability conditions for Problem 2 are more restrictive than those for Problem 1.

Example 3. Find a solution (8) of Eqn. (7) with

$$E = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} -1 & 1 & 0 \\ 1 & 2 & -1 \\ 0 & 2 & 1 \end{bmatrix},$$

$$B = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad U(s) = \begin{bmatrix} 1 & -1 & s \\ 0 & -\alpha & 0 \\ 0 & s-2 & -1 \end{bmatrix}. \tag{33}$$

In this case the assumptions (9) and (26) are satisfied since

$$\operatorname{rank}[Es - A, B] = \operatorname{rank} \begin{bmatrix} 1 & -1 & s & 0 \\ -1 & -2 & 1 & 1 \\ 0 & s-2 & -1 & 0 \end{bmatrix} = 3$$

for all finite $s \in \mathbb{C}$,

$$\operatorname{rank}[E, B] = \operatorname{rank} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} = 3$$

and the matrix

$$D = Es - U(s) = \begin{bmatrix} -1 & 1 & 0 \\ 0 & \alpha & 0 \\ 0 & 2 & 1 \end{bmatrix} \tag{34}$$

is real.

The orthogonal matrices $P, Q \in \mathbb{R}^{3\times 3}$ transforming (33) to (28) have the forms

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \tag{35}$$

and

$$\tilde{E} = PEQ = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$\tilde{A} = PAQ = \begin{bmatrix} \tilde{A}_1 \\ \tilde{A}_2 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 2 \\ -1 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix},$$

$$\tilde{B} = PB = \begin{bmatrix} \tilde{B}_1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -- \\ 0 \\ 0 \end{bmatrix}. \tag{36}$$

Using (30), (34) and (35), we obtain

$$\tilde{D} = PDQ = \begin{bmatrix} \tilde{D}_1 \\ \tilde{D}_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & \alpha \\ \hline -1 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix}. \tag{37}$$

From (36) and (37) it follows that the condition (31) is satisfied and Eqn. (7) with (33) has a solution X, Y satisfying (8).

Using (32), (36) and (37), we obtain

$$\begin{aligned} X = K &= \tilde{B}_1^{-1}[\tilde{A}_1 - \tilde{D}_1]Q^{-1} \\ &= [1, -1, 2-\alpha] \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \\ &= [1, 2-\alpha, -1] \end{aligned} \tag{38}$$

It is easy to check that (38) and $Y = I_3$ satisfy (7) with (33). ♦

5. Applications

Consider the singular system

$$E\dot{x} = Ax + Bu, \tag{39a}$$

$$y = Cx, \tag{39b}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ and $y \in \mathbb{R}^p$ are the semistate, input and output vectors, respectively, and $E, A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$ with $\det E = 0$. It is assumed that $\operatorname{rank} C = p$ and (2) holds.

The singular system

$$E\hat{x} = A\hat{x} - Bu - K(C\hat{x} - y), \quad \hat{x}(0) = \hat{x}_0, \\ \hat{x} \in \mathbb{R}^m, \quad K \in \mathbb{R}^{n\times p} \tag{40}$$

is called a full-order perfect observer of the system (39) if and only if $\hat{x}(t) = x(t)$ for $t > 0$ and any initial conditions x_0 and $\hat{x}_0$ of (39) and (40).

It was shown (Kaczorek, 2000) that there exists a full-order perfect observer (40) of the system (39) if the system is completely observable, i.e.

$$\operatorname{rank} \begin{bmatrix} Es - A \\ C \end{bmatrix} = n \tag{41a}$$

for all finite $s \in \mathbb{C}$, and

$$\operatorname{rank} \begin{bmatrix} E \\ C \end{bmatrix} = n. \tag{41b}$$

In this case a matrix K exists such that

$$\det [Es - A + KC] = \alpha, \tag{42}$$

where α is a nonzero scalar independent of s. Note that by a transposition of (42) we obtain (5). Therefore the design problem of the observer (40) for the system (39) has been reduced to Problem 1.

The design problem of reduced-order perfect observers and of perfect functional observers for the system (39) can also be reduced to Problem 1 (Kaczorek, 2000; 2002a).

Consider the singular system (39) with the state-feedback (3). The transfer matrix of the closed-loop system described by (4) and (39b) is given by $T(s) = C[Es - A + BK]^{-1}B$. If $[Es - A + BK] = U(s)$ with $U(s)$ being unimodular, then the transfer matrix $T(s) = CU^{-1}(s)B$ is a polynomial matrix. Therefore, finding a solution (8) of (7) is equivalent to finding a state-feedback gain matrix K such that the closed-loop transfer matrix is polynomial.

6. Concluding Remarks

Two related problems, namely the problem of the infinite eigenvalue assignment and that of the solvability of polynomial matrix equations have been considered. Necessary and sufficient conditions for the existence of solutions to both the problems were established. Relationships between the problems were discussed and some applications from the field of the perfect observer design for singular linear systems were presented. The deliberations were illustrated by numerical examples. With slight modifications the deliberations can be extended to singular discrete-time linear systems. An extension towards two-dimensional linear systems (Kaczorek, 1993) is also possible.

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